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COMMUTATIVITY DEGREE OF A CLASS OF FINITE GROUPS AND CONSEQUENCES

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Abstract

The commutativity degree of a finite group is the probability that two randomly chosen group elements commute. The object of this paper is to compute the commutativity degree of a class of finite groups obtained by semidirect product of two finite abelian groups. As a byproduct of our result, we provide an affirmative answer to an open question posed by Lescot.

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1. Introduction

Let G be a finite group. The *commutativity degree* of G (see [5, 6]) is given by

$$\Pr(G) = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G|^2}.$$

Let $C = \{(x, y) \in G \times G \mid xy = yx\}$. Then it is not difficult to see that $|C| = \sum_{g\in G}|C_G(g)|$, where $C_G(g) = \{h \in G \mid gh = hg\}$ is the centraliser of an element $g \in G$ in G. Thus

$$\Pr(G) = \frac{1}{|G|^2}\sum_{g\in G}|C_G(g)|. \tag{1.1}$$

In [6], Lescot computed the commutativity degrees of dihedral groups (D_{2n}) and quaternion groups ($Q_{2^{n+1}}$) and showed that

$$\Pr(D_{2n}) \to \tfrac{1}{4} \quad \text{and} \quad \Pr(Q_{2^{n+1}}) \to \tfrac{1}{4}$$

as the orders of the groups D_{2n} and $Q_{2^{n+1}}$ tend to infinity. He then asked, 'whether there are other natural families of groups with the same property'. In this paper, we compute the commutativity degree of a class of finite groups obtained by semidirect product of two finite abelian groups and provide an affirmative answer to the above question.

Also, we provide examples of groups having commutativity degree 5/14, using our result. This particular value of the commutativity degree has some interest because Rusin misses out this value in his table. (See [7], where Rusin has computed all possible values of the commutativity degree greater than 11/32.) It may be mentioned here that a general version of the above question was posed and answered by Erovenko and Sury [4].

Recall that if H and K are any two groups and $\theta : K \to \mathrm{Aut}(H)$ is a homomorphism then the cartesian product $H \times K$ forms a group under the binary operation

$$(h_1, k_1)(h_2, k_2) = (h_1\theta(k_1)(h_2), k_1k_2),$$

where $h_i \in H$ and $k_i \in K$, $i = 1, 2$. This group is known as the semidirect product of H by K (with respect to θ), and is denoted by $H \rtimes_\theta K$. In this paper, we consider the semidirect product of a cyclic group of order n by an abelian group of order $2m$.

Let $H = \langle a \mid a^n = 1\rangle$ and K be any abelian group of order $2m$. Notice that K has a subgroup of index 2, and hence there is a nontrivial group homomorphism $\epsilon : K \to \{-1, 1\}$, and so there is a group homomorphism $\theta : K \to \mathrm{Aut}(H)$ so that $\theta(k)(a) = a^{\epsilon(k)}$ for all $k \in K$. We consider the group $H \rtimes_\theta K$. Note that if $n = 1$ or 2 then θ becomes trivial and hence the corresponding semidirect product becomes a direct product; therefore we take $n \geq 3$. Also, note that if $H = \langle a \mid a^n = 1\rangle$, $K = \langle b \mid b^2 = 1\rangle$ and $\theta(b)(a) = a^{-1}$, then $H \rtimes_\theta K \cong D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1}\rangle$. If n is odd, $K = \langle b \mid b^4 = 1\rangle$ and $\theta(b)(a) = a^{-1}$, then $H \rtimes_\theta K$ is isomorphic to the dicyclic group $Q_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, bab^{-1} = a^{-1}\rangle$. Moreover, if $m \neq 1, 2$ then $H \rtimes_\theta K$ is not isomorphic to any dihedral or dicyclic group.

We have the following main theorem.

Theorem 1.1. *Let $H = \langle a \mid a^n = 1\rangle$ and K be any abelian group of order $2m$. Consider the homomorphism $\theta : K \to \mathrm{Aut}(H)$ defined as $\theta(k)(a) = a^{\epsilon(k)}$ for all $k \in K$, where $\epsilon : K \to \{-1, 1\}$ is a nontrivial group homomorphism. Then*

$$\Pr(H \rtimes_\theta K) = \begin{cases} \dfrac{n+3}{4n} & \text{if } n \text{ is odd,} \\ \dfrac{n+6}{4n} & \text{if } n \text{ is even.} \end{cases}$$

2. Proof of Theorem 1.1

Let $G = H \rtimes_\theta K$. To prove the theorem, we shall calculate the size of the centraliser $C_G((a^x, k))$ for any $a^x \in H$ and $k \in K$. For $x, r \in \{0, 1, \ldots, n-1\}$ and for $k, \ell \in K$,

$$(a^x, k)(a^r, \ell) = (a^{x+\epsilon(k)r}, k\ell) \quad \text{and} \quad (a^r, \ell)(a^x, k) = (a^{r+\epsilon(\ell)x}, \ell k).$$

Since K is abelian, (a^r, ℓ) is in $C_G((a^x, k))$ if and only if $x + \epsilon(k)r = r + \epsilon(\ell)x \pmod{n}$, or equivalently

$$x(1 - \epsilon(\ell)) = r(1 - \epsilon(k)) \pmod{n}. \tag{2.1}$$

Let x and r vary over $N_n = \{0, 1, \ldots, n-1\}$, a ring under arithmetic (mod n).

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Case 1. n odd.

Subcase 1(a). $\epsilon(k) = 1$.

Equation (2.1) is just $x(1 - \epsilon(\ell)) = 0 \pmod n$. This holds if and only if $\epsilon(\ell) = 1$, or $\epsilon(\ell) = -1$ and $x = 0$ (since $2x = 0$ in the ring N_n if and only if $x = 0$, because 2 is invertible in N_n, when n is odd). Thus, when $x \neq 0$, $\epsilon(\ell)$ must be 1 (this holds for exactly m elements ℓ of K), and r is arbitrary, while, when $x = 0$, both r and ℓ are arbitrary. Hence

$$|C_G((a^x, k))| = \begin{cases} mn & \text{if } \epsilon(k) = 1 \text{ and } x \neq 0, \\ 2mn & \text{if } \epsilon(k) = 1 \text{ and } x = 0. \end{cases}$$

Subcase 1(b). $\epsilon(k) = -1$.

Equation (2.1) is now $x(1 - \epsilon(\ell)) = 2r \pmod n$. Again, since 2 is invertible in N_n, given any of the $2m$ ℓ's in K, the last equation determines r. Hence

$$|C_G((a^x, k))| = 2m \quad \text{if } \epsilon(k) = -1.$$

Case 2. n even.

Subcase 2(a). $\epsilon(k) = 1$.

Equation (2.1) is again just $x(1 - \epsilon(\ell)) = 0 \pmod n$. This holds if and only if $\epsilon(\ell) = 1$, or $\epsilon(\ell) = -1$ and $x = 0$ or $x = n/2$. Thus, when $x \neq 0, n/2$, $\epsilon(\ell)$ must be 1, and r is arbitrary, while, when $x = 0$ or $n/2$, both r and ℓ are arbitrary. Hence

$$|C_G((a^x, k))| = \begin{cases} mn & \text{if } \epsilon(k) = 1 \text{ and } x \neq 0, n/2, \\ 2mn & \text{if } \epsilon(k) = 1 \text{ and } x = 0 \text{ or } x = n/2. \end{cases}$$

Subcase 2(b). $\epsilon(k) = -1$.

Equation (2.1) is now $x(1 - \epsilon(\ell)) = 2r \pmod n$. Since n is even, the map $r \mapsto 2r \pmod n$ is a 2-to-1 map from N_n to the set of even integers in N_n, of which $x(1 - \epsilon(\ell)) \pmod n$ is one. So given any of the $2m$ ℓ's in K, the last equation holds for exactly two r's in N_n. Hence

$$|C_G((a^x, k))| = 4m \quad \text{if } \epsilon(k) = -1.$$

In Case 1, the three different values of $|C_G(g)|$ calculated occur for, respectively, $(n-1)m$, m and mn elements g. Hence

$$\sum_{g \in G} |C_G(g)| = (n-1)m \cdot mn + m \cdot 2mn + mn \cdot 2m = m^2 n(n+3).$$

In Case 2, the three different values of $|C_G(g)|$ occur for, respectively, $(n-2)m$, $2m$ and mn elements g. Hence

$$\sum_{g \in G} |C_G(g)| = (n-2)m \cdot mn + 2m \cdot 2mn + mn \cdot 4m = m^2 n(n+6).$$

The result follows from (1.1).

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3. Some consequences

By Theorem 1.1,

$$\Pr(H \rtimes_\theta K) \to \tfrac{1}{4} \quad \text{as } n \to \infty.$$

This answers the question posed by Lescot [6], as cited above.

Also, putting $n = 7$ and 14 in Theorem 1.1,

$$\Pr(H \rtimes_\theta K) = \tfrac{5}{14}.$$

In [7], Rusin determined all possible values of the commutativity degree greater than 11/32 and classified all finite groups having those values as commutativity degree. Surprisingly, he misses out the value 5/14. Here we are giving two classes of finite groups, namely $H \rtimes_\theta K$, where $H = \langle a \mid a^7 = 1\rangle$ and $\langle a \mid a^{14} = 1\rangle$, and K is any abelian group of even order, having commutativity degree 5/14. It may be mentioned here that recently, the author together with Das [3] has pointed out the following fact:

$$\Pr(G) = \frac{5}{14} \quad \text{if and only if} \quad G' = C_7, G' \cap Z(G) = \{1\} \text{ and } \frac{G}{Z(G)} \cong D_{14},$$

where G' denotes the commutator subgroup of G and C_7 denotes the cyclic group of order seven.

We conclude the paper with the following discussion.

Let $|\mathrm{Cent}(G)| = |\{C_G(x) \mid x \in G\}|$, that is, the number of distinct centralisers in G. A finite group G is called an n-centraliser group if $|\mathrm{Cent}(G)| = n$, and a primitive n-centraliser group if

$$|\mathrm{Cent}(G/Z(G))| = |\mathrm{Cent}(G)| = n.$$

In [2], Belcastro and Sherman studied n-centraliser groups for some n and asked about the existence of n-centraliser groups for any n other than 2 and 3. By counting the number of distinct centralisers of Q_{4m}, Ashrafi [1] answered this question affirmatively. Note that by counting the number of distinct centralisers of $C_n \rtimes_\theta C_{2m}$, where $C_n = \langle a \mid a^n = 1\rangle$, $C_{2m} = \langle b \mid b^{2m} = 1\rangle$ and $\theta(b)(a) = a^{-1}$,

$$|\mathrm{Cent}(C_n \rtimes_\theta C_{2m})| = \begin{cases} n+2 & \text{if } n \text{ is odd,} \\ \dfrac{n}{2}+2 & \text{if } n \text{ is even.} \end{cases}$$

This also answers affirmatively the questions posed by Belcastro and Sherman, cited above. Also, if n is odd then $(C_n \rtimes_\theta C_{2m})/Z(C_n \rtimes_\theta C_{2m}) \cong C_n \rtimes_\theta C_2$. Therefore, if n is odd then $C_n \rtimes_\theta C_{2m}$ provides examples of primitive $(n+2)$-centraliser groups.

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