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REAL IDEALS IN POINTFREE RINGS OF CONTINUOUS FUNCTIONS

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Abstract

Real ideals of the ring $\mathfrak{R}L$ of real-valued continuous functions on a completely regular frame L are characterized in terms of cozero elements, in the manner of the classical case of the rings $C(X)$. As an application, we show that L is realcompact if and only if every free maximal ideal of $\mathfrak{R}L$ is hyper-real—which is the precise translation of how Hewitt defined realcompact spaces, albeit under a different appellation. We also obtain a frame version of Mrówka's theorem that characterizes realcompact spaces.

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1. Introduction

Real ideals in classical rings $C(X)$ of real-valued continuous functions on a Tychonoff space X have very transparent characterizations in terms of zero-sets [11, Theorem 5.14] which often simplify computations with these types of ideals. The intent of this note is to extend these characterizations to the rings $\mathfrak{R}L$ of real-valued continuous functions on a completely regular frame L. There are some noteworthy consequences of results obtained en route to the characterizations, and also of the characterizations themselves. They include the following:

(a) a characterization of realcompact frames as precisely those L for which every free maximal ideal of $\mathfrak{R}L$ is hyper-real;

(b) a frame analogue of Mrówka's [17] characterization of realcompact spaces which states that

> *a Tychonoff space X is realcompact if and only if for each $p \in \beta X \setminus X$ there exists a function $f \in C(\beta X)$ such that $f(p) = 0$ and $f(x) > 0$ for all $x \in X$;*

(c) a characterization of pseudocompact frames as those L for which every maximal ideal of $\mathfrak{R}L$ is real.

The first of these results deserves some elaboration. In his celebrated paper [13], Hewitt *defined* realcompact spaces (under the appellation 'Q-spaces') to be those completely regular spaces X such that every free maximal ideal of the ring $C(X)$ is hyper-real. He then characterized them, in modern parlance, as precisely those X such that every z-ultrafilter on X with the countable intersection property is fixed. It is the frame-theoretic articulation of this characterization which has generally been adopted as the definition of realcompact frames. Actually, in pointfree topology realcompactness has always been defined by a condition which is a frame version of one or other characterization of spatial realcompactness, such as:

(1) *a completely regular frame L is realcompact if any σ-proper maximal ideal of* Coz L *is completely proper*;

as in [9], or

(2) *a completely regular frame L is realcompact if every ring homomorphism $\mathfrak{R}L \to \mathbb{R}$ is a point evaluation*,

as in [4, 6]. These definitions are equivalent and are 'conservative' in the sense that a completely regular space X is realcompact if and only if the frame $\mathfrak{O}X$ of open sets of X is realcompact.

The result in (a) above is the exact translation to frames of Hewitt's original definition of realcompact spaces as it appears in [13]. Although this result has hitherto not appeared stated explicitly as above, Banaschewski has shown us how it can be proved without the description of real ideals given here. His elegant proof (which is choice-free, to boot) is outlined in Remark 4.2.

2. Preliminaries

Here we collect a few facts about frames and their rings of real-valued continuous functions that will be relevant for our discussion, and fix notation. For the general theory of frames we refer to [14, 18]. Recall that a *frame* is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by 1_L and 0_L respectively, dropping the subscripts if L is clear from the context. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$.

We denote the 'completely below' relation by $\prec\!\prec$. All frames considered here are assumed to be completely regular. A *point* of L is an element p such that $p < 1$ and $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$. The points of any regular frame are precisely those elements which are maximal below the top. We denote the set of all points of L by $\mathrm{Pt}(L)$.

The *right adjoint* of a frame homomorphism h is denoted by h_*. If $h : L \to M$ is an onto frame homomorphism (between regular frames) and $p \in \mathrm{Pt}(L)$, then either $h(p) = 1$ or $h(p) \in \mathrm{Pt}(M)$. Indeed, suppose that $h(p) < 1$. Let $y \in M$ be

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such that $h(p) \leq y < 1$. Then $p \leq h_*(y) < 1$, so that maximality gives $p = h_*(y)$, and hence $h(p) = hh_*(y) = y$. Therefore $h(p) \in \mathrm{Pt}(M)$.

Pointfree function rings can be studied starting with $\mathfrak{O}\mathbb{R}$, as in [2], or starting with the *frame of reals* $\mathfrak{L}(\mathbb{R})$, as in [3]. We follow the latter approach. The frame $\mathfrak{L}(\mathbb{R})$ is defined by generators, which are pairs (p, q) of rationals, and the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$;
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$;
(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$;
(R4) $1_{\mathfrak{L}(\mathbb{R})} = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

A *continuous real-valued function* on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$. The ring $\mathfrak{R}L$ has as its elements continuous real-valued functions on L, with operations determined by the operations of $\mathbb{Q}$ viewed as a lattice-ordered ring as follows.

For $\diamond \in \{+, \cdot, \wedge, \vee\}$ and $\alpha, \beta \in \mathfrak{R}L$,

$$\alpha \diamond \beta = \bigvee\{\alpha(r, s) \wedge \beta(t, u) \mid \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\},$$

where $\langle \cdot, \cdot \rangle$ denotes the open interval in $\mathbb{Q}$, and the given condition means that $x \diamond y \in \langle p, q\rangle$ for any $x \in \langle r, s\rangle$ and $y \in \langle t, u\rangle$.

For any $\alpha \in \mathfrak{R}L$ and $p, q \in \mathbb{Q}$,

$$(-\alpha)(p, q) = \alpha(-q, -p),$$

and for any $r \in \mathbb{R}$, the *constant function* $\mathbf{r}$ is the member of $\mathfrak{R}L$ given by

$$\mathbf{r}(p, q) = \begin{cases} 1 & \text{if } p < r < q, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathfrak{R}L$ becomes an archimedean f-ring with identity, and is therefore *reduced*, meaning that it has no nonzero nilpotent element. Furthermore, the correspondence $L \mapsto \mathfrak{R}L$ is functorial, where, for any frame homomorphism $h : L \to M$, the ℓ-ring homomorphism $\mathfrak{R}h : \mathfrak{R}L \to \mathfrak{R}M$ is given by $\mathfrak{R}h(\alpha) = h \cdot \alpha$, the centre dot designating composition.

An important link between a frame and its ring of real-valued continuous functions is given by the *cozero map* $\mathrm{coz} : \mathfrak{R}L \to L$ defined by

$$\mathrm{coz}\,\varphi = \bigvee\{\varphi(p, 0) \vee \varphi(0, q) \mid p, q \in \mathbb{Q}\} = \varphi((-, 0) \vee (0, -)),$$

where, for any $r \in \mathbb{Q}$,

$$(-, r) = \bigvee\{(p, r) \mid p < r \text{ in } \mathbb{Q}\} \quad \text{and} \quad (r, -) = \bigvee\{(r, q) \mid q > r \text{ in } \mathbb{Q}\}.$$

The cozero map has several known properties (see [2, 3]) that we shall use freely.

A *cozero element* of L is an element of the form $\mathrm{coz}\,\varphi$ for some $\varphi \in \mathfrak{R}L$. The *cozero part* of L, denoted $\mathrm{Coz}\,L$, is the sublattice of L consisting of all cozero elements of L. Because we assume the axiom of choice (in all its guises) throughout, $\mathrm{Coz}\,L$ is, for us, a sub-σ-frame of L. General properties of cozero elements and cozero parts of frames can be found in [8]. Here we highlight the following:

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(a) if $a, b \in \operatorname{Coz} L$ and $a \prec\!\!\prec b$, then there exists $s \in \operatorname{Coz} L$ such that $a \wedge s = 0$ and $s \vee b = 1$;

(b) if (a_n) is a sequence in $\operatorname{Coz} L$ with $\bigvee a_n = 1$, then there is a sequence (b_n) in $\operatorname{Coz} L$ (called a *shrinking* of (a_n)) such that $b_n \prec\!\!\prec a_n$ for each n, and $\bigvee b_n = 1$.

The first result follows from [8, Corollary 3], and the second is proved (more generally) in [7, Proposition 4]. It should be noted though that, regarding the latter result, in the cited paper the rather below relation is used instead of the completely below relation. However, in a regular σ-frame the two relations coincide, modulo countable dependent choice.

An ideal I of $\operatorname{Coz} L$ is *σ-proper* if, for any countable $S \subseteq I$, $\bigvee S < 1$. It is *completely proper* if $\bigvee I < 1$, and it is a *σ-ideal* if it is closed under countable joins. A frame L is *realcompact* if any σ-proper maximal ideal of $\operatorname{Coz} L$ is completely proper. Recall that an element φ of $\mathfrak{R}L$ is said to be *bounded* if $\varphi(p, q) = 1$ for some $p, q \in \mathbb{Q}$. The subring of $\mathfrak{R}L$ consisting of bounded elements is denoted by $\mathfrak{R}^*L$. A frame L is said to be *pseudocompact* if $\mathfrak{R}L = \mathfrak{R}^*L$.

As in the classical case, call an ideal Q of $\mathfrak{R}L$ a *z-ideal* if, for any $\alpha, \beta \in \mathfrak{R}L$, $\operatorname{coz} \alpha = \operatorname{coz} \beta$ and $\beta \in Q$ imply that $\alpha \in Q$. By the rules of the coz function, it is immediate that Q is a z-ideal if and only if $\operatorname{coz} \alpha \leq \operatorname{coz} \beta$ and $\beta \in Q$ imply that $\alpha \in Q$. Every maximal ideal M of $\mathfrak{R}L$ is a z-ideal. To see this, suppose that $\operatorname{coz} \alpha = \operatorname{coz} \beta$ and $\beta \in M$. If $\alpha \notin M$, then the ideal of $\mathfrak{R}L$ generated by M and α is the entire ring. Therefore $\mathbf{1} = \mu + \varrho\alpha$, for some $\mu \in M$ and $\varrho \in \mathfrak{R}L$. But then this implies (the false statement) that M contains the invertible element $\mu^2 + \beta^2$. This element is invertible because

$$1 = \operatorname{coz}(\mu + \varrho\alpha) \leq \operatorname{coz} \mu \vee \operatorname{coz}(\varrho\alpha) \leq \operatorname{coz} \mu \vee \operatorname{coz} \beta = \operatorname{coz}(\mu^2 + \beta^2).$$

Finally, there are several ways of realizing βL, the Stone-Čech compactification of L. We shall regard it as the frame of regular ideals of $\operatorname{Coz} L$. We denote the right adjoint of the join map $\sigma_L : \beta L \to L$ by r_L, and recall that $r_L(a) = \{c \in \operatorname{Coz} L \mid c \prec\!\!\prec a\}$.

3. Real and hyper-real ideals of $\mathfrak{R}L$

We adhere to the standard terminology and practice in partially ordered fields such as can be found in [11, Ch. 5]. In particular, given a maximal ideal M of $\mathfrak{R}L$, we consider the field $\mathfrak{R}L/M$ as a partially ordered field with $\geq$ defined by

$$\alpha + M \geq 0 \Leftrightarrow \alpha - \beta \in M \quad \text{for some } \beta \geq \mathbf{0} \text{ in } \mathfrak{R}L.$$

Recall that an ideal I of an ℓ-ring is said to be an *ℓ-ideal* if whenever $|a| \leq |b|$ and $b \in I$, then $a \in I$. Every maximal ideal M of $\mathfrak{R}L$ is an ℓ-ideal because, if $|\alpha| \leq |\beta|$ with $\beta \in M$, then

$$\operatorname{coz} \alpha = \operatorname{coz}(|\alpha|) \leq \operatorname{coz}(|\beta|) = \operatorname{coz} \beta,$$

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implying that $\alpha \in M$ as M is a z-ideal. Of course this already follows from [12, Lemma 1.1] since $\mathfrak{R}L$ has bounded inversion. Consequently, by [11, Theorem 5.3], if M is a maximal ideal of $\mathfrak{R}L$, then the partial order on $\mathfrak{R}L/M$ is given by

$$\alpha + M \geq 0 \Leftrightarrow \alpha - |\alpha| \in M,$$

and is, in fact, a total order. We provide what is, for our purposes, a more convenient criterion for an element $\alpha + M$ of $\mathfrak{R}L/M$ to be positive based on cozero elements. This criterion is preceded by the following auxiliary result[1].

LEMMA 3.1. *For any* $\varphi \in \mathfrak{R}L$, $\operatorname{coz}(\varphi - |\varphi|) = \varphi(-, 0)$.

PROOF. Since $\varphi \leq |\varphi|$, $\varphi - |\varphi| \leq \mathbf{0}$, and therefore $\operatorname{coz}(\varphi - |\varphi|) = (\varphi - |\varphi|)(-, 0)$. Since $\varphi - |\varphi| \leq \varphi$, $\varphi(-, 0) \leq (\varphi - |\varphi|)(-, 0)$. It therefore remains to show that $(\varphi - |\varphi|)(-, 0) \leq \varphi(-, 0)$. For any $\alpha, \beta \in \mathfrak{R}L$,

$$(\alpha + \beta)(-, 0) = \bigvee \{\alpha(-, s) \wedge \beta(-, -s) \mid s \in \mathbb{Q}\},$$

as has been observed in [3, p. 42]. Therefore, in light of the fact that $\alpha(-, q) = (-\alpha)(-q, -)$, for any $q \in \mathbb{Q}$ and $\alpha \in \mathfrak{R}L$,

$$\begin{aligned}(\varphi - |\varphi|)(-, 0) &= \bigvee \{\varphi(-, s) \wedge (-|\varphi|)(-, -s) \mid s \in \mathbb{Q}\} \\ &= \bigvee \{\varphi(-, s) \wedge |\varphi|(s, -) \mid s \in \mathbb{Q}\} \\ &\leq \varphi(-, 0) \vee \bigvee \{\varphi(-, s) \wedge |\varphi|(s, -) \mid s \in \mathbb{Q}, s > 0\}.\end{aligned}$$

Now, as shown in [2, p. 13], for any $V \in \mathfrak{L}(\mathbb{R})$,

$$|\varphi|(V) = \bigvee_{U \vee (-U) \subseteq V} \varphi(U).$$

Thus, for any $s \in \mathbb{Q}$ with $s > 0$,

$$|\varphi|(s, -) = \bigvee \{\varphi(p, q) \mid \langle p, q \rangle \vee \langle -q, -p \rangle \subseteq \langle s, \infty \rangle\},$$

so that

$$\begin{aligned}\varphi(-, s) \wedge |\varphi|(s, -) &= \bigvee \{\varphi(-, s) \wedge \varphi(p, q) \mid \langle p, q \rangle \vee \langle -q, -p \rangle \subseteq \langle s, \infty \rangle\} \\ &= \bigvee \{\varphi((-, s) \wedge (p, q)) \mid \langle p, q \rangle \vee \langle -q, -p \rangle \subseteq \langle s, \infty \rangle\}.\end{aligned}$$

Call an element (p, q) of $\mathfrak{L}(\mathbb{R})$ *admissible* if $\langle p, q \rangle \vee \langle -q, -p \rangle \subseteq \langle s, \infty \rangle$. In order for (p, q) to be admissible and contribute nontrivially to the join above, we must have $s < p$ or $s < -q$. But now if $s < p$, then $(-, s) \wedge (p, q) = 0_{\mathfrak{L}(\mathbb{R})}$; and if $s < -q$, then

$$(-, s) \wedge (p, q) = (p, q) \leq (-, -s) \leq (-, 0).$$

[1] I thank Rick Ball for pointing out a minor slip in the original proof.

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Thus, in either case,

$$\varphi((-, s) \wedge (p, q)) \leq \varphi(-, 0).$$

It follows therefore that

$$\bigvee\{\varphi(-, s) \wedge |\varphi|(s, -) \mid s \in \mathbb{Q}, s > 0\} \leq \varphi(-, 0),$$

and hence $(\varphi - |\varphi|)(-, 0) \leq \varphi(-, 0)$. □

The criterion, alluded to above, for the positivity of an element of the field $\mathfrak{R}L/M$ is given by the following result.

COROLLARY 3.2. *For any $\alpha \in \mathfrak{R}L$ and M a maximal ideal of $\mathfrak{R}L$, $\alpha + M \geq 0$ if and only if $\alpha(-, 0) \leq \operatorname{coz} \beta$, for some $\beta \in M$.*

PROOF. Suppose that $\alpha + M \geq 0$. Then $\alpha - |\alpha| \in M$. Put $\beta = \alpha - |\alpha|$. By Lemma 3.1, $\operatorname{coz} \beta = \operatorname{coz}(\alpha - |\alpha|) = \alpha(-, 0)$. Thus, $\alpha(-, 0) \leq \operatorname{coz} \beta$.

Conversely, suppose that $\alpha(-, 0) \leq \operatorname{coz} \beta$ for some $\beta \in M$. By Lemma 3.1, $\operatorname{coz}(\alpha - |\alpha|) = \alpha(-, 0) \leq \operatorname{coz} \beta$, so that $\alpha - |\alpha| \in M$ since M is a z-ideal. Therefore $\alpha + M \geq 0$. □

We note that, for any maximal ideal M, the field $\mathfrak{R}L/M$ always contains a subfield isomorphic to the field of real numbers. The mapping $r \mapsto \mathbf{r}$ is a one–one ring homomorphism $\mathbb{R} \to \mathfrak{R}L$. Thus, the map

$$\Psi : \mathbb{R} \to \mathfrak{R}L/M \qquad \text{given by } \Psi(r) = \mathbf{r} + M$$

is a one–one ring homomorphism. It is clear that it is a ring homomorphism. It is one–one because, for any $r \in \mathbb{R}$, $\Psi(r) = 0$ implies that $\mathbf{r} \in M$, so that $\mathbf{r}$ is not invertible, and hence $r = 0$. Now, as usual, M is *real* if $\mathfrak{R}L/M \cong \mathbb{R}$, and *hyper-real* otherwise. The discussion preceding these definitions shows that

> *a maximal ideal M is hyper-real if and only if $\mathfrak{R}L/M$ contains an infinitely large element if and only if it contains an infinitely small element.*

The next step towards our main goal (Proposition 3.6) is to characterize infinitely large elements of $\mathfrak{R}L/M$. Let $\alpha \in \mathfrak{R}L$ and $a \in L$. We say that α is *bounded on* $\uparrow a$ if the composite $\mathfrak{L}(\mathbb{R}) \xrightarrow{\alpha} L \xrightarrow{-\vee a} \uparrow a$ is a bounded element of $\mathfrak{R}(\uparrow a)$. Otherwise, we say that α is *unbounded on* $\uparrow a$. Note that α is bounded on $\uparrow a$ if and only if $a \vee \alpha(-n, n) = 1_L$ for some $n \in \mathbb{N}$.

In one of the implications in the following result we shall use the fact that if M is a maximal ideal of $\mathfrak{R}L$, then $\operatorname{coz}[M]$ is a maximal ideal of $\operatorname{Coz} L$. To see this, let $c \in \operatorname{Coz} L$ be such that $c \vee \operatorname{coz} \alpha \neq 1$, for each $\alpha \in M$. Pick $\gamma \in \mathfrak{R}L$ such that $c = \operatorname{coz} \gamma$. Suppose, by way of contradiction, that $\gamma \notin M$. By maximality, this implies that $\mathbf{1} = \varphi + \tau\gamma$, for some $\varphi \in M$ and $\tau \in \mathfrak{R}L$. But then this implies that $1 \leq \operatorname{coz} \varphi \vee c$, contrary to the nature of c. It follows therefore that $\gamma \in M$, whence we have $c \in \operatorname{coz}[M]$, proving that $\operatorname{coz}[M]$ is a maximal ideal of $\operatorname{Coz} L$.

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LEMMA 3.3. *Let $\alpha \geq \mathbf{0}$ in $\mathfrak{R}L$ and M be a maximal ideal of $\mathfrak{R}L$. The following are equivalent:*

(1) *$\alpha + M$ is infinitely large;*
(2) *$\alpha \notin M$ and $\alpha(-, n) \in \operatorname{coz}[M]$ for each $n \in \mathbb{N}$;*
(3) *α is unbounded on $\uparrow c$ for each $c \in \operatorname{coz}[M]$.*

PROOF. (1) $\Rightarrow$ (2): suppose that $\alpha + M$ is infinitely large. Since $\alpha + M \neq 0$, it follows that $\alpha \notin M$. Now let $n \in \mathbb{N}$. Then $\alpha + M \geq \mathbf{n} + M$ since $\alpha + M$ is infinitely large. Therefore $(\alpha - \mathbf{n}) + M \geq 0$, and hence there is a positive $\tau \in \mathfrak{R}L$ such that $(\alpha - \mathbf{n}) - \tau \in M$. Take $\rho \in M$ such that $\alpha - \mathbf{n} = \tau + \rho$. Thus, $\alpha \geq \rho + \mathbf{n}$, and therefore

$$\begin{aligned}\alpha(-, n) &\leq (\rho + \mathbf{n})(-, n)\\ &\leq (\rho + \mathbf{n})((-, n) \vee (n, -))\\ &= \operatorname{coz}((\rho + \mathbf{n}) - \mathbf{n}) \quad \text{by [2, Lemma 3.2.1]}\\ &= \operatorname{coz} \rho\\ &\in \operatorname{coz}[M].\end{aligned}$$

Since $\operatorname{coz}[M]$ is an ideal of $\operatorname{Coz} L$ and $\alpha(-, n) \in \operatorname{Coz} L$, it follows that $\alpha(-, n) \in \operatorname{coz}[M]$.

(2) $\Rightarrow$ (1): let $n \in \mathbb{N}$. Observe that, for any $\alpha \in \mathfrak{R}L$,

$$\begin{aligned}(\alpha - \mathbf{n})(-, 0) &= \bigvee\{\alpha(-, s) \wedge (-\mathbf{n})(-, -s) \mid s \in \mathbb{Q}\}\\ &= \bigvee\{\alpha(-, s) \wedge \mathbf{n}(s, -) \mid s \in \mathbb{Q}\}\\ &= \bigvee\{\alpha(-, s) \mid s \in \mathbb{Q}, s < n\}\\ &= \alpha(-, n),\end{aligned}$$

the last but one step because $\mathbf{n}(s, -) = 0_{\mathfrak{L}(\mathbb{R})}$ if $s \geq n$, and $\mathbf{n}(s, -) = 1_{\mathfrak{L}(\mathbb{R})}$ if $s < n$. Now suppose that the stated condition holds. Consider any $n \in \mathbb{N}$. By Lemma 3.1,

$$\operatorname{coz}((\alpha - \mathbf{n}) - |\alpha - \mathbf{n}|) = (\alpha - \mathbf{n})(-, 0) \leq \alpha(-, n) \in \operatorname{coz}[M].$$

Therefore $(\alpha - \mathbf{n}) - |\alpha - \mathbf{n}| \in M$ since M is a z-ideal. Thus, $(\alpha - \mathbf{n}) + M \geq 0$, and hence $\alpha + M \geq \mathbf{n} + M$. This shows that $\alpha + M$ is infinitely large.

(2) $\Rightarrow$ (3): if not, let $c \in \operatorname{coz}[M]$ be such that α is bounded on $\uparrow c$. Pick $n \in \mathbb{N}$ such that $c \vee \alpha(-n, n) = 1$. Since $\alpha(-n, n) \leq \alpha(-, n)$, this implies that $1 = c \vee \alpha(-, n) \in \operatorname{coz}[M]$; a contradiction since $\operatorname{coz}[M]$ is a proper ideal.

(3) $\Rightarrow$ (2): let $n \in \mathbb{N}$, and consider any $c \in \operatorname{coz}[M]$. Since α is not bounded on $\uparrow c$, $\alpha(-n, n) \vee c < 1$. Since $\alpha(-n, n) \in \operatorname{Coz} L$ and $\operatorname{coz}[M]$ is a maximal ideal of $\operatorname{Coz} L$, it follows that $\alpha(-n, n) \in \operatorname{coz}[M]$. But now $\alpha(-, n) = \alpha(-n, n)$ since $\alpha \geq \mathbf{0}$, so (2) holds. □

We need two more lemmas. The first, distilled from [1, Proposition 3.13], serves only to facilitate certain calculations in proving the second, which is germane to our goal.

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LEMMA 3.4. *Let α and β be elements of $\mathfrak{R}L$. Then, for any $q \in \mathbb{Q}$,*

$$(\alpha + \beta)(q, -) = \bigvee_{r+s=q} (\alpha(r, -) \wedge \beta(s, -)).$$

Of course in [1] this result is stated for $q \in \mathbb{R}$ and the summands r, s also in $\mathbb{R}$. Clearly the result also holds as we have restated it, keeping in mind that, in our case, the summands r, s are also restricted to come from $\mathbb{Q}$.

In order to state the second lemma we require some background. Suppose that (φ_n) is a sequence of positive elements of $\mathfrak{R}L$. The set

$$\{(\varphi_1 \wedge \mathbf{2^{-1}}) + \cdots + (\varphi_n \wedge \mathbf{2^{-n}}) \mid n \in \mathbb{N}\}$$

has a supremum in the poset $\mathfrak{R}L$ (see [6, Section 6] and [19, Lemma 4]). This supremum is denoted by

$$\sum_{n=1}^{\infty}(\varphi_n \wedge \mathbf{2^{-n}}).$$

The property of this supremum which we require is given by the following lemma.

LEMMA 3.5. *Let (φ_n) be a sequence of positive elements of $\mathfrak{R}L$, and put*

$$\varphi = \sum_{n=1}^{\infty}(\varphi_n \wedge \mathbf{2^{-n}}).$$

Then, for any $m \in \mathbb{N}$,

$$\varphi\left(\frac{1}{2^m}, -\right) \leq \operatorname{coz} \varphi_1 \vee \cdots \vee \operatorname{coz} \varphi_m.$$

PROOF. Write $\varphi = \alpha + \beta$ with

$$\alpha = (\varphi_1 \wedge \mathbf{2^{-1}}) + \cdots + (\varphi_m \wedge \mathbf{2^{-m}}) \quad \text{and} \quad \beta = \sum_{j=m+1}^{\infty} (\varphi_j \wedge \mathbf{2^{-j}}).$$

Since, for each $k \geq m + 1$,

$$\sum_{j=m+1}^{k} (\varphi_j \wedge \mathbf{2^{-j}}) \leq \frac{\mathbf{1}}{\mathbf{2^{m+1}}} + \cdots + \frac{\mathbf{1}}{\mathbf{2^k}} \leq \frac{\mathbf{1}}{\mathbf{2^m}},$$

it follows that $\beta \leq \mathbf{1/2^m}$. For brevity, let us put $q = 1/2^m$. Now, by Lemma 3.4,

$$\begin{aligned}
\varphi(q, -) &= \bigvee_{r+s=q} (\alpha(r, -) \wedge \beta(s, -)) \\
&= \bigvee_{r+s=q;\, s<q} (\alpha(r, -) \wedge \beta(s, -)) \vee \bigvee_{r+s=q;\, s\geq q} (\alpha(r, -) \wedge \beta(s, -)) \\
&\leq \bigvee_{r+s=q;\, s<q} \alpha(r, -) \vee \bigvee_{r+s=q;\, s\geq q} (\alpha(r, -) \wedge \mathbf{q}(s, -)) \\
&= \bigvee_{r+s=q;\, s<q} \alpha(r, -) \quad \text{since } \mathbf{q}(s, -) = 0 \text{ for } s \geq q
\end{aligned}$$

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$$
\begin{aligned}
&\leq \alpha(0, -) \quad \text{since } s < q \text{ and } r + s = q \text{ imply that } r > 0 \\
&= \operatorname{coz} \alpha \quad \text{since } \alpha \geq \mathbf{0} \\
&= \operatorname{coz}(\varphi_1 \wedge \mathbf{2^{-1}}) \vee \cdots \vee \operatorname{coz}(\varphi_m \wedge \mathbf{2^{-m}}) \\
&= \operatorname{coz} \varphi_1 \vee \cdots \vee \operatorname{coz} \varphi_m,
\end{aligned}
$$

as required. □

We are now equipped to characterize real ideals. Observe that if $\alpha + M$ is infinitely large, then $\alpha + M \geq 0$, and so $\alpha + M = |\alpha| + M$. Thus, if $\alpha + M$ is infinitely large, we may assume, without loss of generality, that $\alpha \geq \mathbf{0}$.

PROPOSITION 3.6. *The following are equivalent for a maximal ideal M of $\mathfrak{R}L$.*

(1) *M is real.*
(2) *$\operatorname{coz}[M]$ is a σ-ideal.*
(3) *$\operatorname{coz}[M]$ is σ-proper.*

PROOF. Since $\operatorname{coz}[M]$ is a maximal ideal of $\operatorname{Coz} L$ whenever M is a maximal ideal of $\mathfrak{R}L$, we see that the equivalence of (2) and (3) is a special case of [9, Lemma 1].

Next, we show that that (2) implies (1). So assume that $\operatorname{coz}[M]$ is a σ-ideal but M is hyper-real. Then $\alpha + M$ is infinitely large for some $\alpha \geq \mathbf{0}$ in $\mathfrak{R}L$. By Lemma 3.3, $\alpha(-, n) \in \operatorname{coz}[M]$, for each $n \in \mathbb{N}$. Since $\operatorname{coz}[M]$ is a σ-ideal, it contains the element

$$
\bigvee_{n \in \mathbb{N}} \alpha(-, n) = \alpha\Big(\bigvee_{n \in \mathbb{N}} (-, n)\Big) = \alpha(1_{\mathfrak{L}(\mathbb{R})}) = 1,
$$

which is false because $\operatorname{coz}[M]$ is a proper ideal of $\operatorname{Coz} L$.

Finally, assume that (2) fails. We will show that (1) fails. Take $\varphi_n \geq \mathbf{0}$ in M for each $n \in \mathbb{N}$ such that $\bigvee \operatorname{coz} \varphi_n \notin \operatorname{coz}[M]$. Let φ be the element of $\mathfrak{R}L$ defined by

$$
\varphi = \sum_{n=1}^{\infty} (\varphi_n \wedge \mathbf{2^{-n}}).
$$

By Lemma 3.5,

$$
\varphi(2^{-n}, -) \leq \operatorname{coz} \varphi_1 \vee \cdots \vee \operatorname{coz} \varphi_m,
$$

for each $n \in \mathbb{N}$. Now, by Lemma 3.1,

$$
\begin{aligned}
\operatorname{coz}((\mathbf{2^{-n}} - \varphi) - |\mathbf{2^{-n}} - \varphi|) &= (\mathbf{2^{-n}} - \varphi)(-, 0) \\
&= (\varphi - \mathbf{2^{-n}})(0, -) \\
&= \varphi(2^{-n}, -) \\
&\leq \operatorname{coz} \varphi_1 \vee \cdots \vee \operatorname{coz} \varphi_n \\
&\in \operatorname{coz}[M].
\end{aligned}
$$

It then follows from Corollary 3.2 that $(\mathbf{2^{-n}} - \varphi) + M \geq 0$, so that

$$
0 \leq \varphi + M \leq \mathbf{2^{-n}} + M \leq \frac{\mathbf{1}}{\mathbf{n}} + M. \tag{†}
$$

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That $\varphi + M \geq 0$ follows from the fact that $\varphi \geq \mathbf{0}$. Since $\operatorname{coz}\varphi = \bigvee \operatorname{coz}\varphi_n \notin \operatorname{coz}[M]$, it follows that $\varphi \notin M$, and hence $\varphi + M > 0$. Since (†) holds for every $n \in \mathbb{N}$ and $\varphi + M > 0$, we deduce that $\varphi + M$ is infinitely small. Therefore (1) does not hold. □

As in the classical case, we say that an ideal Q of $\mathfrak{R}L$ is *fixed* if $\bigvee \operatorname{coz}[Q] < 1$, and we say that it is *free* otherwise. The foregoing characterization shows that *every fixed maximal ideal of* $\mathfrak{R}L$ *is real.*

A noteworthy corollary to the above proposition is a characterization of hyper-real ideals in terms of properties of υL, the realcompact coreflection of L. Let us recall how υL is constructed (see [9] or [16] for details). The regular Lindelöf coreflection of L, denoted λL, is the frame of σ-ideals of $\operatorname{Coz} L$ (see [15]). The frame υL is constructed in the following manner. For any $t \in L$, let $[t] = \{c \in \operatorname{Coz} L \mid c \leq t\}$. The map $\ell : \lambda L \to \lambda L$ given by

$$\ell(J) = \left[\bigvee J\right] \wedge \bigwedge \{P \in \operatorname{Pt}(\lambda L) \mid J \leq P\}$$

is a nucleus. The frame υL is defined to be $\operatorname{Fix}(\ell)$.

The map

$$\beta L \to \lambda L \quad \text{given by } I \mapsto \langle I \rangle_\sigma,$$

where $\langle \cdot \rangle_\sigma$ signifies σ-ideal generation in $\operatorname{Coz} L$, is an onto frame homomorphism (see [9]). Thus, there is an onto frame homomorphism

$$\beta L \longrightarrow \upsilon L \quad \text{given by } I \mapsto \ell(\langle I \rangle_\sigma).$$

We characterize hyper-real ideals of $\mathfrak{R}L$ in terms of this map. We have thus far not needed to know what maximal ideals of $\mathfrak{R}L$ look like. For the next result we do need an explicit description (see [10]). For each $I \in \beta L$, the ideal $\mathbf{M}^I$ of $\mathfrak{R}L$ is defined by

$$\mathbf{M}^I = \{\varphi \in \mathfrak{R}L \mid r_L(\operatorname{coz}\varphi) \subseteq I\}.$$

Then maximal ideals of $\mathfrak{R}L$ are precisely the ideals $\mathbf{M}^I$ for $I \in \operatorname{Pt}(\beta L)$. Let us also (mimicking the classical case) introduce the following notation. For any $I \in \beta L$, set

$$\mathbf{A}^I = \{c \in \operatorname{Coz} L \mid r_L(c) \subseteq I\}.$$

Clearly, $\operatorname{coz}[\mathbf{M}^I] = \mathbf{A}^I$, so that $\mathbf{A}^I$ is an ideal of $\operatorname{Coz} L$. The latter can also be shown directly making use of the fact that, for any $a, b \in \operatorname{Coz} L$, $r_L(a \vee b) = r_L(a) \vee r_L(b)$. As observed in the preliminaries, if $I \in \operatorname{Pt}(\beta L)$, then either $\langle I \rangle_\sigma = 1_{\lambda L}$ or $\langle I \rangle_\sigma \in \operatorname{Pt}(\lambda L)$.

COROLLARY 3.7. *For any point I of βL the following conditions are equivalent:*

(1) $\mathbf{M}^I$ *is hyper-real;*
(2) I *is not σ-proper;*
(3) $\langle I \rangle_\sigma = 1_{\lambda L}$*;*
(4) $\ell(\langle I \rangle_\sigma) = 1_{\upsilon L}$.

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PROOF. (1) ⇒ (2): if $\mathbf{M}^I$ is hyper-real, then, by Proposition 3.6, $\mathbf{A}^I$ is not σ-proper. Take a sequence (s_n) in $\mathbf{A}^I$ such that $\bigvee s_n = 1$. Let (t_n) be a shrinking of (s_n). Since $r_L(s_n) \subseteq I$ and $t_n \prec\!\!\prec s_n$, it follows that $t_n \in I$ for each n, and hence I is not σ-proper.

The implications (2) ⇒ (3) ⇒ (4) are immediate.

(4) ⇒ (1): if $\ell(\langle I\rangle_\sigma) = 1_{\upsilon L}$, then

$$\Big[\bigvee \langle I\rangle_\sigma\Big] = \bigwedge\{P \in \mathrm{Pt}(\lambda L) \mid \langle I\rangle_\sigma \le P\} = 1_{\upsilon L}.$$

Since $I \in \mathrm{Pt}(\beta L)$, either $\langle I\rangle_\sigma = 1_{\lambda L}$ or $\langle I\rangle_\sigma \in \mathrm{Pt}(\lambda L)$. The latter is not possible, otherwise we have

$$1_{\lambda L} = 1_{\upsilon L} = \bigwedge\{P \in \mathrm{Pt}(\lambda L) \mid \langle I\rangle_\sigma \le P\} \le \langle I\rangle_\sigma,$$

which is a contradiction. Therefore we must have $\langle I\rangle_\sigma = 1_{\lambda L}$, which is clearly equivalent to saying that I is not σ-proper. Since, as subsets of $\mathrm{Coz}\, L$, we have that $I \subseteq \mathbf{A}^I$, it follows that $\mathbf{A}^I$ is not σ-proper. Thus $\mathbf{M}^I$ is hyper-real. □

We remark, in passing, that the equivalence of (1) and (4) generalizes the spatial result that, for a point p of βX, the maximal ideal $\mathbf{M}^p$ of $C(X)$ is hyper-real if and only if $p \notin \upsilon X$.

4. Some applications

Using the machinery developed in the foregoing section, we now prove the results about realcompactness and pseudocompactness stated in the introduction. Recall from the preliminaries the definition of realcompact frames that we have adopted. As pointed out in [9], in the presence of the axiom of countable dependent choice (which is weaker than AC, which we have assumed throughout), *L is realcompact if and only if every σ-proper point of βL is completely proper*. Here is the first of the desired results.

PROPOSITION 4.1. *A completely regular frame L is realcompact if and only if every free maximal ideal of $\mathfrak{R}L$ is hyper-real.*

PROOF. Assume that L is realcompact, and let M be a free maximal ideal of $\mathfrak{R}L$. Take a point I of βL such that $M = \mathbf{M}^I$. By freeness, $\bigvee \mathbf{A}^I = 1$. For any $a \in \mathbf{A}^I$, $r_L(a) \subseteq I$, so that, by complete regularity, we have $a = \bigvee r_L(a) \le \bigvee I$. Therefore $\bigvee I = 1$. By realcompactness, I is not σ-proper, and hence, by Corollary 3.7, $\mathbf{M}^I$ is hyper-real.

Conversely, let I be a point of βL with $\bigvee I = 1$. Then $\mathbf{M}^I$ is hyper-real, by hypothesis, and therefore I is not σ-proper, by Corollary 3.7. Consequently, L is realcompact. □

REMARK 4.2. When we showed this proof to Banaschewski, after much rumination he produced a choice-free proof which does not require the explicit description of real ideals given here. Here is an outline of his very delectable argument. Recall that L is realcompact if and only if any ring homomorphism $\varphi : \mathfrak{R}L \to \mathbb{R}$ is $\mathfrak{R}\xi$ for some frame homomorphism $\xi : L \to \mathbf{2}$. Denote by **AfR** the category of commutative archimedean f-rings with identity, where the morphisms are ℓ-ring homomorphisms.

Recall that an *archimedean kernel* of an archimedean f-ring A is an ℓ-ideal K such that, for any $a, b \geq 0$ in A, if $(na - b)^+ \in K$ for every n, then $a \in K$. The set $\mathfrak{K}A$ of all archimedean kernels of A is a completely regular frame, and the correspondence $A \mapsto \mathfrak{K}A$ is a functor **AfR** $\to$ **CRegFrm** which is left adjoint to $\mathfrak{R}$: **CregFrm** $\to$ **AfR** (see [5, 6]) with adjunction map

$$\mu_L : \mathfrak{K}(\mathfrak{R}L) \to L \quad \text{given by } J \mapsto \bigvee \operatorname{coz}[J].$$

Now, any real maximal ideal of $\mathfrak{R}L$ is a maximal archimedean kernel since any ring homomorphism $\mathfrak{R}L \to \mathbb{R}$ is, in fact, an ℓ-ring homomorphism. Conversely, for any maximal $Q \in \mathfrak{K}(\mathfrak{R}L)$, $\mathfrak{R}L/Q \cong \mathbb{R}$ because it is a totally ordered archimedean ring with identity, containing $\mathbb{R}$ and hence equal to $\mathbb{R}$ (up to isomorphism), so that Q is a real maximal ideal of $\mathfrak{R}L$.

Let L be realcompact, and consider any maximal $Q \in \mathfrak{K}(\mathfrak{R}L)$. Then there is a ring homomorphism $\varphi : \mathfrak{R}L \to \mathbb{R}$ with $\operatorname{Ker}(\varphi) = Q$, and, by realcompactness, a frame homomorphism $\xi : L \to 2$ such that $\varphi = \mathfrak{R}\xi$ (using the fact that $\mathfrak{R}\mathbf{2} \cong \mathbb{R}$). Now, if $\gamma \in Q$ then $\xi\gamma = 0$, and hence $\xi(\operatorname{coz}\gamma) = \operatorname{coz}(\xi\gamma) = 0$, showing that $\xi(\mu_L(Q)) = 0$ and therefore $\mu_L(Q) < 1$, saying that Q is fixed.

Conversely, given any $\varphi : \mathfrak{R}L \to \mathbb{R}$, let $Q = \operatorname{Ker}(\varphi)$, so that $\mu_L(Q) < 1$ by the present hypothesis. Since $Q \in \operatorname{Pt}(\mathfrak{K}(\mathfrak{R}L))$ with $\mu_L(Q) < 1$, we have that $\mu_L(Q) \in \operatorname{Pt}(L)$, and so there is a frame homomorphism $\xi : L \to \mathbf{2}$ such that $\xi(\mu_L(Q)) = 0$, and hence $\xi\mu_L = \zeta$ for the frame homomorphism $\zeta : \mathfrak{K}(\mathfrak{R}L) \to \mathbf{2}$ determined by Q. Now the aim is to show that $\varphi = \mathfrak{R}\xi$, and for this it suffices to see that $\varphi(\gamma) = 0$ if and only if $\xi\gamma = 0$ for any $\gamma \in \mathfrak{R}L$ because, for any $r \in \mathbb{R}$, $\varphi(\mathbf{r}) = r = \xi\mathbf{r}$. If $\varphi(\gamma) = 0$, then $\gamma \in Q$, and so $\operatorname{coz}\gamma \leq \mu_L(Q)$, and therefore

$$\operatorname{coz}(\xi\gamma) = \xi(\operatorname{coz}\gamma) \leq \xi\mu_L(Q) = \zeta(Q) = 0,$$

showing that $\xi\gamma = 0$. Conversely, if $\xi\gamma = 0$, then $\xi(\operatorname{coz}\gamma) = 0$, and since $J = \{\alpha \in \mathfrak{R}L \mid \operatorname{coz}\alpha \leq \operatorname{coz}\gamma\}$ is an archimedean kernel, it follows that

$$\zeta(J) = \xi\mu_L(J) = \xi(\operatorname{coz}\gamma) = 0,$$

thus $J \subseteq Q$, by the definition of ζ, while $\gamma \in J$ then shows that $\varphi(\gamma) = 0$. This completes the proof.

Next, we present a characterization of realcompact frames which, as stated in the introduction, is a frame version of Mròwka's [17] characterization of realcompact spaces.

PROPOSITION 4.3. *A completely regular frame L is realcompact if and only if for every $I \in \operatorname{Pt}(\beta L)$ with $\bigvee I = 1$, there exists $\varphi \in \mathfrak{R}(\beta L)$ such that $\operatorname{coz}\varphi \leq I$ and $\bigvee \varphi(0, -) = 1$.*

PROOF. ($\Rightarrow$): suppose that L is realcompact, and let I be a point of βL with $\bigvee I = 1$. By realcompactness, there is a sequence (c_n) in I such that $\bigvee c_n = 1$. For each n

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choose $\gamma_n \geq \mathbf{0}$ in $\mathfrak{R}L$ such that $\operatorname{coz}\gamma_n = c_n$. Let $\gamma = \sum_{n=1}^{\infty}(\gamma_n \wedge \mathbf{2^{-n}})$. Then $\mathbf{0} \leq \gamma \leq \mathbf{1}$, so that γ is bounded. Since $\sigma_L : \beta L \to L$ is a C^*-quotient map (see [2] for details), there exists $\varphi \in \mathfrak{R}(\beta L)$ such that $\sigma_L \cdot \varphi = \gamma$. Now Lemma 3.5 shows that, for any $n \in \mathbb{N}$,

$$\gamma(2^{-n}, -) \leq c_1 \vee \cdots \vee c_n;$$

upon invoking the equality $\sigma_L \cdot \varphi = \gamma$, this reduces to

$$\sigma_L(\varphi(2^{-n}, -)) \leq c_1 \vee \cdots \vee c_n,$$

so that, in view of r_L being the right adjoint of σ_L,

$$\varphi(2^{-n}, -) \leq r_L(c_1 \vee \cdots \vee c_n) \leq I,$$

the latter inequality holding because $c_1 \vee \cdots \vee c_n \in I$. Since this is true for every n,

$$\varphi(0, -) = \bigvee_{n \in \mathbb{N}} \varphi(2^{-n}, -) \leq I.$$

On the other hand, $\gamma(-, 0) = 0$ since $\gamma \geq \mathbf{0}$, and so

$$\bigvee \varphi(-, 0) = (\sigma_L \cdot \varphi)(-, 0) = \gamma(-, 0) = 0,$$

which implies that $\varphi(-, 0) = 0_{\beta L}$. It follows therefore that $\operatorname{coz}\varphi \leq I$, establishing the first part of the desired result. The second part holds because, in light of the equality $\gamma(0, -) = \operatorname{coz}\gamma$ (as $\gamma \geq \mathbf{0}$),

$$\bigvee \varphi(0, -) = (\sigma_L \cdot \varphi)(0, -) = \gamma(0, -) = \operatorname{coz}\gamma = \bigvee c_n = 1.$$

($\Leftarrow$): let I be a point of βL such that $\bigvee I = 1$. Take $\varphi \in \mathfrak{R}(\beta L)$ with the hypothesized features. Now, $\operatorname{coz}\varphi$ is a cozero element of the Lindelöf frame βL. Therefore it is a Lindelöf element by [8, Corollary 4]. Thus, in light of the fact that

$$\operatorname{coz}\varphi = \bigvee \{r_L(c) \mid c \in \operatorname{coz}\varphi\},$$

there is a sequence (s_n) in $\operatorname{coz}\varphi$ such that

$$\operatorname{coz}\varphi = \bigvee_{n \in \mathbb{N}} r_L(s_n).$$

Applying the frame homomorphism σ_L to this gives

$$\bigvee s_n = \bigvee \operatorname{coz}\varphi \geq \bigvee \varphi(0, -) = 1,$$

which completes the proof since the sequence (s_n) is in I as $\operatorname{coz}\varphi \subseteq I$, by hypothesis. □

We conclude by giving a characterization of pseudocompact frames in terms of real ideals. This result also extends a similar one for spaces.

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Proposition 4.4. *A completely regular frame L is pseudocompact if and only if every maximal ideal of $\mathfrak{R}L$ is real.*

Proof. Suppose that L is not pseudocompact. Then there is an unbounded function $\varphi \geq \mathbf{0}$ in $\mathfrak{R}L$. For each $n \in \mathbb{N}$, let $a_n = \varphi(-n, n)$. Then $a_n \in \operatorname{Coz} L$ and $a_n \leq a_{n+1} < 1$ for each n. Let $I \subseteq \operatorname{Coz} L$ be defined by

$$I = \{z \in \operatorname{Coz} L \mid z \leq a_k \text{ for some } k \in \mathbb{N}\}.$$

Then I is a proper ideal of Coz L, and is therefore contained in some maximal ideal J of Coz L. The set

$$M = \{\alpha \in \mathfrak{R}L \mid \operatorname{coz} \alpha \in J\}$$

is a maximal ideal of $\mathfrak{R}L$. That it is an ideal is easy to check using the rules of the coz map. To see maximality, let $\tau \notin M$. Then $\operatorname{coz} \tau \notin J$, and so there exists $c \in J$ such that $\operatorname{coz} \tau \vee c = 1$. If $\gamma \in \mathfrak{R}L$ is such that $\operatorname{coz} \gamma = c$, then $\operatorname{coz}(\tau^2 + \gamma^2) = 1$, which implies that the ideal generated by τ and M is the entire ring. Now note that $a_n \in \operatorname{coz}[M]$, for each n. Furthermore, $\varphi \notin M$, otherwise we have $\operatorname{coz} \varphi \in \operatorname{coz}[M]$, and hence

$$1 = \operatorname{coz} \varphi \vee \varphi(-1, 1) \in \operatorname{coz}[M],$$

contrary to the fact that M is proper. Since $\varphi \geq \mathbf{0}$, $\varphi(-, 0) = 0$, and therefore, for every $n \in \mathbb{N}$,

$$\varphi(-, n) = \varphi(-n, n) \in \operatorname{coz}[M].$$

By Lemma 3.3, it follows that $\varphi + M$ is infinitely large, so that M is hyper-real.

Conversely, suppose that $\mathfrak{R}L$ has a hyper-real maximal ideal, M, say. Take $\alpha \geq \mathbf{0}$ in $\mathfrak{R}L$ such that $\alpha + M$ is infinitely large. By Lemma 3.3, α is unbounded on $\uparrow c$, for some $c \in \operatorname{coz}[M]$. This clearly implies α is an unbounded function in $\mathfrak{R}L$. Therefore L is not pseudocompact. □

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