QUADRATIC SYSTEMS WITH A WEAK FOCUS

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In this paper we study the number and the relative position of the limit cycles of a plane quadratic system with a weak focus. In particular, we prove the limit cycles of such a system can never have $(2, 2)$-distribution, and that there is at most one limit cycle not surrounding this weak focus under any one of the following conditions:

   (i)   the system has at least 2 saddles in the finite plane,

   (ii)   the system has more than 2 finite singular points and more than 1 singular point at infinity,

   (iii)   the system has exactly 2 finite singular points, more than 1 singular point at infinity, and the weak focus is itself surrounded by at least one limit cycle.

1. INTRODUCTION

This paper discusses quadratic systems with a weak focus and a strong focus. Without loss of generality, we may suppose that $O(0, 0)$ is the strong focus and $N(0, 1)$ the weak focus. Then, by [6], the quadratic system can be written in the form

$$(1) \qquad \begin{aligned} dx/dt &= -y - mx + lx^2 + mxy + y^2, \\ dy/dt &= x(1 + ax + by). \end{aligned}$$

If $m = 0$ then $N(0, 1)$ and $O(0, 0)$ are both weak foci (or centres), and (1) has no limit cycle. Therefore

$$(2) \qquad 0 < |m| < 2, \qquad b + 1 < 0.$$

Without loss of generality we can suppose

$$(3) \qquad a > 0.$$

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In fact, if $a = 0$ then (1) has no limit cycle [2], and if $a < 0$ then by the change of variables $t \to -t$, $x \to -x$ we obtain $a > 0$. In the following we always suppose that (2) and (3) are satisfied when we discuss (1).

Except N and O, the coordinates (x, y) of the other finite singular points of (1) are determined by the following equations:

(4) $(lb^2 + a^2 - abm)x^2 + [a(b + 2) - mb(b + 1)]x + b + 1 = 0,$

(5) $y = -(1 + ax)/b.$

The discriminant of (4) is $b^2\Delta$, where

(6) $\Delta := [a - m(b + 1)]^2 - 4l(b + 1).$

The following results, which we state as lemmas, are proved in [8].

LEMMA 1. *If $ma(b + 2l) \geqslant 0$ or $a - mb \leqslant 0$, then (1) has no limit cycle surrounding O.*

LEMMA 2. *If $M < 0$, or if $m > 0$ and $lb^2 + a^2 - abm > 0$, then (1) has at most one limit cycle surrounding O.*

In order to consider the infinite singular points of (1), let

(7) $v = x/y, \qquad z = 1/y, \qquad dt/d\tau = z.$

Then (1) becomes

(8)
$$dz/d\tau = -z(vz + av^2 + bv),$$
$$dv/d\tau = -av^3 - (b - l)v^2 + mv + 1 + z(-v^2 - mv - 1).$$

Thus $z = 0$, $v = \lambda$ is an infinite singular point if and only if λ satisfies the equation

(9) $\varphi(\lambda) := a\lambda^3 + (b - l)\lambda^2 - m\lambda - 1 = 0.$

LEMMA 3. *Suppose $m > 0$ and put*

(10) $\Delta_0 := 4am^3 + (b - l)^2 m^2 + 18a(b - l)m - 27a^2 + 4(b - l)^3.$

If $\Delta_0 > 0$ then (9) has three real roots $\lambda_3 < \lambda_2 < 0 < \lambda_1$ and $\varphi'(\lambda_3) > 0$. If $\Delta_0 = 0$ then (9) has a simple real root $\lambda_1 > 0$, a double real root $\lambda_2 = \lambda_3 < 0$ and $\varphi'(\lambda_3) = 0$. If $\Delta_0 < 0$ then (9) has a unique real root λ_1 and $\lambda_1 > 0$. In all cases we have $\varphi'(\lambda_1) > 0$.

PROOF: By elementary algebra, (10) is the discriminant of (9). Since $\varphi(0) = -1$, $\varphi(+\infty) = +\infty$, the cubic equation (9) has a real root $\lambda_1 > 0$. If $\Delta_0 < 0$ then λ_1 is the unique real root of (9). If $\Delta_0 > 0$ then, since $\varphi'(0) = -m < 0$, the other two real roots are negative. Similarly if $\Delta_0 = 0$ the double root is negative. The rest of the lemma is obvious.

2. Main results

Lemma 2 can be given a more geometrical equivalent form. It follows from (4) that $lb^2 + a^2 - abm > 0$ if and only if (1) has four finite singular points which form a convex quadrilateral. If $m < 0$ and (1) has a limit cycle surrounding O then, by Lemma 1, $a - mb > 0$ and $b + 2l > 0$, hence $l > 0$. Therefore $lb^2 + a^2 - abm > 0$. By Berlinskii's Theorem [7], this is equivalent to assuming that the system (1) has two saddles in addition to the two foci. Thus, by Lemma 2, we have

THEOREM A. *Suppose a quadratic system has 4 finite singular points: a strong focus, a weak focus, and two saddles. Then the system has at most one limit cycle surrounding the strong focus.*

This paper will discuss the remaining case of a quadratic system with a strong focus, a weak focus and at most one saddle. That is, we consider the system (1) when, in addition to (2) and (3), we also have

$$(11) \qquad m > 0, \qquad lb^2 + a^2 - abm \leqslant 0.$$

Suppose λ is a real root of (9), and let

$$(12) \qquad h(\xi) = (\lambda^2 + m\lambda + 1) - (-2a\lambda^2 + 2l\lambda - b\lambda + m)\xi.$$

Then, by (9), we have

$$(13) \qquad (-2a\lambda^2 + 2l\lambda - b\lambda + m)\lambda = (1 + b)\lambda^2 - (\lambda^2 + m\lambda + 1) - 1 < 0.$$

Thus the coefficient of ξ in (12) is nonzero and $h(x - \lambda y) = 0$ represents a line. Since

$$1 - \frac{-(\lambda^2 + m\lambda + 1)}{\lambda(-2a\lambda^2 + 2l\lambda - b\lambda + m)} = \frac{(b+1)\lambda^2 - 1}{\lambda(-2a\lambda^2 + 2l\lambda - b\lambda + m)} > 0,$$

the intersection point of the line $h(x - \lambda y) = 0$ and the y-axis is between N and O. Let

$$(14) \qquad D := \{(x, y) \mid h(x - \lambda y) \geqslant 0\}$$

be the closed half-plane bounded by this line which contains the origin. In this paper we prove the following results, which deal with four possible situations.

THEOREM 1. *Suppose (1) satisfies (11) and $\Delta_0 < 0$. If in D, besides the singular point O, there is at least one finite singular point of (1) (this singular point cannot be N), then (1) has at most one limit cycle surrounding the strong focus O.*

THEOREM 2. *Suppose (1) satisfies (11) and $\Delta_0 < 0$. If in D, besides the singular point O, there is no finite singular point of (1), then (1) has an odd number of limit*

cycles surrounding the strong focus O (a cycle of multiplicity k being counted as k cycles).

THEOREM 3. Suppose (1) satisfies (11) and $\Delta_0 \geqslant 0$, $\Delta \geqslant 0$. Then (1) has at most one limit cycle surrounding the strong focus O.

THEOREM 4. Suppose (1) satisfies (11) and $\Delta_0 \geqslant 0$, $\Delta < 0$. Then, if there is a limit cycle surrounding the weak focus, there is at most one limit cycle surrounding the strong focus.

Since any limit cycle of a quadratic system surrounds a focus, Theorem A corresponds to condition (i) in the Abstract, and it follows from Lemma 3 that Theorem 3, in conjunction with Lemma 2, corresponds to (ii). Similarly, Theorem 4 in conjunction with Lemma 2 corresponds to (iii).

In [4] the authors study quadratic systems with a 3rd-order weak focus and a unique, simple infinite singular point (which implies that there are exactly two finite singular points, the second being a strong focus). Our Theorems 1 and 2 also deal with the case of a unique infinite singular point. However, besides a weak focus and a strong focus, the system may have other finite singular points, the weak focus need not be of 3rd-order, and the unique infinite singular point need not be simple. Thus Theorems 1 and 2 in this paper considerably extend the statement in [4] concerning the number of limit cycles surrounding the strong focus.

In [5] the authors study some quadratic systems with two finite singular points — a weak focus and a strong focus — and three infinite singular points. Theorem 4 of this paper provides additional information about the distribution of limit cycles of such a system.

The limit cycles of a quadratic system are said to have (i, j)-*distribution* if there are exactly i limit cycles surrounding one focus and exactly j limit cycles surrounding another focus (multiple limit cycles being counted according to their multiplicity).

It follows at once from Lemma 2 and Theorems 1 — 4 that if a quadratic system with a weak focus and a strong focus has more than one limit cycle surrounding the strong focus, then either the number of such limit cycles is odd or there is no limit cycle surrounding the weak focus. In particular,

THEOREM 5. *If a quadratic system has a weak focus and a strong focus, its limit cycles cannot have (2, 2) distribution.*

3. PROOFS OF THE THEOREMS

First of all we introduce the following lemmas.

LEMMA 4. *Suppose the system*

$$(15) \qquad dx/dt = y - F(x), \qquad dy/dt = -g(x)$$

satisfies the following conditions:

(i) $g(x) \in C^1$ *and* $F(x) \in C^2$ *for* $x \in (x_{02}, x_{01})$, *where* $x_{02} < 0 < x_{01}$;

(ii) $xg(x) > 0$ *for* $x \in (x_{02}, x_{01})$ *and* $x \neq 0$;

(iii)   *there exists an* x_0, *with* $x_{02} < x_0 < 0$, *such that* $f(x) < 0$ *for* $x \in$ (x_{02}, x_0) *and* $f(x) > 0$ *for* $x \in (x_0, x_{01})$, *where* $f(x) = F'(x)$;

(iv) $[f(x)/g(x)]' < 0$ *for* $x \in (x_{02}, x_0)$ *and* $x \in (0, x_{01})$.

Then the system (15) *has at most one limit cycle in the strip*

$$D_1 := \{(x, y) \mid x_{02} < x < x_{01}, -\infty < y < +\infty\}.$$

If it exists, it must be an unstable cycle.

This lemma follows from Theorem 6.4 of [7] and also from Theorem 1 of [3]. Moreover, the last Theorem shows that if a limit cycle exists, it must have a positive characteristic exponent.

LEMMA 5. *Suppose* (15) *satisfies conditions (i) and (ii) in Lemma 4, and in addition* $(f(x) = F'(x))$

(iii)' $f(0) = 0$, $g'(0) > 0$;

(iv)' $[f(x)/g(x)]' > 0$ *(or* < 0 *) for* $x \in (x_{02}, 0) \cup (0, x_{01})$.

Then (15) *has no limit cycle entirely contained in the strip* D_1.

PROOF: If (15) has a limit cycle entirely contained in D_1 then, by Filippov's transformation — see Theorem 5.4 of [7] — the simultaneous equations

$$F(x_1) = F(x_2), \qquad G(x_1) = G(x_2),$$

where $G(x) = \int_0^x g(\xi)d\xi$, have a solution with $x_{02} < x_2 < 0$ and $0 < x_1 < x_{01}$. For each x with $0 \leqslant x \leqslant x_1$ there is a unique $x' = \varphi(x)$ with $x_2 \leqslant x' \leqslant 0$ such that $G[\varphi(x)] = G(x)$. Evidently $\varphi(x)$ is a decreasing function with $\varphi(0) = 0$, $\varphi(x_1) = \varphi(x_2)$. Since the function $\psi(x) = F[\varphi(x)] - F(x)$ vanishes for $x = 0$ and $x = x_1$, its derivative vanishes at a point $\xi_1 \in (0, x_1)$. Since $\varphi'(x) = g(x)/g[\varphi(x)]$, it follows that

$$f[\varphi(\xi_1)]/g[\varphi(\xi_1)] = f(\xi_1)/g(\xi_1).$$

Thus the simultaneous equations

$$f(\xi_1)/g(\xi_1) = f(\xi_2)/g(\xi_2), \qquad G(\xi_1) = G(\xi_2)$$

have a solution with $x_2 < \xi_2 < 0$, $0 < \xi_1 < x_1$. But (iii)' implies that $f(x)/g(x)$ has a finite limit as $x \to 0$. Consequently, by a similar argument, the derivative of the function

$$f[\varphi(x)]/g[\varphi(x)] - f(x)/g(x)$$

vanishes at a point $\eta_1 \in (0, \xi_1)$. Thus, putting $\eta_2 = \varphi(\eta_1)$, we obtain

$$(f/g)'(\eta_2)\varphi'(\eta_1) = (f/g)'(\eta_1).$$

Since $\varphi'(\eta_1) < 0$, and, by (iv)', $(f/g)'(\eta_2)$ has the same sign as $(f/g)'(\eta_1)$, this is a contradiction. The proof is complete. □

LEMMA 6. *If λ is a real root of (9), then*

$$(16) \qquad lb^2 + a^2 - abm = (a\lambda + b)(b^2\lambda^2 + a\lambda - mb\lambda - b)/\lambda^2.$$

Moreover, if $m > 0$ and $lb^2 + a^2 - abm \leqslant 0$, then

$$(17) \qquad (m + 2l\lambda - a\lambda^2)\lambda < 0.$$

PROOF: Using (9), we obtain the left side of (16) from the right side by direct calculation. In addition, it is clear that $l < 0$ if $m > 0$ and $lb^2 + a^2 - abm \leqslant 0$. Therefore, by (9) we have

$$(m + 2l\lambda - a\lambda^2)\lambda = (b + l)\lambda^2 - 1 < 0.$$

The proof is complete. □

PROOF OF THEOREM 1: Take $\lambda = \lambda_1 > 0$, where λ_1 is defined as in Lemma 3. (For simplicity λ_1 is rewritten as λ in the proofs of Theorems 1 and 2.) Putting

$$(18) \qquad \xi = x - \lambda y, \qquad \eta = y,$$

(1) is changed into

$$(19) \qquad \begin{aligned} d\xi/dt &= -(m + \lambda)\xi + (l - a\lambda)\xi^2 - h(\xi)\eta, \\ d\eta/dt &= (\xi + \lambda\eta)[1 + a\xi + (a\lambda + b)\eta], \end{aligned}$$

where $h(\xi)$ is defined as in (12). If $h(\xi) = 0$ we have, by (13),

$$(20) \qquad \xi = \xi^* := (\lambda^2 + m\lambda + 1)/(-2a\lambda^2 + 2l\lambda - b\lambda + m) < 0.$$

The region D in the (x, y)-plane (see (14)) becomes the region $h(\xi) \geqslant 0$ in the (ξ, η)-plane, namely

$$(14)' \qquad D' := \{(\xi, \eta) \mid \xi \geqslant \xi^*\}.$$

In the (ξ, η) coordinate system, the coordinates of the weak focus N are $(-\lambda, 1)$. Since $N \notin D$, we have $-\lambda < \xi^*$. Moreover, since

$$(21) \qquad d\xi/dt \mid_{\xi=\xi^*} = [(1 + b)\lambda^2 - 1]\xi^*/\lambda^2(-2a\lambda^2 + 2l\lambda - b\lambda + m) < 0,$$

the vertical line $\xi = \xi^*$ is a line without contact, there is no finite singular point on it, and a trajectory of (19) which intersects this line must cross it from right to left (thus leaving D').

Furthermore, we use the transformations (see Theorem 15.15 in [7]):

(22)
$$\eta_1 = -(m + \lambda)\xi + (l - a\lambda)\xi^2 - h(\xi)\eta, \qquad x = \xi;$$
$$\eta_2 = \eta_1 |1 - x/\eta^*|^\gamma, \qquad x = x, \qquad d\tau/dt = |1 - x/\xi^*|^{-\gamma},$$

where

(23)
$$\gamma = \left(m + 2l\lambda - a\lambda^2\right)/\left(2a\lambda^2 - 2l\lambda + b\lambda - m\right) < 0,$$

by (13) and (17). If t represents τ again, then (19) is changed into

(24)
$$dx/dt = \eta_2, \qquad d\eta_2/dt = -g(x) - f(x)\eta_2,$$

where

(25)
$$f(x) = |1 - x/\xi^*|^\gamma (x + \lambda)f_1(x)/\lambda h(x),$$
$$f_1(x) = m(\lambda^2 + m\lambda + 1) - \left(b + 2l + ml\lambda - am\lambda^2\right)x,$$
$$g(x) = |1 - x/\xi^*|^{2\gamma} (x + \lambda)x g_1(x)/\lambda h(x),$$
$$g_1(x) = \left(\lambda^2 + m\lambda + 1\right) + \left(-2b\lambda^2 + m\lambda + a\lambda - mb\lambda + 2\right)x/\lambda$$
$$+ \left(b^2\lambda^2 + a\lambda - mb\lambda - b\right)x^2/\lambda^2.$$

If

(26)
$$y = y_2 + \int_0^x f(\xi)d\xi, \qquad x = x,$$

then the system (1) is finally changed into

(27)
$$dx/dt = y - F(x), \qquad dy/dt = -g(x),$$

where

.(28)
$$F(x) = \int_0^x f(\xi)d\xi.$$

By calculation, it is easy to see that

(29)
$$g_1(\xi^*) = \left(\lambda^2 + m\lambda + 1\right)[(1 + b)\lambda^2 - 1](a\lambda + b)/\lambda^2\left(-2a\lambda^2 + 2l\lambda - b\lambda + m\right)^2.$$

Hence $g_1(x)$ has the linear factor $h(x)$ if and only if $a\lambda + b = 0$. Moreover, in this situation it is easy to see by direct calculation that

$$(30) \qquad\qquad g_1(x) = (1 + ax)h(x),$$

and hence

$$(25)' \qquad\qquad g(x) = |1 - x/\xi^*|^{2\gamma}(x + \lambda)(1 + ax)x/\lambda.$$

We consider separately two possibilities.

(i) $lb^2 + a^2 - abm < 0$. Then $a\lambda + b < 0$ and $g_1(\xi^*) > 0$, by (16) and (29) respectively. The zeros of $g_1(x)$ are the ξ-coordinates of the finite singular points of (19) other than N and 0. Since the coefficients of $g_1(x)$ are all positive, and by hypothesis there is at least one zero greater than ξ^*, it follows that the quadratic $g_1(x)$ has two real zeros x_1, x_2 satisfying

$$(31) \qquad\qquad \xi^* < x_2 \leqslant x_1 < 0.$$

Since a limit cycle of (19) which surrounds O cannot surround any other singular point, it is sufficient to show that (27) satisfies the hypothesis of Lemma 4, with $x_{02} = x_1$, $x_{01} = +\infty$.

Conditions (i) and (ii) are obviously satisfied. If $f(x) = 0$ then either $x = -\lambda < \xi^*$ or

$$(32) \qquad\qquad x = x_0 := m(\lambda^2 + m\lambda + 1)/(b + 2l + ml\lambda - am\lambda^2).$$

Moreover $x_0 < 0$, since $l < 0$. Since obviously no limit cycle surrounds O if $x_0 \leqslant x_1$, we may assume $x_1 < x_0$. Then condition (iii) is satisfied.

To verify condition (iv) we evaluate the derivative of

$$w(x) := \log[f(x)/g(x)] = \log[f_1(x)/xg_1(x)\,|1 - x/\xi^*|^\gamma].$$

Evidently

$$\begin{aligned}
(33) \qquad w'(x) &= f_1'(x)/f_1(x) - 1/x - g_1'(x)/g_1(x) - \gamma h'(x)/h(x) \\
&= -m(\lambda^2 + m\lambda + 1)/xf_1(x) - M(x)/g_1(x)h(x) \\
&= -W(x)/xf_1(x)g_1(x)h(x),
\end{aligned}$$

where

$$(34) \qquad\qquad W(x) = m(\lambda^2 + m\lambda + 1)g_1(x)h(x) + xf_1(x)M(x),$$

$$(35) \qquad\qquad M(x) = g_1'(x)h(x) + \gamma g_1(x)h'(x).$$

Since for $x \in (x_1, x_0) \cup (0, +\infty)$ we have $x f_1(x) > 0$ and $g_1(x) h(x) > 0$, we need only show that also $M(x) > 0$.

We have either $x_2 = x_1$ and $M(x_1) = 0$, or $x_2 < x_1$ and

$$(36) \qquad M(x_2) < 0 < M(x_1).$$

Consequently it is sufficient to show that the leading coefficient of the quadratic $M(x)$ is positive. Since the leading coefficients of $g_1(x)$ and $h(x)$ are positive, this is equivalent to showing that $\gamma + 2 > 0$. But

$$(37)$$
$$(\gamma + 2)\lambda(2a\lambda^2 - 2l\lambda + b\lambda - m) = (m + 2l\lambda - a\lambda^2)\lambda + 2\lambda(2a\lambda^2 - 2l\lambda + b\lambda - m)$$
$$= 3a\lambda^3 + 2(b - l)\lambda^2 - m\lambda = \lambda\varphi'(\lambda),$$

by (9). Thus $\gamma + 2$ has the same sign as $\lambda\varphi'(\lambda)$, by (13). Hence $\gamma + 2 > 0$, and condition (iv) of Lemma 4 is also satisfied. This proves the theorem in case (i).

(ii) $lb^2 + a^2 - abm = 0$. Then $a\lambda + b = 0$, by (16), and $g(x)$ is given by (25)$'$. Since by hypothesis $g(x)$ vanishes for some non-zero $x > \xi^*$, it follows that $-1/a > \xi^*$. Since

$$\xi^* + 1/a = [a(b + 2) - mb(b + 1)]/ab(-2a\lambda^2 + 2l\lambda - b\lambda + m),$$

this is equivalent to $a(b + 2) - mb(b + 1) < 0$. It may now be verified, as in case (i), that all the conditions of Lemma 4 are satisfied with $x_{02} = x_1 = -1/a$, $x_{01} = +\infty$. This proves Theorem 1. □

PROOF OF THEOREM 2: By (8), at the infinite singular point $z = 0$, $v = \lambda$, the coefficient matrix of the system of linear approximation is

$$J = \begin{pmatrix} -(a\lambda^2 + b\lambda) & 0 \\ -(\lambda^2 + m\lambda + 1) & -\varphi'(\lambda) \end{pmatrix},$$

where $\varphi(\lambda)$ is again given by (9). As in the proof of the preceding theorem, we consider separately two cases.

(i) $lb^2 + a^2 - abm < 0$. Then $a\lambda + b < 0$ and $\varphi'(\lambda) > 0$, by (16) and Lemma 3 respectively. Thus the unique infinite singular point $z = 0$, $v = \lambda$ is a saddle, situated at the infinite ends of the line without contact $h(x - \lambda y) = 0$. By (21), a trajectory of (1) must leave D when it intersects the line $h(x - \lambda y) = 0$. In addition, the unique singular point O in D is a stable focus, since $m > 0$. Thus D, with the point O omitted, and the equator form a generalised Bendixson annular region. Therefore there is an odd number of limit cycles surrounding O in D (see Figure 1). This proves the theorem in case (i).
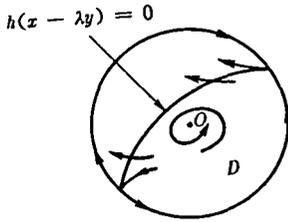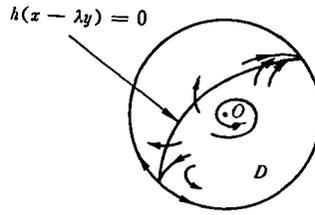
Figure 1   Figure 2

(ii) $lb^2 + a^2 - abm = 0$. Then $a\lambda + b = 0$, one of the eigenvalues at the unique infinite singular point $z = 0$, $v = \lambda$ is zero and the other is negative. We use the same method as in Theorem 65 of [1] to discuss the singular point $z = 0$, $v = \lambda$ of (8). Let

$$\xi = z, \qquad \eta = v - \lambda$$

and

$$x = \xi, \qquad y = -(\lambda^2 + m\lambda + 1)\xi + \rho\eta, \qquad t = \rho\tau,$$

where

(38) $$\rho := -\varphi'(\lambda) = (b + 2l)\lambda + m.$$

Then (8) is changed into

(8)′
$$dx/dt = P(x, y) = b[\rho + a(\lambda^2 + m\lambda + 1)]x^2/a\rho^2 + bxy/\rho^2 + \cdots,$$
$$dy/dt = Q(x, y) = y + \cdots,$$

where $\cdots$ denotes terms of higher degree. Solving $Q(x, y) = 0$, we get $y = \Psi(x) = O(x^2)$. Substituting this in the first equation of (8)′, we obtain

$$P(x, \Psi(x)) = b[\rho + a(\lambda^2 + m\lambda + 1)]x^2/a\rho^2 + \cdots.$$

By direct calculation,

$$b[\rho + a(\lambda^2 + m\lambda + 1)] = a(b + 2) - mb(b + 1).$$

If $a(b + 2) - mb(b + 1) > 0$ then, by Theorem 65 of [1], the singular point $O'(0, 0)$ of (8)′ is a saddle-node and the phase-portrait in its neighbourhood is given by Figure 3. But the signs of t and τ are different, since $\rho < 0$ by (38).

     Hence the phase-portrait in the neighbourhood of the singular point $z = 0$, $v = \lambda$ of (8) is given by Figure 4. By a similar argument to that in case (i), it can now be proved that there is an odd number of limit cycles surrounding O (see Figure 2).
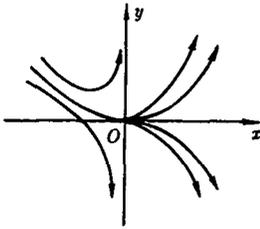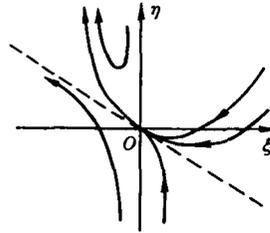
Figure 3   Figure 4

If $a(b+2) - mb(b+1) = 0$, then from (ii) in the proof of Theorem 1 the system has no finite singular points besides N and O. The sum of the indices of these two finite singular points is 2, so the index of the unique infinite singular point ($z = 0$, $v = \lambda$) is -1. A singular point, one of whose eigenvalues is zero and whose index is -1, can only be a saddle. It follows, as in (i), that there is an odd number of limit cycles surrounding O.

If $a(b+2) - mb(b+1) < 0$, then from (ii) in the proof of Theorem 1, there is a finite singular point besides O in D, which is contrary to hypothesis. Since all cases have been discussed, this completes the proof of Theorem 2. $\qquad\square$

PROOF OF THEOREM 3: Take $\lambda = \lambda_3 < 0$, where λ_3 is defined as in Lemma 3. (For simplicity, λ_3 is rewritten as λ in the proofs of Theorems 3 and 4.) As in the proof of Theorem 1, we obtain the system (27) by the transformations (18), (22) and (26), where $g(x)$, $g_1(x)$ and $f(x)$, $f_1(x)$ are the same as in (25), with the same meaning of ξ^*. However, by (13) we now have $\xi^* > 0$. Since the points N and O are on different sides of the line $h(x - \lambda y) = 0$, we also have $\xi^* < -\lambda$.

We again denote the real zeros of $g_1(x)$ by x_1, x_2, where $x_2 \leqslant x_1$, and we define x_0 by (32). Since there are no limit cycles around O if $x_0 \geqslant \xi^*$, we may assume $x_0 < \xi^*$. But

$$\xi^* - x_0 = (\lambda^2 + m\lambda + 1)\rho/\lambda(-2a\lambda^2 + 2l\lambda - b\lambda + m)(b + 2l + ml\lambda - am\lambda^2),$$

where $\rho = (b + 2l)\lambda + m > 0$ since $l < 0$. Hence, $b + 2l + ml\lambda - am\lambda^2 < 0$, by (13), and $x_0 < 0$, by (32). Thus

$$(39) \qquad\qquad x_0 < 0 < \xi^* < -\lambda.$$

Since $a\lambda + b < 0$ and $lb^2 + a^2 - abm \leqslant 0$, by (16) we have

$$(40) \qquad\qquad b^2\lambda^2 + a\lambda - mb\lambda - b \geqslant 0.$$

We consider separately two cases.

(i) $b^2\lambda^2 + a\lambda - mb\lambda - b > 0$. By (25), $g_1(x)$ is a quadratic polynomial with positive leading coefficient. Moreover $g_1(x)$ is not divisible by $h(x)$, since $g_1(\xi^*) > 0$ by (29). Thus the zeros of $g_1(x)$ are just the ξ-coordinates of the finite singular points of (19) other than N and O. By hypothesis such zeros exist. We denote them by x_1, x_2, where $x_2 \leqslant x_1$. Let us discuss the relationships between x_2, x_1, x_0, 0 and ξ^*.

If x_2 or x_1 lies in the interval $(x_0, 0)$ then, as in the discussion above, it can be seen that there is no limit cycle surrounding O. Thus this case can be excluded. Since $g_1(0) > 0$, $g_1(\xi^*) > 0$, it follows that x_2 and x_1 must both lie in one of the following three intervals:

$$\text{(41)} \qquad\qquad (-\infty, x_0), \qquad (0, \xi^*), \qquad (\xi^*, +\infty).$$

We now show that x_2 and x_1 cannot both belong to $(0, \xi^*)$.

Consider the position of the singular points of (1). Since $l < 0$, we have $m^2 - 4l > 0$, Thus the vertical isocline

$$\text{(42)} \qquad -y - mx + lx^2 + mxy + y^2 = (y + mx)(y - 1) + lx^2 = 0$$

is a hyperbola, which has no locus in the region

$$(y + mx)(y - 1) < 0.$$

The horizontal isocline

$$\text{(43)} \qquad\qquad\qquad y = -(1 + ax)/b$$

is a line with positive slope, crossing the y-axis between N and O. Since by hypothesis the discriminant $\Delta \geqslant 0$, (42) and (43) must intersect. Let R and M denote their points of intersection (with $R = M$ if $\Delta = 0$). Together with N and O, they do not form a convex quadrilateral. Hence R and M are either both on the arc I, which is on the right side of N, or both on the arc II, which is on the left side of O (see Figure 5).
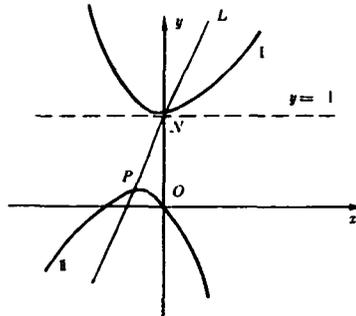

Figure 5

If R and M are on I, then after making the transformation

$$\xi = x - \lambda y, \qquad \eta = y,$$

it is easy to see that $\xi_R, \xi_M > -\lambda$, which means $-\lambda < x_2 \leqslant x_1$.

Suppose R and M are on II. On the line

$$L: m(y-1) + (b+2l)x = 0$$

the divergence of (1) is zero. It is a line with positive slope which passes through N. The line L intersects the x-axis when $x = m/(b+2l)$, hence to the right of the point $(x = m/l)$ where the arc II intersects the x-axis. Consequently the line L intersects the arc II at some point P. Since a limit cycle surrounding O cannot intersect the line $1 + ax + by = 0$, and since it must intersect the line L, there is no limit cycle surrounding O if R or M is to the right of P (or coincides with P) on the arc II. Thus we may assume that R and M are both to the left of P on the arc II. Then the sign of the divergence of (1) at R and M is opposite to its sign at O. After the series of transformations (18), (22) and (26), the sign of the divergence of (27) at R and M is still opposite to its sign at O. But in the strip $x_0 < x < \xi^*$ the divergence of (27) is of constant sign. Hence x_2 and x_1 are not both in the interval $(0, \xi^*)$. Thus we need only consider the following two situations:

(44) $-\infty < x_2 \leqslant x_1 < x_0 < 0 < \xi^*,$

or

(45) $-\infty < x_0 < 0 < \xi^* < x_2 \leqslant x_1.$

Put $x_{01} = \xi^*$ and let $x_{02} = x_1$ or $-\infty$ according as (44) or (45) holds. We need only show that the conditions of Lemma 4 are satisfied by the system (27). Conditions (i)–(iii) are abviously satisfied. To verify condition (iv) we use (33)–(35), as in the proof of Theorem 1. Since $xf_1(x) > 0$ and $g_1(x)h(x) > 0$, it is sufficient to show that $M(x) > 0$ for $x_{02} < x < x_{01}$. Since the leading coefficient of $g_1(x)$ is positive and the leading coefficient of $h(x)$ is now negative, the coefficient of x^2 in $M(x)$ has the opposite sign to $\gamma + 2$. Hence, by (37), (13) and Lemma 3, the coefficient of x^2 in $M(x)$ is positive if $\Delta_0 > 0$ and zero if $\Delta_0 = 0$. If $x_2 < x_1$, then $M(x_2) < 0 < M(x_1)$ if (44) holds and $M(x_2) > 0 > M(x_1)$ if (45) holds. If $x_2 = x_1$, then $M(x_2) = 0$ and $M'(x_2) = g_1''(x_2)h(x_2)$ is positive or negative according as (44) or (45) holds. In every case it follows immediately that $M(x) > 0$ for $x_{02} < x < x_{01}$. This completes the proof in case (i).

(ii) $b^2\lambda^2 + a\lambda - mb\lambda - b = 0$. Then $g_1(x)$ is linear and has a unique zero x_1. Just as in (i), we see that either $x_1 < x_0$ or $\xi^* < x_1$. Put $x_{01} = \xi^*$ and let $x_{02} = x_1$ or $-\infty$ according as $x_1 < x_0$ or $x_1 > \xi^*$. Then it is easy to check that the conditions of Lemma 4 are all satisfied. This completes the proof of Theorem 3. $\square$

NOTE. From the proof of Theorem 1 we see that $\Delta_0 \geqslant 0$ is also permitted, although this case is included in Theorem 3. For clarity, we expressed these theorems as above.

PROOF OF THEOREM 4: To consider the uniqueness of the limit cycle surrounding O we again use the system (27). As in the beginning of the proof of Theorem 3 we have $0 < \xi^* < -\lambda$ and we may assume $x_0 < 0$. Thus

(46) $$-\infty < x_0 < 0 < \xi^* < -\lambda < +\infty.$$

To use the fact that there is now a limit cycle surrounding N we make the change of variables

(47) $$x \to x + \lambda, \qquad \eta_2 \to \eta_2$$

in (24). Then (24) becomes

(24)′
$$dx/dt = \eta_2,$$
$$d\eta_2/dt = -\overline{g}(x) - \overline{f}(x)\eta_2,$$

where $\overline{g}(x) = g(x - \lambda)$, $\overline{f}(x) = f(x - \lambda)$. Then $\overline{g}(0) = 0$, $\overline{f}(0) = 0$ and

$$\overline{g}'(0) = |1 + \lambda/\xi^*|^{2\gamma}(-\lambda)g_1(-\lambda)/\lambda h(-\lambda).$$

Since $h(-\lambda) < 0$ and $g_1(-\lambda) > 0$, we have $\overline{g}'(0) > 0$. Hence conditions (i), (ii) and (iii)′ of Lemma 5 are satisfied by the system (15) formed from (24)′. Since the conclusion of Lemma 5 does not hold, it follows that condition (iv)′ is not satisfied. Thus $[f(x)/g(x)]'$ is not of constant sign in the interval $\xi^* < x < +\infty$. Since $f_1(x) > 0$, $g_1(x) > 0$ and $h(x) < 0$, this is equivalent, by (33)–(35), to saying that $W(x)$ is not of constant sign for $\xi^* < x < +\infty$.

By Lemma 4, in order to prove the theorem it is sufficient to show that $W(x) > 0$ for $x \in (-\infty, x_0) \cup (0, \xi^*)$. We have

(48) $$M(\xi^*) = \gamma g_1(\xi^*)h'(\xi^*) > 0,$$

(49) $$M'(\xi^*) = (a\lambda^2 + b\lambda)g_1'(\xi^*),$$

(50) $$M''(x) = M''(\xi^*) = (\gamma + 2)g_1''(\xi^*)h'(\xi^*) \geqslant 0.$$

If $g_1'(\xi^*) \leqslant 0$, then $M'(\xi^*) \leqslant 0$; hence $M(x) > 0$ for $x \in (-\infty, \xi^*)$ and $W(x) > 0$ for $x \in (-\infty, x_0) \cup (0, \xi^*)$. Consequently we may assume that $g_1'(\xi^*) > 0$, and thus $M'(\xi^*) > 0$. We have also

$$M(0) = g_1'(0)h(0) + \gamma g_1(0)h'(0) = (\lambda^2 + m\lambda + 1)(a\lambda^2 + a - mb) > 0$$
and
$$M'(0) = g_1''(0)h(0) + (\gamma + 1)g_1'(0)h'(0)$$
$$= (\gamma + 1)g_1'(\xi^*)h'(0) - \gamma g_1''(0)h(0) > 0,$$
since
$$\gamma + 1 = (a\lambda^2 + b\lambda)/(2a\lambda^2 - 2l\lambda + b\lambda - m) < 0.$$

Since $M'(x)$ is linear, it is positive throughout the interval $[0, \xi^*]$, and hence $M(x) > 0$ for $x \in [0, \xi^*]$. It follows at once that $W(x) > 0$ for $x \in (0, \xi^*)$.

It remains to prove that $W(x) > 0$ for $x \in (-\infty, x_0)$. We have

$$W(x_0) = m(\lambda^2 + m\lambda + 1)g_1(x_0)h(x_0) > 0,$$
$$W'(x_0) = m(\lambda^2 + m\lambda + 1)(1 - \gamma)g_1(x_0)h'(x_0) < 0,$$
$$W(\xi^*) = \xi^* f_1(\xi^*)M(\xi^*) > 0,$$
$$W'(\xi^*) = m(\lambda^2 + m\lambda + 1)(1 + \gamma)g_1(\xi^*)h'(\xi^*) > 0.$$

Thus $W'(x)$ vanishes for some $x \in (x_0, \xi^*)$. Since $W(x_1) < 0$ for some $x_1 \in (\xi^*, +\infty)$, it is clear that $W'(x)$ also vanishes for some $x \in (\xi^*, x_1)$. Since the coefficient of x^4 in $W(x)$ is positive if $\Delta_0 > 0$ and zero if $\Delta_0 = 0$, $W'(x)$ also vanishes for some $x \in (x_1, +\infty)$ if $\Delta_0 > 0$. Since all zeros of $W'(x)$ are accounted for, it follows that $W'(x) < 0$ for $x \in (-\infty, x_0)$. Hence $W(x) > 0$ for $x \in (-\infty, x_0)$. This completes the proof of Theorem 4. ⬚

Suppose a quadratic system has a weak focus and a strong focus, besides these, it has two real or complex finite singular points p and q (or $p = q$, or one of them lies at the equator). The line L connecting p and q must intersect the segment joining the two foci, say at P. (The definitions for L in the other cases are similar, and are left to the reader). We say that the quadratic system is *positively normalised* if the trajectory at P crosses from the side of L containing the strong focus to the side of L containing the weak focus.

THEOREM B. *Suppose the finite singular points of a quadratic system include a strong focus, a weak focus, and at most one saddle. Suppose also that the system is positively normalised. Then if the strong focus is unstable, it has no limit cycle surrounding it.*

PROOF: By hypothesis and [6], the system can be written in the form (1) with (2), (3), $m < 0$ and $lb^2 + a^2 - abm \leqslant 0$. And the line L is $1 + ax + by = 0$. Then we must have either $l \leqslant 0$ or $a - bm \leqslant 0$. The proof is complete by Lemma 1. ⬚

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