J. Austral. Math. Soc. (*Series A*) 39 (1985), 146–158

NOTES ON CONGRUENCES ON REGULAR SEMIGROUPS. I

R. J. KOCH and B. L. MADISON

(Received 18 November 1982)

Communicated by T. E. Hall

Abstract

Four properties of congruences on a regular semigroup S are studied and compared. Let $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{D}$ denote Green's relations and let $V = \{(a, b) \in S \times S | a$ and b are mutually inverse$\}$. A congruence ρ on S is (1) *rectangular* provided $\rho \cap \mathscr{D} = (\rho \cap \mathscr{L}) \circ (\rho \cap \mathscr{R})$, (2) *V-commuting* provided $\rho \circ V = V \circ \rho$, (3) $(\mathscr{L}, \mathscr{R})$*-commuting* provided $\mathscr{L} \circ \rho = \rho \circ \mathscr{L}$ and $\mathscr{R} \circ \rho = \rho \circ \mathscr{R}$, and (4) *idempotent-regular* provided each idempotent ρ-class is a regular subsemigroup of S.

A rectangular congruence is $(\mathscr{L}, \mathscr{R})$-commuting and a V-commuting congruence is idempotent-regular. If ρ is idempotent-regular and $(\mathscr{L}, \mathscr{R})$-commuting then ρ is V-commuting. Examples and conditions are given to show what other implications among the four properties hold. In addition to characterizations of the properties, these are studied in the presence of other conditions on S. For example, if S is a stable regular semigroup, then each congruence under $\mathscr{D}$ is rectangular.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 20 M 15; secondary 20 M 20.

We introduce several properties of congruences on regular semigroups and study their inter-relationships. Let ρ be a congruence on a regular semigroup S.

(1) ρ is *rectangular* if whenever $a\rho b$ and $a\mathscr{D}b$, there exists $x, y \in S$ with $a\mathscr{R}x\mathscr{L}b\mathscr{R}y\mathscr{L}a$ and $a\rho x\rho y\rho b$.

(2) ρ is *V-commuting* if $\rho \circ V = V \circ \rho$ where $V = \{(a, b) \in S \times S$: a and b are mutually inverse$\}$.

(3) ρ is $(\mathscr{L}, \mathscr{R})$*-commuting* if $\rho \circ \mathscr{L} = \mathscr{L} \circ \rho$ and $\rho \circ \mathscr{R} = \mathscr{R} \circ \rho$.

(4) ρ is *idempotent-regular* if $I^2 = I \in S/\rho$ implies that the ρ-class $\bar{I}$ is a regular subsemigroup of S.

The second author was supported by dual funding from NSF-EPSCOR Grant ISP8011447 and the State of Arkansas.

Properties (2) and (3) were introduced by the authors [6]. Property (4) was introduced by Nambooripad [9]. These notions have been applied by one of the authors in [7].

Among the results that we establish is the fact that if S is a stable regular semigroup, then each congruence under $\mathscr{D}$ is rectangular. We give various characterizations of properties (1), (2), and (3), some of which involve the "fullness" of multiplication of certain equivalence classes. We characterize the property $\rho \circ V = V \circ \rho$ in terms of properties of the decomposition induced by ρ on the set of idempotents. We discuss the class (VC) of semigroups on which each congruence is V-commuting and show that (VC) contains the full transformation semigroups. Also, we present several examples to delineate the results. We are indebted to K. Byleen for an observation about the four-spiral semigroup (Example 5.3) and to J. D. Lawson for the initial part of Example 5.2.

1. Notation and conventions

We assume throughout the paper that S is a regular semigroup. Let ρ be a congruence on S. We denote the natural map $S \to S/\rho$ by ρ or ϕ and for $C \in S/\rho$ we denote $\phi^{-1}(C)$ by $\overline{C}$ to indicate it as a subset of S rather than as an element of S/ρ. Elements of S/ρ will be represented by upper case letters with E, F, I, and K being idempotents. Elements of S will be lower case letters with e, f, i, j, k, and h being idempotents.

The notation $p \perp q$ indicates that p and q are mutually inverse elements, and $V = \{(p, q) \colon p \perp q\}$. For a relation $\mathscr{K}$ on S, we will use the notation $(a, b) \in \mathscr{K}$ and $a \mathscr{K} b$ interchangeably and $\mathscr{K}_x$ will represent the $\mathscr{K}$-class of x. Terminology from Clifford and Preston [3] will be used, and we make free use of Green's translational lemmas and the results of Miller and Clifford on the algebra of $\mathscr{D}$-classes.

2. Rectangular congruences

As above, a congruence ρ on a regular semigroup S is said to be rectangular provided $\rho \cap \mathscr{D} = (\rho \cap \mathscr{L}) \circ (\rho \cap \mathscr{R})$. We note that $\rho \cap \mathscr{D} \supset (\rho \cap \mathscr{L}) \circ (\rho \cap \mathscr{R})$ always.

PROPOSITION 2.1. *Any congruence ρ on a semigroup S satisfies the following*:
(a) $(\rho \cap \mathscr{R}) \circ \mathscr{L} = \mathscr{L} \circ (\rho \cap \mathscr{R})$;
(b) $(\rho \cap \mathscr{L}) \circ \mathscr{R} = \mathscr{R} \circ (\rho \cap \mathscr{L})$;
(c) $(\rho \cap \mathscr{L}) \circ (\rho \cap \mathscr{R}) = (\rho \cap \mathscr{R}) \circ (\rho \cap \mathscr{L})$.

https://doi.org/10.1017/S1446788700022424 Published online by Cambridge University Press

PROOF. We show $(\rho \cap \mathscr{R}) \circ \mathscr{L} \subset \mathscr{L} \circ (\rho \cap \mathscr{R})$. The other containments follow similarly. Let $(a, b) \in (\rho \cap \mathscr{R}) \circ \mathscr{L}$. Then there is x so that $a(\rho \cap \mathscr{R})x\mathscr{L}b$. Since $b\mathscr{L}x$ there is a t so that $b = tx$. Hence $ta(\rho \cap \mathscr{R})tx = b$ and $a\mathscr{L}ta$. Thus $(a, b) \in \mathscr{L} \circ (\rho \cap \mathscr{R})$.

PROPOSITION 2.2. *Let ρ be a congruence on a regular semigroup S and $\rho \subset \mathscr{D}$. If ρ is rectangular, then $\rho \circ \mathscr{L} = \mathscr{L} \circ \rho$ and $\rho \circ \mathscr{R} = \mathscr{R} \circ \rho$, that is, ρ is $(\mathscr{L}, \mathscr{R})$-commuting.*

PROOF. Since ρ is rectangular and $\rho \subset \mathscr{D}$, we have $\rho \subset (\mathscr{L} \cap \rho) \circ (\mathscr{R} \cap \rho) = (\mathscr{R} \cap \rho) \circ (\mathscr{L} \cap \rho)$. Therefore $\rho \circ \mathscr{L} \subset (\mathscr{L} \cap \rho) \circ (\mathscr{R} \cap \rho) \circ \mathscr{L} = (\mathscr{L} \cap \rho) \circ \mathscr{L} \circ (\mathscr{R} \cap \rho) \subset \mathscr{L} \circ (\mathscr{R} \cap \rho) \subset \mathscr{L} \circ \rho$. The other parts are similar.

EXAMPLE 2.3. It follows from [3, Theorem 10.58] that if X is a finite set and if ρ is a congruence on the transformation semigroup $\mathscr{T}_X$ on X, then ρ is rectangular. If, however, X is infinite, then ρ might not be rectangular. For instance, let $X = N = \{1, 2, 3, 4, \ldots\}$, and take $\rho = \{(a, b) | dr(a, b) < \infty\}$ where $D(a, b) = \{x \in N | a(x) \neq b(x)\}$ and $dr(a, b) = \max\{|a(D(a, b))|, |b(D(a, b))|\}$. For some $(a, b) \in \rho \cap \mathscr{D}$ we will show that there is no $c \in \mathscr{T}_N$ so that $a(\mathscr{R} \cap \rho)c(\mathscr{L} \cap \rho)b$, that is, there is no $c \in \mathscr{T}_N$ so that $\ker(a) = \ker(c)$, $c(N) = b(N)$, $dr(a, c) < \infty$, and $dr(c, b) < \infty$. Choose $b\colon N \to N$ to be the identity and $a\colon N \to N$ by $a(1) = a(2) = 1$ and $a(n) = n$ for $n > 3$. Suppose there is $c \in \mathscr{T}_N$ satisfying the above. Since b is one-to-one and $dr(b, c) < \infty$, then $D(b, c)$ will be finite. Hence there is M so that for $n \geqslant M$, $c(n) = n$. Hence c maps $\{1, 2, \ldots, M\}$ onto $\{1, 2, \ldots, M\}$, but $c(1) = c(2)$. This is a contradiction. Hence ρ is not rectangular.

A congruence ρ on a regular semigroup S is said to be *$\mathscr{H}$-covering* provided for each $a \in S$, $\phi(H_a) = H_{\phi(a)}$.

THEOREM 2.4. *If a congruence ρ on a regular semigroup S is $\mathscr{H}$-covering, then ρ is rectangular. If $\rho \subset \mathscr{D}$, the converse holds.*

PROOF. Suppose ρ is $\mathscr{H}$-covering. We show $(\rho \cap \mathscr{D}) \subset (\mathscr{L} \cap \rho) \circ (\mathscr{R} \cap \rho)$. Let $(a, b) \in \rho \cap \mathscr{D}$ and $t \in R_b \cap L_a$. Since $t\mathscr{L}a\rho b\mathscr{R}t$ we have $\phi(t)\mathscr{H}\phi(a)$, and therefore $\phi(H_t) = H_{\phi(t)} = H_{\phi(a)} = \phi(H_a)$. So, there exists $x \in H_t$ with $\phi(x) = \phi(a)$. Now $(a, x) \in \mathscr{L} \cap \rho$ and $(x, b) \in \mathscr{R} \cap \rho$, so $(a, b) \in (\mathscr{L} \cap \rho) \circ (\mathscr{R} \cap \rho)$.

Suppose ρ is rectangular and $\rho \subset \mathscr{D}$. We show that for $a \in S$, $H_{\phi(a)} \subset \phi(H_a)$; the other containment is clear. Let $X \in H_{\phi(a)}$ and $\phi(a) = A$. By Theorem 2.2, ρ is $(\mathscr{L}, \mathscr{R})$-commuting. Since $X\mathscr{L}A$, there exists (by [4]) $x \in \bar{X}$ and $p \in \bar{A}$ with $x\mathscr{L}p$. Then $a\rho p\mathscr{L}x$ implies there is q so that $a\mathscr{L}q\rho x$. Similarly, since $X\mathscr{R}A$, there is

$y \in \bar{X}$ and $r \in \bar{A}$ with $y\mathscr{R}r$. Then $a\rho r\mathscr{R}y$ implies there is s so that $a\mathscr{R}s\rho y$. Hence $q\mathscr{L}a\mathscr{R}s$ with $q, s \in \bar{X}$. Since ρ is rectangular, there is $w \in \bar{X}$ with $q\mathscr{L}w\mathscr{R}s$. Thus $w \in H_a$ and $X = \phi(w) \in \phi(H_a)$.

COROLLARY 2.5. (1) *If S is combinatorial (that is, if $\mathscr{H}$ is trivial) and $\rho \subset \mathscr{D}$ is rectangular, then S/ρ is combinatorial.*

(2) *If S/ρ is combinatorial, then ρ is rectangular.*

(3) *If ρ and σ are rectangular and $\sigma \subset \rho \subset \mathscr{D}$, then ρ/σ is rectangular.*

(4) *If $\sigma \subset \rho \subset \mathscr{D}$ and σ and ρ/σ are rectangular, then ρ is rectangular.*

We omit the straightforward arguments.

PROPOSITION 2.6. *If ρ is a congruence on a regular semigroup S and $\rho \subset \mathscr{L}$, then ρ is rectangular.*

PROOF. From the definition, if $(a, b) \in \rho$ then $a\mathscr{L}b$ and $a(\rho \cap \mathscr{L})b(\rho \cap \mathscr{R})b$.

We recall that a semigroup S is *stable* if and only if $Sa \subset Sab$ implies $Sa = Sab$ and $dS \subset cdS$ implies $dS = cdS$. It is known [1] that S is stable if and only if S does not contain a bicycle semigroup. Stable semigroups include finite (or compact) semigroups.

THEOREM 2.7. *Let $\rho \subset \mathscr{D}$ be a congruence on the stable regular semigroup S. Then*

(1) *ρ is rectangular;*

(2) *if $I^2 = I \in S/\rho$, then $\bar{I}$ is a completely simple subsemigroup of S;*

(3) *S/ρ is stable.*

PROOF. We first show that ρ is rectangular. Let $(x, y) \in \rho$ and choose $e^2 = e\mathscr{R}x$. Let $I = \phi(e)$. Then there exists $t \in S$ so that $xt = e$. Since $\rho \subset \mathscr{D}$, we know there is a $\mathscr{D}$-class D so that $\bar{I} \cup \{x, y\} \subset D$. We can assume $t \in D$ by replacing t by txt if necessary. In fact, $xt = (xt)(xt) \in Stxt$ and $txt \in Sxt$ yield $xt\mathscr{L}txt$. Hence we can assume $xt\mathscr{L}t$. Since $xt\rho yt$ and S is stable $yte \in R_{yt} \cap L_e \cap \bar{I}$ [1, Corollary 1.1(5)]. Hence since $e = e^2 \in R_x \cap L_{yte}$, $R_{yt} = R_{yte}$, and $yt \in R_y \cap L_t$ [1, Corollary 1.1(5)], we have $ytex = ytx \in R_y \cap L_x \cap \phi^{-1}\phi(x)$. Thus ρ is rectangular. Since $a \in \bar{I}$ implies $a, a^2 \in \bar{I} \subset D_a$, we have $a^2 \in R_a \cap L_a = H_a$ [1, Corollary 1.1(5)] is a group. It follows directly that the idempotent class $\bar{I}$ intersects H_a in a group. Hence $\bar{I}$ is a rectangular union of groups and is completely simple.

Suppose S/ρ is not stable; then S/ρ contains a bicyclic semigroup $C(P, Q)$ with $QP = I$, the identity. Since $P \perp Q$ in S/ρ there exists $p \in \bar{P}$ and $q \in \bar{Q}$ with

https://doi.org/10.1017/S1446788700022424 Published online by Cambridge University Press

$p \perp q$ in S. If $qp = e$, then $\phi(qe) = QI = Q = \phi(q)$. Thus, $qe\rho q$. Since $\rho \subset \mathscr{D}$, $qe \in D_q$, and by stability, $qe \in R_q \cap L_e = R_e \cap L_e = H(e)$ [1]. Thus $Q = \phi(qe)$ belongs to a group, a contradiction.

3. V-commuting congruences

Let ρ be a congruence on a regular semigroup S and $V = \{(a, b) \in S \times S | a \perp b\}$. Recall that ρ is V-commuting provided $\rho \circ V = V \circ \rho$ and that ρ is idempotent-regular provided each idempotent ρ-class is a regular subsemigroup of S. We note that either containment, $\rho \circ V \subset V \circ \rho$ or $V \circ \rho \subset \rho \circ V$, implies the other. We will use this without comment. If $A, B \in S/\rho$ with $A\mathscr{R}B$, we say $ht(A) \leqslant ht(B)$ if for each $\mathscr{R}$-class R of S with $R \cap \bar{A} \neq \varnothing$, we have $R \cap \bar{B} \neq \varnothing$. Similarly if $A\mathscr{L}C$ in S/ρ, we say $w(A) \leqslant w(C)$ if for each $\mathscr{L}$-class L of S with $L \cap \bar{A} \neq \varnothing$ we have $L \cap \bar{B} \neq \varnothing$. The following results give several characterizations of V-commuting congruences and point out relationships between V-commuting and idempotent-regular congruences.

In the following, $\rho_E = \rho \cap E \times E$, $\mathscr{R}_E = \mathscr{R} \cap E \times E$, and $\mathscr{L}_E = \mathscr{L} \cap E \times E$.

THEOREM 3.1. *Let ρ be a congruence on a regular semigroup S. The following* (1)–(8) *are equivalent.*

(1) $\rho \circ V = V \circ \rho$.

(2) *If $A \perp B$ in S/ρ and $a \in \bar{A}$, then there is $b \in \bar{B}$ so that $a \perp b$.*

(3) $\rho(V_x) = V_{\rho(x)}$ *for each $x \in S$.*

(4)(a) *If $A\mathscr{R}I = I^2$ in S/ρ and $a \in \bar{A}$ then there is $e^2 = e \in \bar{I}$ so that $a\mathscr{R}e$ in S.*

(b) *If $A\mathscr{L}F = F^2$ in S/ρ and $a \in \bar{A}$ then there is $f^2 = f \in \bar{F}$ so that $a\mathscr{L}f$ in S,*

(c) *ρ is idempotent-regular.*

(5)(a) *If $F\mathscr{L}A\mathscr{R}I$, $I^2 = I$, $F^2 = F$ in S/ρ then $\bar{I}\bar{A} = \bar{A}\bar{F} = \bar{A}$.*

(b) *ρ is idempotent-regular.*

(6)(a) *If $I^2 = I\mathscr{R}J = J^2$ in S/ρ then $ht(I) = ht(J)$, and if $I^2 = I\mathscr{L}J = J^2$ in S/ρ then $w(I) = w(J)$.*

(b) *ρ is idempotent-regular.*

(7)(a) *$\rho \circ \mathscr{R} \cap E \times E = \mathscr{R} \circ \rho \cap E \times E$, and $\rho \circ \mathscr{L} \cap E \times E = \mathscr{L} \circ \rho \cap E \times E$.*

(b) *ρ is idempotent-regular.*

(8) $\rho_E \circ \mathscr{R}_E = \mathscr{R}_E \circ \rho_E$ *and* $\rho_E \circ \mathscr{L}_E = \mathscr{L}_E \circ \rho_E$.

PROOF. (1) → (2). Let $A \perp B$ in S/ρ and choose $a \in \bar{A}$. By Hall [4], there are $a_0 \in \bar{A}$ and $b_0 \in \bar{B}$ so that $a_0 \perp b_0$. Since $a\rho a_0 \perp b_0$, by (1) there is b so that $a \perp b\rho b_0$. Hence $b \in \bar{B}$ and $a \perp b$.

(2) → (3). That $\rho(V_x) \subset V_{\rho(x)}$ is clear. To show the reverse, let $\rho(x) = A$ and $A \perp B$. Then $x \in \bar{A}$ and by (2), there is $b \in \bar{B}$ with $x \perp b$. Hence $B = \rho(b)$ and $b \in V_x$.

(3) → (4). Let $F\mathscr{L}A\mathscr{R}I$ and $a \in \bar{A}$. If A' is an inverse of A relative to I and F, then $A' \in V_{\rho(a)} = \rho(V_a)$. So $A' = \rho(x)$ for some $x \perp a$. Then $ax\mathscr{R}a$ and $ax \in \bar{I}$. Let $e = ax$; then (4)(a) holds. Similarly (4)(b) holds. To see that ρ is idempotent-regular, note that $I \perp I$. By (3), each element of $\bar{I}$ has an inverse in $\bar{I}$, so $\bar{I}$ is a regular subsemigroup.

(4) → (5). Let $F\mathscr{L}A\mathscr{R}I$ and $I^2 = I$, $F^2 = F$ in S/ρ. Clearly $\bar{I}\bar{A} \subset \bar{A}$. To show $\bar{A} \subset \bar{I}\bar{A}$, let $a \in \bar{A}$. By (4)(a) there exists $x \in \bar{I}$ with $a\mathscr{R}x$. Since ρ is idempotent-regular there exists $e = e^2 \in \bar{I} \cap R_x$; then $a\mathscr{R}e$, so $a = ea$. Hence $a \in \bar{I}\bar{A}$. The other arguments go similarly.

(5) → (6). Let $I^2 = I\mathscr{R}J = J^2$. We show that if $t \in \bar{I}$, then $R_t \cap \bar{J} \neq \varnothing$. This will give $ht(I) \leqslant ht(J)$ and a symmetric argument will complete showing $ht(I) = ht(J)$. Let $t \in \bar{I}$ and using (5)(b) choose $e^2 = e\mathscr{L}t$ with $e \in \bar{I}$. By (5)(a), $\bar{J}\bar{I} = \bar{I}$ and there are $x \in \bar{J}$, $y \in \bar{I}$ so that $e = xy$. Then $e = e^3 = e(xy)e = (ex)(ye)$, so $ex\mathscr{R}e$ with $ex \in \bar{J}$ since $\bar{I}\bar{J} = \bar{J}$. Now $e\mathscr{R}ex$ implies $t = te\mathscr{R}tex \in \bar{J}$. Hence $R_t \cap J \neq \varnothing$.

(6) → (7). We show $\rho \circ \mathscr{R} \cap E \times E \subset \mathscr{R} \circ \rho \cap E \times E$. The other containments follow similarly. If $(e, f) \in \rho \circ \mathscr{R} \cap E \times E$, there is x so that $e\rho x\mathscr{R}f$. Then $\rho(e)\mathscr{R}\rho(f)$ and by (6)(a), $ht\rho(e) = ht\rho(f)$. Thus there is t so that $e\mathscr{R}x\rho f$. Thus $(e, f) \in \mathscr{R} \circ \rho \cap E \times E$.

(7) → (8). Let $(e, g) \in \rho_E \circ \mathscr{R}_E$. Then there exists $f = f^2$ with $e\rho f\mathscr{R}g$. By (7)(a) there exists x with $e\mathscr{R}x\rho g$. By (7)(b) there exists $h = h^2\rho g$ with $x\mathscr{R}h$. Then $e\mathscr{R}h\rho g$, so $(e, g) \in \mathscr{R}_E \circ \rho_E$. The remaining arguments are similar.

(8) → (1). Let $(a, b) \in \rho \circ V$; then there exists x with $a\rho x \perp b$. Let $e^2 = e\mathscr{R}a$, and let $K = \phi(e)$, $I = \phi(bx)$, and $J = \phi(xb)$. Then $J\mathscr{R}K$, so there exist idempotents $j \in \bar{J}$ and $h \in \bar{K}$ with $j\mathscr{R}k$ [4]. Now $e\rho k\mathscr{R}j$, so by (8) there exists $i = i^2$ with $e\mathscr{R}i\rho j$. Similarly there exists $f^2 = f\mathscr{L}a$ with $f \in \bar{I}$. Let a' be the inverse of a relative to f and i. Since $\phi(b)$ and $\phi(a')$ are both inverse to $\phi(a)$ relative to I and J, we have $a'\rho b$. Hence $a \perp a'\rho b$ and (1) follows.

REMARK 1. The equivalences of (1), (2), and (3) hold in a more general setting, using essentially similar arguments. We may replace V by any relation $\mathscr{K}$ that enjoys the lifting property, that is, if $A\mathscr{K}B$ in S/ρ, then there are $a \in \bar{A}$ and $b \in \bar{B}$ so that $A\mathscr{K}b$ in S. From [4] we know that $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{D}$ satisfy this lifting property.

REMARK 2. Although we will not make use of it, V can be described as follows. These are equivalent: (1) $a \perp b$ (2) $ab \in R_a \cap L_b \cap E$ (3) $(a, b) \in \mathscr{L} \circ \Delta_E \circ \mathscr{R} \cap m^{-1}(E)$, where $\Delta_E = \{(e, e): e \in E\}$ and m: $S \times S \to S$ is the multiplication on S. We omit the straightforward arguments. Thus, $V = m^{-1}(R_a \cap L_b \cap E) = \mathscr{L} \circ \Delta_F \circ \mathscr{R} \cap m^{-1}(E)$.

REMARK 3. We see from (8) of Theorem 3.1 that a congruence ρ commutes with V if and only if ρ induces a decomposition of E in which the classes are height and width compatible. Recall [12] that two congruences α and β are θ-equivalent ($\alpha\theta\beta$) if α and β induce the same decomposition of E. We see then that If $\alpha \circ V = V \circ \alpha$ and $\alpha\theta\beta$, then $\beta \circ V = V \circ \beta$.

REMARK 4. A regular semigroup S is said to be V-regular, [10] or [11], if for all $a, b \in S$ we have $V_{ab} \subset V_b V_a$, and weakly V-regular, [8], if for all $a, b \in S$ we have $V_{ab} \subset V_b S V_a$. Theorem 1 of [8] and Proposition 2.4 of [9] show that if S is weakly V-regular and ρ is a congruence on S then (2) of Theorem 3.1 holds. Hence $\rho \circ V = V \circ \rho$.

REMARK 5. If $\rho \subset \mathscr{L}$, then $\rho \circ V = V \circ \rho$ [6].

COROLLARY 3.2. *If ρ is rectangular, idempotent-regular, and under $\mathscr{D}$, then $\rho \circ V = V \circ \rho$.*

PROOF. This follows from (7) → (1) and Proposition 2.2.

The property $\rho \circ V = V \circ \rho$ holds for a rather large class of congruences on regular semigroups. For example, every congruence ρ on an inverse semigroup satisfies $\rho \circ V = V \circ \rho$. The following more general result holds.

THEOREM 3.3. *If ρ is a congruence on a regular semigroup S and S/ρ is an inverse semigroup, then $\rho \circ V = V \circ \rho$.*

PROOF. If $(a, b) \in \rho \circ V$, then there is $x \in S$ so that $a\rho x \perp b$. Choose $y \in S$ so that $a \perp y$. Hence $\rho(a) \perp \rho(y)$ and $\rho(a) = \rho(x) \perp \rho(b)$. Since S/ρ is inverse, then $\rho(b) = \rho(y)$ and $a \perp y\rho b$. Hence $(a, b) \in V \circ \rho$.

EXAMPLE 3.4. Let S be the band $\{e, f, g\}$ with binary operation as follows:

	e	f	g
e	e	f	f
f	e	f	f
g	e	f	g

Note that $D_g = \{g\}$, $D_e = D_f = R_e = R_f = \{e, f\}$, $L_e = \{e\}$, and $L_f = \{f\}$. The relation $\rho = \{(f, g), (g, f)\} \cup \Delta$ is a congruence on S. We observe that ρ is rectangular and idempotent-regular, but $\rho \not\subset \mathscr{D}$ and ρ commutes with none of $\mathscr{R}$, $\mathscr{V}$, and $\mathscr{D}$.

The examples given in 5.2 are regular completely semisimple semigroups which have congruences that do not commute with V. The example given in Example 5.3 has a congruence $\rho \subset \mathscr{D}$ but $\rho \circ V \neq V \circ \rho$; in fact, ρ is not idempotent-regular.

As was stated in Remark 4 above, if S is V-regular, then each congruence ρ on S commutes with V. For X any set, it is known, [10] or [11], that the full transformation semigroup $\mathscr{T}_X$ is V-regular. We present the following straightforward argument that any congreunce ρ on $\mathscr{T}_X$ commutes with V. We give five lemmas, omitting the routine proofs of the first four.

LEMMA 3.5. *If $a \in \mathscr{T}_X$ then $a' \perp a$ if and only if* (1) *a' is a section for a on aX, that is, $aa'(x) = x$ for each $x \in aX$ and* (2) *$a'X \subset a'aX$.*

LEMMA 3.6. *If ρ and τ are V-commuting congruences on a regular semigroup S then $(\rho \cup \tau) \circ V = V \circ (\rho \cup \tau)$.*

LEMMA 3.7. *If I is an ideal of a regular semigroup S and ρ_I is the associated Rees congruence, then $\rho_I \circ V = V \circ \rho_I$.*

LEMMA 3.8. *If ρ_I is a Rees congruence and ρ is a V-commuting congruence on a regular semigroup S, then $(\rho \cap \rho_I) \circ V = V \circ (\rho \cap \rho_I)$.*

LEMMA 3.9. *If ρ is a difference rank congruence on $\mathscr{T}_X$, then $\rho \circ V = V \circ \rho$.*

PROOF. Let $|X| = p \geqslant q \geqslant \aleph_0$ and $\rho = \{(a, b) | dr(a, b) < q\}$, where $D(a, b) = \{x \in X | a(x) \neq b(x)\}$ and $dr(a, b) = \max\{|a(D(a, b))|, |b(D(a, b))|\}$. Let $(a, b) \in V \circ \rho$ and choose a' so that $a \perp a'\rho b$ and set $D = D(a', b)$. Define $b' \in \mathscr{T}_X$ as follows. For $y \in bX \setminus (bD \cup a'D)$, set $b'(y) = a(y)$. Since $a \perp a'$, using Lemma 3.5, we have $y = a'a(y)$ and since $y \notin a'D$ it follows that $a(y) \notin D$. Hence $y = a'a(y) = ba(y) = bb'(y)$. Thus b' is a section for b on $bX \setminus (bD \cup a'D)$. Hence $y = a'a(y) = ba(y) = bb'(y)$. Thus b' is a section for b on $bX \setminus (bD \cup a'D)$. For $y \in bD \cup (bX \cap a'D)$, choose $b'(y)$ so that $bb'(y) = y$. For $y \in a'D \setminus bX$, choose $x \in D$ with $y = a'(x)$ and set $b'(y) = b'b(x)$. Now, b' satisfies the conditions of Lemma 3.5 on $bX \cup a'D = a'X \cup bD$. Since $dr(a', b) < q$, the symmetric difference of $a'X$ and bX has cardinality less than q; notationally we use $a'X =_q bX$. It follows that $aa'X =_q abX$. From the above defining of b' on bX, we have $b'bX =_q abX$. Thus $aa'X =_q b'bX$. Define b' on $X \setminus (bX \cup a'X)$ by $b'(x) = a(x)$ if $a(x) \in b'bX$ and $b'(x) = z_0$ for some $z_0 \in b'bX$ otherwise. Now $b' \perp b$ since b' is a section for b on bX and $b'X \subset b'bX$. Further, since $|a'D| < q$ and $aX \subset aa'X =_q b'bX$ we know $dr(b', a) < q$. Thus $a\rho a' \perp b$, and $\rho \circ V = V \circ \rho$ follows.

https://doi.org/10.1017/S1446788700022424 Published online by Cambridge University Press

THEOREM 3.10. *If ρ is any congruence on $\mathcal{T}_X$, then $\rho \circ V = V \circ \rho$.*

PROOF. If the primary cardinal number of ρ is infinite, then by a result of Malcev [3, Theorem 10.72] there are ideals $I_0, \ldots, I_n$ and difference rank congruences $\delta_1, \ldots, \delta_{n+1}$ so that $\rho = \rho_{I_0} \cup (\rho_{I_1} \cap \delta_1) \cup \cdots \cup (\rho_{I_n} \cap \delta_n) \cup \delta_{n+1}$. The result follows from this case from the previous lemmas.

If the primary cardinal number of ρ is finite, namely n, then [3, Theorem 10.68] we can write $\rho = i \cup [\sigma \cap (D_n \times D_n)] \cup [I_n \times I_n]$ where D_n is the $\mathcal{D}$-class of all elements of $\mathcal{T}_X$ with rank n, I_n is the principal ideal of $\mathcal{T}_X$ consisting of elements of rank less than n, σ is a congruence on I_{n+1}/I_n, and i is the identity congruence. Again, due to a result of Malcev [3, Theorem 10.58], we know that $\sigma \subset \mathcal{H}$ and the $\mathcal{H}$-classes in I_{n+1}/I_n coincide with the $\mathcal{H}$-classes of $\mathcal{T}_X$. Let $(a, b) \in \rho \circ V$. There exists $c \in \mathcal{T}_X$ so that $a\rho c \perp b$. Now $a\rho c$ implies one of $a = c, (a, c) \in I_n \times I_n$, or $(a, c) \in \sigma \subset \mathcal{H}$. In case $a = c$ or $(a, c) \in I_n \times I_n$ we easily see $(a, b) \in V \circ \rho$. If $(a, c) \in \sigma \cap (D_n \times D_n)$ then in I_{n+1}/I_n since $\sigma \subset \mathcal{H}$, we know there is $d \in I_{n+1} \setminus I_n$ so that $a \perp d\rho b$ [6, Corollary 2.3]. Hence $(a, b) \in V \circ \rho$ and $\rho \circ V = V \circ \rho$.

Let VC denote the class of regular semigroups with the property that if S is in VC and ρ is a congruence on S then $\rho \circ V = V \circ \rho$. From the above, it follows that VC contains inverse semigroups and full transformation semigroups. More generally, using Theorem 1 of [8] and Theorem 3.1 we see that any weakly V-regular semigroup is in VC. Example 3.4 shows that VC does not contain all bands and hence not all orthodox semigroups. The examples in 5.2 are completely semisimple semigroups (= stable semigroups) that are not in VC. Example 5.3 shows that a bisimple regular semigroup may not be in VC.

THEOREM 3.11. *The class VC is closed under quotients.*

PROOF. Let σ be a congruence on a semigroup S in VC, and let $\bar{\rho}$ be a congruence on S/σ. Then $\rho = \{(a, b) \in S \times S | (\sigma(a), \sigma(b)) \in \bar{\rho}\}$ is a congruence on S with $\sigma \subset \rho$. Further $\bar{\rho} = \rho/\sigma = \{(\sigma(a), \sigma(b)) | (a, b) \in \rho\}$. Let $(\sigma(a), \sigma(b)) \in V \circ \bar{\rho}$. Then $\sigma(a) \perp \sigma(x)\bar{\rho}\sigma(b)$ for some $\sigma(x) \in S/\sigma$. Using 3.1, we have that $\sigma \circ V = V \circ \sigma$ and $\sigma(a) \perp \sigma(x)$ imply there exists $x' \in S$ with $x\sigma x'$ and $a \perp x'$. Now $\sigma(x) = \sigma(x')\bar{\rho}\sigma(b)$ implies $x'\rho b$, and $a \perp x'\rho b$ implies there is $y \in S$ so that $a\rho y \perp b$. Thus $\sigma(a)\bar{\rho}\sigma(y) \perp \sigma(b)$ and $(\sigma(a)\sigma(b)) \in \bar{\rho} \circ V$. Hence $\bar{\rho} \circ V = V \circ \bar{\rho}$.

THEOREM 3.12. *Let σ and ρ be congruences on a regular semigroup S with $\sigma \subset \rho$.*
(1) If $V \circ \sigma = \sigma \circ V$ and $V \circ \rho/\sigma = \rho/\sigma \circ V$ then $\rho \circ V = V \circ \rho$.
(2) If $V \circ \sigma = \sigma \circ V$ and $V \circ \rho = \rho \circ V$ then $\rho/\sigma \circ V = V \circ \rho/\sigma$.

https://doi.org/10.1017/S1446788700022424 Published online by Cambridge University Press

PROOF. To prove (1), let $(a, b) \in \rho \circ V$. There is $x \in S$ so that $a\rho x \perp b$. Thus $\sigma(a)(\rho/\sigma)\sigma(x) \perp \sigma(b)$. Since ρ/σ commutes with V, there is $\sigma(y)$ so that $\sigma(a) \perp \sigma(y)(\rho/\sigma)\sigma(b)$. By Theorem 3.1 there is z with $z\sigma y$ such that $a \perp z$. Since $\sigma(y)(\rho/\sigma)\sigma(b)$ and $b\rho y\sigma z$, we have $b\rho z$ and $a \perp z$. Hence $(a, b) \in V \circ \rho$.

To prove (2) let $(\sigma(a), \sigma(b)) \in \rho/\sigma \circ V$ and choose x so that $\sigma(a)(\rho/\sigma)\sigma(x) \perp \sigma(b)$. Thus $a\rho x$ and there is z with $x\sigma z$ and $z \perp b$. Hence $a\rho z \perp b$ and there is w so that $a \perp w\rho b$. Thus $\sigma(a) \perp \sigma(w)(\rho/\sigma)\sigma(b)$ and $(\sigma(a), \sigma(b)) \in V \circ \rho/\sigma$.

We finish this section by giving a result that is related to Theorem 2.7. It gives a class of congruences that commute with V and are more restrictive than idempotent-regular congruences.

THEOREM 3.13. *Let ρ be a congruence on a regular semigroup S so that if $I^2 = I \in S/\rho$ then $\bar{I}$ is a completely simple subsemigroup of S. Then $\rho \subset \mathscr{D}$ and ρ is rectangular.*

PROOF. We first show that $\rho \circ V = V \circ \rho$ by verifying that part (6) of Theorem 3.1 holds. It is clear that ρ is idempotent-regular, and we show that if $I^2 = I\mathscr{R}J = J^2$ in S/ρ then $ht(I) = ht(J)$. The dual condition follows in a similar manner. Let $e^2 = e \in \bar{I}$. From [4] we know that there is $f \in \bar{I}$ and $g = g^2 \in \bar{J}$ with $f \leqslant e$ (that is, $fe = ef = f$) and $f\mathscr{R}g$. Since $\bar{I}$ is completely simple, $e = f$ and $e\mathscr{R}g$. By this and the symmetric argument, $ht(I) = ht(J)$.

To see that $\rho \subset \mathscr{D}$, let $A \in S/\rho$ and choose $I^2 = I\mathscr{R}A$. Then from part 4(a) of Theorem 3.1, there is $e^2 = e \in \bar{I}$ so that $a\mathscr{R}e$. Since $\bar{I}$ is completely simple, there is a $\mathscr{D}$-class D so that $\bar{I} \subset D$. It follows that $A \subset D$.

To see that ρ is rectangular, let $a, b \in \bar{A}$ where $A\mathscr{R}I = I^2$ in S/ρ. There are $e, f \in \bar{I}$ with $e^2 = e\mathscr{R}a$ and $f^2 = f\mathscr{R}b$. Since $\bar{I}$ is completely simple there exists $g^2 = g \in \bar{I} \cap R_f \cap L_e$. Thus since $e \in L_g \cap R_a$ and $g \in L_e \cap R_b$ we have $ga \in R_g \cap L_a$ and $be \in R_e \cap L_b$. Now $a\rho b$ and $g\rho e$ imply $ga\rho be$ and thus ρ is rectangular.

4. $(\mathscr{L}, \mathscr{R})$-commuting congruences

A congruence ρ on a semigroup S is said to be $(\mathscr{L}, \mathscr{R})$-commuting provided $\rho \circ \mathscr{L} = \mathscr{L} \circ \rho$ and $\rho \circ \mathscr{R} = \mathscr{R} \circ \rho$. From Proposition 2.2, if $\rho \subset \mathscr{D}$ and ρ is rectangular, then ρ is $(\mathscr{L}, \mathscr{R})$-commuting, and Example 3.4 shows that without $\rho \subset \mathscr{D}$ this is not true. An example below (4.2) shows that $(\mathscr{L}, \mathscr{R})$-commutativity does not imply rectangularity. The following theorem gives four characterizations of $(\mathscr{L}, \mathscr{R})$-commutativity among idempotent-regular congruences, that is, those where idempotent classes are regular subsemigroups.

THEOREM 4.1. *Let* ρ *be an idempotent-regular congruence on a regular semigroup* S. *These are equivalent.*

(1) *If* $A \perp B$, $I = AB$, $F = BA$, *and* $A\mathscr{R}I\mathscr{L}B\mathscr{R}F\mathscr{L}A$ *in* S/ρ, *then* $\bar{A}\,\bar{B} = \bar{I}$ *and* $\bar{B}\bar{A} = \bar{F}$.

(2) *If* $K = K^2$ *and* $A\mathscr{R}AB\mathscr{L}B\mathscr{R}K\mathscr{L}A$ *in* S/ρ, *then* $\bar{A}\,\bar{B} = \overline{AB}$.

(3) *If* $I^2 = I$, $F^2 = F$, *and* $F\mathscr{L}A\mathscr{R}I$ *in* S/ρ, *then* $ht(I) = ht(A)$ *and* $w(F) = w(A)$.

(4) $\rho \circ \mathscr{R} = \mathscr{R} \circ \rho$ *and* $\rho \circ \mathscr{L} = \mathscr{L} \circ \rho$.

(5) $\phi(R_x) = R_{\phi(x)}$ *and* $\phi(L_x) = L_{\phi(x)}$ *for each* $x \in S$.

PROOF. (1) $\to$ (3). We first show that $ht(I) \leqslant ht(A)$. Choose $B \in S/\rho$ so that $A \perp B$, $AB = I$, and $BA = F$. Let $x \in \bar{I}$. Since ρ is idempotent-regular there is e so that $e\mathscr{R}x$ and $e^2 = e \in \bar{I} = \bar{A}\,\bar{B}$. Then $e = ab = (ea)(be)$ for some $a \in \bar{A}$ and $b \in \bar{B}$. Since $A\mathscr{R}I$ we have $IA = A$ and $ea \in \bar{I}\bar{A} \subset \bar{A}$. Further, $e = (ea)(be)$ implies that $ea\mathscr{R}e$. Thus $ea\mathscr{R}e\mathscr{R}x$ and $ea \in \bar{A}$ yields $ht(I) \leqslant ht(A)$. Similarly, one sees that $w(F) \leqslant w(A)$. In particular, if $I^2 = I\mathscr{R}J = J^2$, then $ht(I) = ht(J)$ and it follows from Theorem 3.1 that $\rho \circ V = V \circ \rho$ and $ht(A) \leqslant ht(I)$. Similarly, one shows that $w(F) = w(A)$.

(3) $\to$ (4). Note first that if $A\mathscr{R}B\mathscr{R}I = I^2$ in S/ρ, then $ht(A) = ht(B) = ht(I)$. Let $(a, b) \in \rho \circ \mathscr{R}$. Then there is x so that $a\rho x\mathscr{R}b$. If $A = \phi(a)$ and $B = \phi(b)$, then $ht(A) = ht(B)$ and there is $y \in \bar{B}$ with $y\rho b$ and $a\mathscr{R}y$. Therefore $(a, b) \in \mathscr{R} \circ \rho$. The other containments follow similarly.

(4) $\to$ (1). We show $\bar{I} \in \bar{A}\,\bar{B}$. Let $x \in \bar{I}$ and choose $e = e^2 \in \bar{I}$ so that $e\mathscr{R}x$. Using the Remark 1 following Theorem 3.1 with (4), we see that there are $a \in \bar{A}$ and $b \in \bar{B}$ with $a\mathscr{R}e\mathscr{L}b$. Thus $ba \in \bar{F}$ and there is $h^2 = h \in \bar{F}$ so that $ba\mathscr{L}h$. Since $a \in L_h \cap R_e$, let a' be the inverse of a with respect to h and e. If $\phi(a') = A'$ then $A'\mathscr{L}I$ and $A'\mathscr{R}F\mathscr{R}B$. Hence $AA' = I = AB$ implies $A' = B$. Hence $x = ex = (aa')x = a(a'x) \in \bar{A}\,\bar{B}$ since $a'x \in \bar{B}\,\bar{I} \subset \bar{B}$.

(1) $\to$ (2). Let $I^2 = I\mathscr{R}A$ and let A' be an inverse of A relative to K and I. Note that $A'(AB) = B$ implies $\bar{A'}\;\overline{AB} \subset \bar{B}$. As in the proof that (1) $\to$ (3) we argue that $ht(AB) \leqslant ht(I)$ and if $a \in AB$ there exists $e^2 = e \in \bar{I}$ with $e\mathscr{R}a$. Thus $a = ea \in \bar{I}\,\overline{AB}$. Hence $\overline{AB} = \bar{I}\,\overline{AB} = \overline{AA'}\,\overline{AB} = \bar{A}\,\bar{A'}\,\overline{AB} \subset \bar{A}\;\bar{B} \subset \overline{AB}$. Hence (2) is established.

(2) $\to$ (1). This is clear.

(4) $\leftrightarrow$ (5). This follows as (1) $\leftrightarrow$ (3) in Theorem 3.1 using Remark 1 after 3.1.

EXAMPLE 4.2. Let $C(p, q)$ be the bicyclic semigroup and let $\rho(p^m q^n) = n - m$. Then ρ: $C(p, q) \to Z$ is a homomorphism. Note that ρ is not rectangular since $\rho(p) = \rho(p^2 q)$ but $\rho(p) \neq \rho(pq)$. Further, ρ is not $(\mathscr{L}, \mathscr{R})$-commuting, since $1\rho pq\mathscr{L}q$ but $1\mathscr{L}x\rho q$ does not hold for any $x \in C(p, q)$. However, the congruence

https://doi.org/10.1017/S1446788700022424 Published online by Cambridge University Press

of the homomorphism $\alpha\rho$ where α: $Z \to Z$ (modulo n) does commute with $\mathscr{L}$ and $\mathscr{R}$ and is not rectangular. Both these congruences on $C(p, q)$ are contained in $\mathscr{D}$ and commute with V.

5. Idempotent-regular congruences

Nambooripad [9] has posed the problems of characterizing idempotent-regular congruences, and of characterizing regular semigroups which have only idempotent-regular congruences. Results from previous sections touch on these problems, and we present some further information.

THEOREM 5.1. *Let σ and ρ be congruences on a regular semigroup S with $\sigma \subset \rho$.*
(1) *If ρ is idempotent-regular, then so is ρ/σ.*
(2) *If $\sigma \circ V = V \circ \rho$ and ρ/σ is idempotent-regular, then ρ is idempotent-regular.*

PROOF. Let $I = I^2$ be in $(S/\sigma)/(\rho/\sigma)$ and $\sigma(a)$ be in $\bar{I}$, the pre-image of I in S/σ. Then a is in I^*, the pre-image of $\bar{I}$ in S. Now I^* is an idempotent ρ-class and is hence a regular subsemigroup. Thus there is $b \in I^*$ so that $a \perp b$; then $\sigma(a) \perp \sigma(b)$ and $\bar{I}$ is regular. This proves (1). To prove (2) let $I^2 = I \in S/\rho$ with $x \in I^*$ the ρ-class mapping to I. Then $\sigma(x)$ has an inverse $\sigma(y)$ in $\sigma(I^*)$. Since $\sigma \circ V = V \circ \sigma$, using Theorem 3.1, we see that there is z with $z\sigma y$ and $z \perp x$. Hence $z \in I^*$ and it follows that ρ is idempotent-regular.

EXAMPLE 5.2. This is a class of examples of regular semigroups with congruences which are not idempotent-regular but which are rectangular and $(\mathscr{L}, \mathscr{R})$-commuting. These contructions extend an example suggested to the authors by J. D. Lawson. We give a special case and illustrate how it generalizes.

Let S be the rectangular band $f_1\mathscr{R}f_2\mathscr{L}f_3\mathscr{R}f_4\mathscr{L}f_1$ and T the partial semigroup $e_1\mathscr{R}x\mathscr{L}e_3\mathscr{R}y\mathscr{L}e_1$ with $e_1^2 = e_1$, $e_3^2 = e_3$, $xy = e_3$, $yx = e_1$, $xe_3 = x$, and $e_3y = y$. The mapping $e_1 \to f_1$, $x \to f_2$, $e_3 \to f_3$, and $y \to f_4$ is a partial homomorphism, and the semigroup T^* formed by extending [3, p. 138] S by $T \cup \{0\}$ is a regular semigroup. The congruence with idempotent classes $\{e_1, f_1\}$, $\{x, f_2\}$, $\{e_3, f\}$, and $\{y, f_4\}$ is not idempotent-regular since $\{x, f_2\}$ and $\{y_1, f_4\}$ are not regular subsemigroups of T^*.

A generalization of the above is obtained as follows. Let $S_0, S_1, \ldots, S_n$ be (for simplicity, $\mathscr{H}$-trivial) rectangular stable regular $\mathscr{D}$-classes (partial semigroups) so that S_i is a sub-rectangle of S_{i-1} and if the xy-entry of S_i is an idempotent, then so is the xy-entry of S_{i-1}. Assume that S_0 is a band. Then the inclusion mappings

$$S_n \xrightarrow{i_n} S_{n-1} \xrightarrow{i_{n-1}} \cdots \to S_2 \xrightarrow{i_2} S_1 \xrightarrow{i_1} S_0$$

are partial homomorphisms. Let $S_i' = S_i \cup \{0\}$ with undefined products set equal to 0, $i = 1, \ldots, n$. Let T_1 be the ideal extension of S_0 by S_1', and for $i = 2, 3, \ldots, n$, let T_i be the ideal extension of S_i' by T_{i-1} using the partial homomorphism induced by the one from S_i to S_{i-1}. Then T_n is a regular stable (completely semisimple) semigroup. If ρ_0 is a congruence on S_0, then define ρ on T_n so that C is a ρ-class if and only if there is a ρ_0-class C_0 in S so that $C = (i_1 i_2 \cdots i_n)^{-1}(C_0)$. It is easy to see that ρ is a congruence on T_n that might not be idempotent-regular.

EXAMPLE 5.3. In [2] the four-spiral semigroup Sp_4 is presented as the free semigroup $\mathscr{F}_X$ on $X = \{a, b, c, d\}$ modulo the congruence generated by $\{(a, a^2), (b, b^2), (c, c^2), (d, d^2), (a, ba), (b, ab), (b, bc), (c, cb), (c, dc), (d, cd), (d, da)\}$. The semigroup Sp_4 is a bisimple, fundamental (in fact, combinatorial) regular idempotent-generated semigroup which is isomorphic to a rectangular band of four subsemigroups, one of which is not regular [2, Theorems 2.4 and 2.7]. Hence S has a congruence ρ (of course $\rho \subset \mathscr{D}$) that is not idempotent-regular.

References

[1] L. W. Anderson, R. P. Hunter and R. J. Koch, 'Some results on stability in semigroups', *Trans. Amer. Math. Soc.* **117** (1965), 521–529.
[2] K. Byleen, J. Meakin and F. Pastijn, 'The fundamental four-spiral semigroup', *J. Algebra* **54** (1978), 6–26.
[3] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7 (Amer. Math. Soc., Providence, R. I., Vol. I, 1961, Vol. II, 1967).
[4] T. E. Hall, 'Congruences and Green's relations on regular semigroups', *Glasgow Math. J.* **13** (1972), 167–175.
[5] T. E. Hall, 'On regular semigroups', *J. Algebra* **24** (1973), 1–24.
[6] R. J. Koch and B. L. Madison, 'Congruences under $\mathscr{L}$ on regular semigroups', *Simon Stevin* **57** (1983), 273–283.
[7] R. J. Koch, 'Comparison of congruences on regular semigroups', *Semigroup Forum* **26** (1983), 295–305.
[8] R. J. Koch, 'On inverses of products in regular semigroups', preprint.
[9] K. S. S. Nambooripad, 'Structure on regular semigroups', *Symposium on regular semigroups*, pp. 101–117 (Northern Illinois University, DeKalb, Illinois, 1979).
[10] K. S. S. Nambooripad and F. Pastijn, 'V-regular semigroups', preprint.
[11] J. A. Onstad, *A study of certain classes of regular semigroups* (Dissertation, University of Nebraska-Lincoln, 1974).
[12] N. R. Reilly and H. E. Scheiblich, 'Congruences on regular semigroups', *Pacific J. Math.* **23** (1967), 349–360.

Louisiana State University
Baton Rouge, Louisiana 70803
U.S.A.

University of Arkansas
Fayetteville, Arkansas 72701
U.S.A.