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# TANGENT SPACES OF A NORMAL SURFACE WITH HYPERELLIPTIC SECTIONS

W. L. EDGE

**1. Introduction.** A rational normal curve $C$ of order $n$ in $[n]$ has, at each point $P$, a nest of osculating spaces

$$[0], [1], [2], \dots, [n-2], [n-1];$$

as $P$ moves on $C$ the $[n-2]$ generates a primal $D_{n-1}$ of order $2n-2$.

Hilbert [3] found the multiplicities on $D_{n-1}$ not only of the $V_{\mu+1}$ generated by $D_{\mu}$ for each lesser value of $\mu$ but also those of all submanifolds common to these various $D_{\mu}$.

A surface $\Phi$ in higher space has, as explained [4] by del Pezzo, a nest of tangent spaces

$$\Omega_0, \Omega_2, \Omega_5, \Omega_9, \dots$$

of respective dimensions

$$0, 2, 5, 9, \dots, \tfrac{1}{2}(k-1)(k+2), \dots;$$

they raise the problem of finding the orders of manifolds generated by them and the multiplicity of each on the higher manifolds to which it belongs: the task does not seem to have been attempted, but it may well be eased if $\Phi$ is rational and normal. It was however noted, in a recent encounter [2] with these $\Omega$, that, contrary to the circumstances for a curve, for some surfaces the nest may grow more slowly. Indeed this possibility, with $\Omega_5$ collapsing to $\Omega_4$, was envisaged [5] by Corrado Segre; with him $\Phi$ need not even be algebraic, let alone rational. This collapse was decreed by Segre as a requirement; but in [2] Castelnuovo's rational normal surface $\Phi$, in $[3p+5]$ with hyperelliptic sections of genus $p$, presented of itself the nest

$$\Omega_0 \subset \Omega_2 \subset \Omega_5 \subset \Omega_8 \subset \dots \subset \Omega_{3p+2} \subset \Omega_{3p+4}$$

where the dimension rises by 3 at each move save the first and last. Henceforth $\Phi$ denotes Castelnuovo's surface.

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$\Phi$ is rational, being mapped [**1**, p.199] on a plane $\pi$ by curves of order $p+3$ with two fixed multiple base points: a node $Y$ and a point $X$ of multiplicity $p+1$. There are, as the mapping makes clear, two pencils of rational curves on $\Phi$; one of conics $\gamma$ mapped by the lines through $X$, the other of curves $\delta$ of order $p+1$ mapped by the lines through $Y$. Through any point $P$ on $\Phi$ pass a single $\gamma$ and a single $\delta$; every $\gamma$ meets every $\delta$ once.

The geometry of $\Phi$ in relation to its ambient space has not, apparently, been studied save in a lecture [**2**] at a Toronto symposium in 1979, where some remarks were made about the nests of tangent spaces. It is, as was then said, sufficient if detailed scrutiny is restricted to the case when $p = 2$; at each point $P$ of $\Phi$, now belonging to an [11], there is a nest

$$P \equiv \Omega_0 \subset \Omega_2 \subset \Omega_5 \subset \Omega_8 \subset \Omega_{10}$$

of spaces, any primes through them cutting $\Phi$ in curves with multiplicities at least 1, 2, 3, 4, 5 at $P$. $\Omega_2$ contains the tangent lines, $\Omega_5$ the osculating planes, $\Omega_8$ the osculating solids at $P$ of branches of all curves on $\Phi$ through $P$. As $P$ varies over $\Phi$ $\Omega_8$ generates a primal $M$ whose equation is [**2**, p. 338] obtainable by eliminating a certain three variables from three equations. This elimination will be carried through in Section 4, where it will be found that $M$ can be given by equating to zero a ten-rowed determinant $\Delta$ whose elements, when not identically zero, are quadratic in the twelve homogeneous coordinates. $\Delta$ evolves by Sylvester's process of dialytic elimination between two quintic polynomials which happen to be first polars of a sextic $\mathfrak{S}$; $\Delta$ need not be displayed in full, but its rank under certain specialisations of $\mathfrak{S}$ can be calculated. These ranks enable one to give, if not the actual multiplicities themselves of various submanifolds on $M$, at least lower bounds for them.

An alternative procedure for finding $\Delta$ uses the null polarity $N$ wherein [**2** p.338] the polar primes of the points of $\Phi$ are its osculating primes $\Omega_{10}$; this is outlined in Section 11.

**2. Castelnuovo's normal surface of order** 12 **and the primal** $M$ **generated by its tangent spaces** $\Omega_8$. $\Phi$ is mapped on $\pi$ by the quintics with a fixed triple point $X$ and a fixed node $Y$. If $X$, $Y$ are two of the vertices of the triangle of reference for homogeneous coordinates $(\xi, \eta, \zeta)$ the parametric form of $\Phi$ is [**2**, p.336]

$$\begin{aligned} &x_0 = \xi^2\eta^3, \quad x_1 = \xi^2\eta^2\zeta, \quad x_2 = \xi^2\eta\zeta^2, \quad x_3 = \xi^2\zeta^3 \\ &y_0 = \xi\eta^3\zeta, \quad y_1 = \xi\eta^2\zeta^2, \quad y_2 = \xi\eta\zeta^3, \quad y_3 = \xi\zeta^4 \\ &z_0 = \eta^3\zeta^2, \quad z_1 = \eta^2\zeta^3, \quad z_2 = \eta\zeta^4, \quad z_3 = \zeta^5 \end{aligned} \tag{2.1}$$

The conics $\gamma$ are mapped by the lines $\eta = \beta\,\zeta$; the plane of such a conic consists of those points obtained by varying $a$, $b$, $c$ in

$$(2.2)\qquad \begin{matrix} a\beta^3 & a\beta^2 & a\beta & a \\ b\beta^3 & b\beta^2 & b\beta & b \\ c\beta^3 & c\beta^2 & c\beta & c \end{matrix}$$

and these planes generate, when $\beta$ varies, a threefold whose order is the number of its intersections with an arbitrary [8]. But this [8] is identified by three linearly independent linear equations in the coordinates; determinantal elimination of $a$, $b$, $c$ from these gives an equation of degree 9 in $\beta$; the planes of the conics $\gamma$ generate a $V_3^9$. This was proved, though in a less elementary way, by Castelnuovo who appealed to a theorem of Segre on the normal space of a planar threefold.

The cubics $\delta$ are mapped by the lines $\xi = \alpha\zeta$; the solid containing such a cubic consists of those points obtained by varying $a$, $b$, $c$, $d$ in

$$(2.3)\qquad \begin{matrix} a\alpha^2 & b\alpha^2 & c\alpha^2 & d\alpha^2 \\ a\alpha & b\alpha & c\alpha & d\alpha \\ a & b & c & d \end{matrix}$$

and these solids generate, when $\alpha$ varies, a fourfold whose order is the number of its intersections with an arbitrary [7]. But this [7] is identified by four linearly independent linear equations in the coordinates; determinantal elimination of $a$, $b$, $c$, $d$ from these gives an equation of degree 8 in $\alpha$; the solids of the cubics $\delta$ generate a $V_4^8$.

Both $V_4^8$ and $V_3^9$ have parts to play in the geometry.

**3.** All sections of $\Phi$ by primes through $\Omega_8(P)$ include [**2**, p.338] both the $\gamma$ and the $\delta$ that pass through $P$. Suppose, then, that $A$ and $B$ are any two points of $\Phi$ neither on the same $\gamma$ nor on the same $\delta$; if $P$ is the intersection of $\gamma_A$ and $\delta_B$, $Q$ that of $\gamma_B$ and $\delta_A$, both $A$ and $B$, and so the whole chord $AB$, are in both $\Omega_8(P)$ and $\Omega_8(Q)$. So the fivefold generated by the chords of $\Phi$ is, at least, nodal on $M$. The multiplicity is indeed higher, as will be seen below. Had $A$ and $B$ been on the same $\gamma$ then $AB$ would have lain, with $A$ and $B$, in an infinity of spaces $\Omega_8$, namely those whose contacts are on $\gamma$; likewise had $A$ and $B$ been on the same $\delta$. This shows both $V_3^9$ and $V_4^8$ to be multiple loci on $M$, as will also be corroborated below.

**4.** Now for the elimination: one has [**2**, p.338] to eject $\xi'$, $\eta'$, $\zeta'$ from

$$(\eta\zeta' - \eta'\zeta)^2(\xi\zeta' - \xi'\zeta)\quad \begin{bmatrix} (\eta\zeta' - \eta'\zeta)(\xi\zeta' - \xi'\zeta) \\ (\eta\zeta' - \eta'\zeta)\zeta \\ (\xi\zeta' - \xi'\zeta)\zeta \end{bmatrix} \quad = 0$$

these being the equations, in $\pi$, of the maps of the sections of $\Phi$ by three linearly independent primes through $\Omega_8(\xi', \eta', \zeta')$. It is understood that the quintic monomials in $\xi$, $\eta$, $\zeta$ would, if the three products were multiplied out, be replaced from (2.1), so one must refrain from any cancellations of these three unprimed letters.

It economises to replace $\xi'/\zeta'$, $\eta'/\zeta'$ by $\alpha$, $\beta$; one has then to eliminate $\alpha$, $\beta$ between

$$(\eta - \beta\zeta)^2(\xi - \alpha\zeta)\begin{bmatrix}(\eta - \beta\zeta)(\xi - \alpha\zeta)\\ (\eta - \beta\zeta)\zeta \\ (\xi - \alpha\zeta)\zeta\end{bmatrix} = 0,$$

which are equivalent to any three of

$$(4.1)\quad (\eta - \beta\zeta)^2(\xi - \alpha\zeta)\begin{bmatrix}(\eta - \beta\zeta)\xi\\ (\eta - \beta\zeta)\zeta\\ (\xi - \alpha\zeta)\eta\\ (\xi - \alpha\zeta)\zeta\end{bmatrix} = 0.$$

The first two of these four equations give

$$(\eta - \beta\zeta)^3\xi^2 = \alpha(\eta - \beta\zeta)^3\xi\zeta = \alpha^2(\eta - \beta\zeta)^3\zeta^2$$

whence, using (2.1),

$$\begin{aligned}\alpha^2:\alpha:1 &= x_0 - 3\beta x_1 + 3\beta^2x_2 - \beta^3x_3:y_0 - 3\beta y_1 + 3\beta^2y_2\\ &\quad - \beta^3y_3:z_0 - 3\beta z_1 + 3\beta^2z_2 - \beta^3z_3\\ &= \mathscr{X}:\mathscr{Y}:\mathscr{Z}, \text{ say,}\end{aligned}$$

so that, whenever (4.1) are satisfied

$$\mathfrak{S} \equiv \mathscr{Z}\mathscr{X} - \mathscr{Y}^2 = 0.$$

One now handles the other two equations (4.1).
If

$$(\xi - \alpha\zeta)^2(\eta - \beta\zeta)^2\eta = 0,$$

$$(\xi^2 - 2\alpha\zeta\xi + \alpha^2\zeta^2)(\eta^3 - 2\beta\eta^2\zeta + \beta^2\eta\zeta^2) = 0,$$

then, by (2.1),

$$\begin{aligned}x_0 - 2\beta x_1 + \beta^2x_2 - 2\alpha(y_0 - 2\beta y_1 + \beta^2y_2)&\\ + \alpha^2(z_0 - 2\beta z_1 + \beta^2z_2) &= 0,\end{aligned}$$

so that

$$q_1 \equiv \mathscr{Z}(x_0 - 2\beta x_1 + \beta^2 x_2) - 2\mathscr{Y}(y_0 - 2\beta y_1 + \beta^2 y_2) + \mathscr{X}(z_0 - 2\beta z_1 + \beta^2 z_2) = 0,$$

a quintic equation for $\beta$. Likewise $(\xi - \alpha\zeta)^2(\eta - \beta\zeta)^2\zeta = 0$ leads to

$$q_2 \equiv \mathscr{Z}(x_1 - 2\beta x_2 + \beta^2 x_3) - 2\mathscr{Y}(y_1 - 2\beta y_2 + \beta^2 y_3) + \mathscr{X}(z_1 - 2\beta z_2 + \beta^2 z_3) = 0,$$

and now one has only to eliminate $\beta$ between the quintics $q_1$ and $q_2$ by Sylvester's dialytic process.

It profits to observe that $q_1$ and $q_2$ are polars of the sextic $\mathfrak{S}$: where polarisation is involved one regards a polynomial in $\beta$ as homogenised, $\beta$ having been replaced by $\beta_1/\beta_2$, and polarises with respect to $\beta_1$ and $\beta_2$. Non-zero constant multipliers happening to obtrude can be discarded as irrelevant. Of course each of $q_1 = 0$, $q_2 = 0$ is a consequence of the other when $\mathfrak{S} = 0$. The condition for $q_1$ and $q_2$ to share a zero is therefore that $\mathfrak{S}$ has a repeated zero, and it may not be necessary to parade the ten-rowed determinant $\Delta$ that emerges from the elimination between $q_1 = 0$ and $q_2 = 0$. But, as each element of $\Delta$ is quadratic in the coordinates, the $\infty^2$ spaces $\Omega_8$ generate a primal $M$ of order 20.

The calculation of multiplicities of sub-manifolds on $M$ is eased by $M$ having a determinantal equation. For the first partial derivatives of $\Delta$ are linear combinations of its first minors, the second partial derivatives of its second minors, and so on. Thus if the coordinates of a point, when substituted in $\Delta$, produce a determinant of rank $\rho$ the multiplicity of the point on $M$ is, at least, $10 - \rho$.

**5. Submanifolds on** $M$. $\mathfrak{S}$ is identically zero at every point of $\Phi$; it is, perhaps less to be expected that, as (2.3) shows, it is identically zero at every point of $V_4^8$. So, on $V_4^8$, both $q_1$ and $q_2$ have every coefficient zero; $\Delta$ becomes the determinant of the zero matrix and has rank 0, so that $V_4^8$ has multiplicity at least ten on $M$. This implies that if a chord of $V_4^8$ does not lie wholly on $M$ it meets $M$ only at the two points of $V_4^8$ that it joins. Such lines, not wholly on $M$, do exist: the join of

$$(x_0, x_1, x_2, x_3; 0, 0, 0, 0; 0, 0, 0, 0)$$

in that solid (2.3) having $\alpha = \infty$ to

$$(0, 0, 0, 0; 0, 0, 0, 0; z_0, z_1, z_2, z_3)$$

in that solid (2.3) having $\alpha = 0$ contains

$$(x_0, x_1, x_2, x_3; 0, 0, 0, 0; z_0, z_1, z_2, z_3)$$

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and here the $x_i$ and $z_i$ need not be such as to endow $\mathfrak{S}$ with a repeated zero. The multiplicity of $V_4^8$ on $M$ is exactly ten.

**6.** Those chords of $V_4^8$ which do lie on $M$ can be identified precisely; they compose two distinct 8-dimensional manifolds.

Any chord of $V_4^8$ consists, by (2.3), of points

$$\begin{matrix} \lambda a\alpha_1^2 + \mu a'\alpha_2^2 & \lambda b\alpha_1^2 + \mu b'\alpha_2^2 & \lambda c\alpha_1^2 + \mu c'\alpha_2^2 & \lambda d\alpha_1^2 + \mu d'\alpha_2^2 \\ \lambda a\alpha_1 + \mu a'\alpha_2 & \lambda b\alpha_1 + \mu b'\alpha_2 & \lambda c\alpha_1 + \mu c'\alpha_2 & \lambda d\alpha_1 + \mu d'\alpha_2 \\ \lambda a + \mu a' & \lambda b + \mu b' & \lambda c + \mu c' & \lambda d + \mu d' \end{matrix}$$

the ratio $\lambda{:}\mu$ varying along the chord. These coordinates produce, on substitution in $\mathfrak{S}$,

$$\lambda\mu(\alpha_1 - \alpha_2)^2(a - 3b\beta + 3c\beta^2 - d\beta^3)(a' - 3b'\beta + 3c'\beta^2 - d'\beta^3).$$

The sextic has a double zero if either of its two factors does; this happens if either $(a, b, c, d)$ is on a tangent of the cubic $\delta$ or $(a', b', c', d')$ is on a tangent of the cubic $\delta'$. Thus the cone of [5]'s joining any generating solid of $V_4^8$ to the tangents of the $\delta$ in which any other such solid cuts $\Phi$ lies wholly on $M$.

The sextic also has a double zero when the two cubics have a common zero. The interpretation of this is that if $A$, $A'$ are points in the solids spanned by $\delta$, $\delta'$, $AA'$ is on $M$ when there is an osculating plane of $\delta$ through $A$ and one of $\delta'$ through $A'$ whose contacts are on the same $\gamma$. In other words: the [5] spanned by planes that osculate any two $\delta$ at their intersections with the same $\gamma$ lies wholly on $M$.

**7.** If, in (2.2), $\beta$ is replaced momentarily by some other letter, say $\psi$, substitution in $\mathfrak{S}$ gives $(ac - b^2)(\psi - \beta)^6$. When $ac = b^2$ (2.2) is on $\Phi$ and $\mathfrak{S}$ identically zero; otherwise, for a point in the plane of a conic $\gamma$ but not on $\gamma$ itself, $q_1$ and $q_2$ are both multiples of $(\psi - \beta)^5$, the rows of $\Delta$ are identical in pairs and $\rho = 5$. The $V_3^9$ generated by the planes of the conics $\gamma$ has multiplicity (at least) 5 on $M$.

**8.** Suppose now that $(\xi_1, \eta_1, \zeta_1)$ and $(\xi_2, \eta_2, \zeta_2)$ are two points on $\Phi$ neither on the same $\gamma$ nor on the same $\delta$; $\xi_1\zeta_2 \neq \xi_2\zeta_1$, $\eta_1\zeta_2 \neq \eta_2\zeta_1$. The points on their join occur on varying $\lambda$, $\mu$ in

$$\ldots x_1 = \lambda\xi_1^2\eta_1\zeta_1 + \mu\xi_2^2\eta_2\zeta_2 \ldots z_3 = \lambda\zeta_1^5 + \mu\zeta_2^5$$

and so make

$$\mathscr{Z} = \lambda\zeta_1^2(\eta_1 - \beta\zeta_1)^3 + \mu\zeta_2^2(\eta_2 - \beta\zeta_2)^3,$$

$$\mathscr{X} = \lambda\xi_1^2(\eta_1 - \beta\zeta_1)^3 + \mu\xi_2^2(\eta_2 - \beta\zeta_2)^3,$$

$$\mathscr{Y} = \lambda\xi_1\zeta_1(\eta_1 - \beta\zeta_1)^3 + \mu\xi_2\zeta_2(\eta_2 - \beta\zeta_2)^3,$$

$$\mathfrak{S} = \lambda\mu(\zeta_1\xi_2 - \zeta_2\xi_1)^2(\eta_1 - \beta\zeta_1)^3(\eta_2 - \beta\zeta_2)^3.$$

The non-zero factors may be dropped, leaving

$$(\eta - \beta\zeta_1)^3(\eta_2 - \beta\zeta_2)^3$$

and now polarisation allows us to take

$$q_1 \equiv (\eta_1 - \beta\zeta_1)^2(\eta_2 - \beta\zeta_2)^2\{2\eta_1\eta_2 - \beta(\eta_1\zeta_2 + \eta_2\zeta_1)\},$$

$$q_2 \equiv (\eta_1 - \beta\zeta_1)^2(\eta_2 - \beta\zeta_2)^2\{\eta_1\zeta_2 + \eta_2\zeta_1 - 2\zeta_1\zeta_2\beta\}.$$

These quintics share two zeros, both repeated; they do not share the remaining one unless

$$2\eta_1\eta_2/(\eta_1\zeta_2 + \eta_2\zeta_1) = (\eta_1\zeta_2 + \eta_2\zeta_1)/2\zeta_1\zeta_2$$

$$(\eta_1\zeta_2 - \eta_2\zeta_1)^2 = 0,$$

and this has been forestalled. The same prohibition debars both $q_1$ and $q_2$ from having a triple zero.

The rank of $\Delta$ for two quintics so related is 6, as can be tested by writing out $\Delta$ for, say, the pair of binary quintics

$$x^2y^2(a_1x + b_1y),\ x^2y^2(a_2x + b_2y);\ a_1a_2b_1b_2(a_1b_2 - a_2b_1) \neq 0.$$

Hence the fivefold of chords of $\Phi$ is (at least) quadruple on $M$.

**9.** Analogous proceedings serve to calculate multiplicities, or at least lower bounds for them, of other manifolds on $M$.

Take the fourfold generated by the tangent planes $\Omega_2$. The coordinates of any point of such a plane result from applying the differential operator $\lambda D_1 + \mu D_2 + \nu D_3$ to those of its contact, and the form of $\mathfrak{S}$ at such a point can be calculated. It is already known that $\mathfrak{S}$ will be identically zero along the line in $\Omega_2(P)$ that is the tangent at $P$ of the $\delta$ passing through $P$, since this tangent is on $V_8^4$. But removal of this factor leaves an $\mathfrak{S}$, not involving the multipliers $\lambda$, $\mu$, $\nu$, which can be interpreted with its polars in the usual way.

Since, at the point $(\xi, \eta, \zeta)$ on $\Phi$,

$$\mathscr{Y} = \zeta\xi(\eta - \beta\zeta)^3$$

its value at a point of $\Omega_2(\xi, \eta, \zeta)$ is, applying $\lambda D_1 + \mu D_2 + \nu D_3$,

$$\lambda\zeta(\eta - \beta\zeta)^3 + 3\mu\zeta\xi(\eta - \beta\zeta)^2 + \nu\xi(\eta - \beta\zeta)^3 - 3\nu\beta\zeta\xi(\eta - \beta\zeta)^2$$
$$= (\lambda\zeta + \nu\xi)(\eta - \beta\zeta)^3 + 3\zeta\xi(\mu - \nu\beta)(\eta - \beta\zeta)^2.$$

Similarly

$$\mathscr{Z} = 2\nu\zeta(\eta - \beta\zeta)^3 + 3\zeta^2(\mu - \nu\beta)(\eta - \beta\zeta)^2,$$
$$\mathscr{X} = 2\lambda\xi(\eta - \beta\zeta)^3 + 3\xi^2(\mu - \nu\beta)(\eta - \beta\zeta)^2,$$

so that

$$\mathfrak{S} \equiv -(\lambda\zeta - \nu\xi)^2(\eta - \beta\zeta)^6.$$

The line of points in $\Omega_2\,(\xi, \eta, \zeta)$ along which $\lambda\zeta = \nu\xi$ will be the tangent of $\delta$ at $(\xi, \eta, \zeta)$; so, as anticipated, drop this multiplier and use

$$\mathfrak{S} \equiv (\eta - \beta\zeta)^6.$$

The situation is the same as with $V_3^9$ in Section 7; the fourfold of tangent planes of $\Phi$ is (at least) quintuple on $M$.

In order to identify the linear constraint on $\lambda, \mu, \nu$ that confines a point of $\Omega_2(\xi, \eta, \zeta)$ to the tangent of $\delta$ one has only to take the intersection of the tangent plane with the solid containing $\delta$. But this solid, by (2.3), is

$$\zeta^2 x_0 = \zeta\xi y_0 = \xi^2 z_0,\ \zeta_2 x_1 = \zeta\xi y_1 = \xi^2 z_1,$$
$$\zeta^2 x_2 = \zeta\xi y_2 = \xi^2 z_2,\ \zeta^2 x_3 = \zeta\xi y_3 = \xi^2 z_3. \tag{9.1}$$

The first of these four pairs of equations requires

$$\zeta^2(\lambda\,.\,2\xi\eta^3 + \mu\,.\,3\xi^2\eta^2) = \zeta\xi(\lambda\eta^3\zeta + \mu\,.\,3\xi\eta^2\zeta + \nu\xi\eta^3)$$
$$= \xi^2\,(\mu\,.\,3\eta^2\zeta^2 + \nu\,.\,2\eta^3\zeta)$$

or, omitting $3\mu\xi^2\eta^2\zeta^2$ from all these numbers,

$$2\lambda\xi\eta^3\xi^2 = \lambda\xi\eta^3\zeta^2 + \nu\xi^2\eta^3\zeta = 2\nu\xi^2\eta^3\zeta$$

which hold when $\lambda\zeta = \nu\xi$. The same constraint is found to satisfy the other three pairs of equations in (9.1).

**10.** While the operator $\lambda D_1 + \mu D_2 + \nu D_3$ serves to identify points spanning $\Omega_2$ one has to use

$$aD_{11} + BD_{22} + cD_{33} + 2dD_{23} + 2eD_{31} + 2fD_{12}$$

to identify points spanning $\Omega_5$. Substitution in $\mathfrak{S}$ will, after some calculations, give information relevant to the multiplicity of $\Omega_5$ on $M$. But there are, in $\Omega_5$, two notable planes: the osculating plane of $\delta$ and the plane of $\gamma$; to lie in either of these planes imposes three linear constraints on $a, b, c, d, e, f$.

The osculating plane is the intersection of $\Omega_5$ and the solid containing $\delta$, and (9.1) are ready to hand. The last pair demands that

$$\zeta^2(2a\zeta^3 + 6c\xi^2\zeta + 12e\xi\zeta^2) = \zeta\xi(12c\xi\zeta^2 + 8e\zeta^3) = \xi^2 20c\zeta^3$$

which are satisfied when $a\zeta^2 = e\zeta\xi = c\xi^2$. The other pairs make their demands too, and it transpires that the conditions for a point of $\Omega_5$ to lie in the osculating plane of $\delta$ are

$$(10.1) \quad a\zeta^2 = e\zeta\xi = c\xi^2, \quad d\xi = f\zeta.$$

When these hold $\mathfrak{S}$ will be identically zero.

The calculations show that one may take

$$\begin{aligned}\mathfrak{S} \equiv\ & (ac - e^2)(\eta - \beta\zeta)^6 \\ & + 6\{ (cf - de)\xi + (ad - ef)\zeta - (ac - e^2)\beta\zeta\}(\eta - \beta\zeta)^5 \\ & + 3(b - 2d\beta + c\beta^2)(c\xi^2 - 2e\zeta\xi + a\zeta^2)(\eta - \beta\zeta)^4 \\ & - 9\{ (c\beta - d)\xi - (e\beta - f)\zeta\}^2(\eta - \beta\zeta)^4\end{aligned}$$

which does vanish identically when all of (10.1) hold.

The equations of the plane of $\gamma$ are, by (2.2),

$$\begin{aligned}\zeta^3 x_0 &= \eta\zeta^2 x_1 = \eta^2\zeta x_2 = \eta^3 x_3, \\ \zeta^3 y_0 &= \eta\zeta^2 y_1 = \eta^2\zeta y_2 = \eta^3 y_3, \\ \zeta^3 x_0 &= \eta\zeta^2 z_1 = \eta^2\zeta z_2 = \eta^3 z_3,\end{aligned}$$

and are found to require of a point in $\Omega_5$

$$b\zeta^2 = d\eta\zeta = c\eta^2, \quad e\eta = f\zeta.$$

These restrictions imply that $cf = de$ so that the multiplier of $(\eta - \beta\zeta)^5$ above becomes six times

$$\begin{aligned}& a(d - c\beta)\zeta - e(f - e\beta)\zeta \\ & = ac(\eta - \beta\zeta) - e^2(\eta - \beta\zeta) = (ac - e^2)(\eta - \beta\zeta).\end{aligned}$$

The multiplier of $(\eta - \beta\zeta)^4$ is the product of $3\zeta^{-2}$ and

$$\begin{aligned}& c(\eta - \beta\zeta)^2(c\xi^2 - 2e\zeta\xi + a\zeta^2) - 3\{c(\eta - \beta\zeta)\xi - e(\eta - \beta\zeta)\zeta\}^2 \\ & = (\eta - \beta\zeta)^2\{ (ac - e^2)\zeta^2 - 2(c\xi - e\zeta)^2\}\end{aligned}$$

so that $\mathfrak{S}$ becomes

$$\{10(ac - e^2) - 6(c\frac{\xi}{\zeta} - e)^2\}(\eta - \beta\zeta)^6,$$

a form which Section 7 leads one to expect.

When $a$, $b$, $c$, $d$, $e$, $f$ are free from restriction both polars of $\mathfrak{S}$ are quintics having $(\eta - \beta\zeta)^3$ for factor: they share a triple root. The rank $\rho$ of $\Delta$ can then be found by using

$$x^3(ax^2 + 2hxy + by^2) \text{ and } x^3(a'x^2 + 2h'xy + b'y^2), \quad aa'bb' \neq 0,$$

the two quadratics not having a common zero; $\rho = 7$. The manifold generated by $\Omega_5$ is (at least) triple on $M$.

**11. The null polarity.** It is perhaps worth while to outline a procedure alternative to the elimination in Section 4; it relies on $\Omega_8$ and $\Omega_2$ being polars of each other in the null polarity $N$ [**2**, p.338], so that the polar primes of three linearly independent points of $\Omega_2$ contain, and determine, $\Omega_8$. The equation of $M$ will be the outcome of eliminating $\xi$, $\eta$, $\zeta$ from three such equations.

Now, on differentiating (2.1), it appears that $\Omega_2$ is spanned by the points whose coordinate vectors are the three left-hand columns $C_1$, $C_2$, $C_3$ in

$$\begin{array}{ccccc}
2\xi\eta^3 & 3\xi^2\eta^2 & \cdot & \cdot & \cdot \\
2\xi\eta^2\zeta & 2\xi^2\eta\zeta & \xi^2\eta^2 & \cdot & \xi^2\eta^2 \\
2\xi\eta\zeta^2 & \xi^2\zeta^2 & 2\xi^2\eta\zeta & \cdot & 2\xi^2\eta\zeta \\
2\xi\zeta^3 & \cdot & 3\xi^2\zeta^2 & \cdot & 3\xi^2\zeta^2 \\
\eta^3\zeta & 3\xi\eta^2\zeta & \xi\eta^3 & \xi\eta^3 & \cdot \\
\eta^2\zeta^2 & 2\xi\eta\zeta^2 & 2\xi\eta^2\zeta & \xi\eta^2\zeta & \xi\eta^2\zeta \\
\eta\zeta^3 & \xi\zeta^3 & 3\xi\eta\zeta^2 & \xi\eta\zeta^2 & 2\xi\eta\zeta^2 \\
\zeta^4 & \cdot & 4\xi\zeta^3 & \xi\zeta^3 & 3\xi\zeta^3 \\
\cdot & 3\eta^2\zeta^2 & 2\eta^3\zeta & 2\eta^3\zeta & \cdot \\
\cdot & 2\eta\zeta^3 & 3\eta^2\zeta^2 & 2\eta^2\zeta^2 & \eta^2\zeta^2 \\
\cdot & \zeta^4 & 4\eta\zeta^3 & 2\eta\zeta^3 & 2\eta\zeta^3 \\
\cdot & \cdot & 5\zeta^4 & 2\zeta^4 & 3\zeta^4
\end{array}$$

The fourth column $C_4$ is $2(\eta C_2 + \zeta C_3) - 3\xi C_1$ with $5\zeta$ cancelled; the fifth is $C_3 - C_4$.

The premultiplication of any of these columns by the row [**2**, p. 338]

$$z_3, \ -3z_2, \ 3z_1, \ -z_0; \ -2y_3, \ 6y_2, \ -6y_1, \ 2y_0; \ x_3, \ -3x_2, \ 3x_1, \ -x_0$$

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gives a prime containing $\Omega_8$. Two such products, using $C_1$ and $C_4$, yield

$$Z\xi = Y\zeta \qquad Y\xi = X\zeta$$

where

$$X = x_3\eta^3 - 3x_2\eta^2\zeta + 3x_1\eta\zeta^2 - x_0\zeta^3,$$
$$Y = y_3\eta^3 - 3y_2\eta^2\zeta + 3y_1\eta\zeta^2 - y_0\zeta^3,$$
$$Z = z_3\eta^3 - 3z_2\eta^2\zeta + 3z_1\eta\zeta^2 - z_0\zeta^3,$$

so that

(11.1) $\xi^2{:}\zeta\xi{:}\zeta^2{:}X{:}Y{:}Z.$

Now premultiply $C_2$; the product lacks $x_0$, $y_0$, $z_0$ and after $\xi$ is eliminated from it by using (11.1) the outcome is seen to be a first polar, the partial derivative with respect to $\eta$, of

$$\mathfrak{S} \equiv ZX - Y^2.$$

In like manner premultiplication of $C_5$ gives the other first polar, so one has to eliminate $\eta/\zeta$ between these two quintics just as in Section 4.

**12. The general situation in** $[3p+5]$. If $p > 2$ the normal surface $\Phi$ is of order $4p+4$ in $[3p+5]$ and, as remarked elsewhere [**2**, p.342], the rectangular array (2.1) is enlarged to one of three rows and $p+2$ columns, the members of the respective rows being, for $i = 0, 1, 2, \ldots, p, p+1$,

$$x_i = \xi^2\eta^{p+1-i}\zeta^i,\ y_i = \xi\eta^{p+1-i}\zeta^{i+1},\ z_i = \eta^{p+1-i}\zeta^{i+2}.$$

The points obtained by varying $a$, $b$, $c$ in

$$\begin{array}{llll} a\beta^{p+1}, & a\beta^p, & \ldots & a \\ b\beta^{p+1}, & b\beta^p, & \ldots & b \\ c\beta^{p+1}, & c\beta^p, & \ldots & c \end{array}$$

are those of a plane meeting $\Phi$ in a conic $\gamma$ mapped on $\pi$ by $\eta = \beta\zeta$; the threefold generated by these planes meets an arbitrary $[3p+2]$ given by three linearly independent linear equations in $3p+3$ points, each corresponding to a zero $\beta$ of a three-rowed determinant with elements of degree $p+1$ in $\beta$. The planes of the $\gamma$ generate, as Castelnuovo knew, a $V_3^{3p+3}$.

The points obtained by varying $a_0, a_1, \ldots . a_{p+1}$ in

$$(12.1)\qquad \begin{array}{llll} a_0\alpha^2, & a_1\alpha^2, & \ldots & a_{p+1}\alpha^2 \\ a_0\alpha\ , & a_1\alpha\ , & \ldots & a_{p+1}\alpha \\ a_0\ \ , & a_1\ \ , & \ldots & a_{p+1} \end{array}$$

are those of a $[p+1]$ meeting $\Phi$ in a rational normal curve $\delta$ mapped on $\pi$ by $\xi = \alpha\zeta$; the $(p+2)$-fold generated by these $[p+1]$ meets an arbitrary $[2p+3]$ given by $p+2$ linearly independent linear equations in $2(p+2)$ points, each corresponding to a zero $\alpha$ of a $(p+2)$-rowed determinant with elements quadratic in $\alpha$. The ambient spaces of the $\delta$ generate a $V_{p+2}^{2p+4}$.

**13.** The nest of tangent spaces is [**2**, p.342]

$$\Omega_0 \subset \Omega_2 \subset \Omega_5 \subset \ldots \subset \Omega_{3p+2} \subset \Omega_{3p+4}$$

and one is concerned with the primal $M$ generated by $\Omega_{3p+2}(P)$ when $P$ traces $\Phi$. So one eliminates $\xi'$, $\eta'$, $\zeta'$ from what (4.1) become when $(\eta - \beta\zeta)^2$ is replaced by $(\eta - \beta\zeta)^p$, and obtains

$$\alpha^2:\alpha:1 = \mathscr{X}:\mathscr{Y}:\mathscr{Z}$$

where

$$\mathscr{X} \equiv \sum_{i=0}^{p+1} \binom{p+1}{i} x_i(-\beta)^i,$$

$$\mathscr{Y} \equiv \sum_{i=0}^{p+1} \binom{p+1}{i} y_i(-\beta)^i,$$

$$\mathscr{Z} \equiv \sum_{i=0}^{p+1} \binom{p+1}{i} z_i(-\beta)^i.$$

Dialytic elimination of $\beta$ between first polars $q_1$, $q_2$ of

$$\mathfrak{S} \equiv \mathscr{Z}\mathscr{X} - \mathscr{Y}^2$$

yields a determinant $\Delta$ of $2(2p+1)$ rows with elements (those other than zeros) quadratic in the $3p+6$ coordinates.

The spaces $\Omega_{3p+2}$ generate a primal $M$ of order $4(2p+1)$. This can also be established by the method of Section 11, using the fact that $\Omega_{3p+2}(\xi, \eta, \zeta)$ and $\Omega_2(\xi, \eta, \zeta)$ are polar spaces in a null polarity if $p$ is even, in a quadric if $p$ is odd [**2**, p.342].

It is manifest from (12.1) that $\mathfrak{S}$ is identically zero, as therefore are $q_1$ and $q_2$, at every point of $V_{p+2}^{2p+4}$, which is thus of multiplicity $4p+2$ on $M$. The multiplicity does not exceed $4p+2$ because there are chords of $V_{p+2}^{2p+4}$ which do not lie wholly on $M$. It appears, just as in Section 6, that

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those chords which do lie on $M$ fill two separate manifolds of dimension $2p+4$. These are composed of

(a) those $[2p+1]$ that join $[p]$'s osculating two $\delta$ at points on the same $\gamma$;

(b) those $[2p+1]$ that join the ambient $[p+1]$ of any $\delta$ to the osculating $[p-1]$'s of any other $\delta$.

**14.** Other matters also run in close analogy to the situation when $p = 2$. For example: at a point on the join of $(\xi_1, \eta_1, \zeta_1)$ and $(\xi_2, \eta_2, \zeta_2)$ on $\Phi$

$$\mathscr{Z} \equiv \lambda\zeta_1^2(\eta_1 - \beta\zeta_1)^{p+1} + \mu\zeta_2^2(\eta_2 - \beta\zeta_2)^{p+1}$$

$$\mathscr{Z}\mathscr{X} - \mathscr{Y}^2 \equiv -(\lambda\zeta - \nu\xi)^2(\eta - \beta\zeta)^{2p+2}.$$

One is led, instead of to two quintics with a pair of repeated roots, to two $(2p+1)$-ics with a pair of $p$-fold roots; $\Delta$ has rank $2p+2$ and the fivefold of chords of $\Phi$ has multiplicity (at least) $2p$ on $M$. And so on.

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*Inveresk House,*
*Musselburgh, Scotland*