Canad. Math. Bull. Vol. **43** (1), 2000 pp. 115–125

Perfect Non-Extremal Riemann Surfaces

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Abstract. An infinite family of perfect, non-extremal Riemann surfaces is constructed, the first examples of this type of surfaces. The examples are based on normal subgroups of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ of level 6. They provide non-Euclidean analogues to the existence of perfect, non-extremal positive definite quadratic forms. The analogy uses the function *syst* which associates to every Riemann surface M the length of a systole, which is a shortest closed geodesic of M.

1 Introduction

(a) In the geometry of numbers, a well-known result of Voronoï [22] states that a positive definite quadratic form is extremal if and only if it is perfect and eutactic (see for example Gruber/Lekkerkerker [10] and Martinet [13] for more on this subject). Extremal positive definite quadratic forms correspond to extremal lattice sphere packings (see in particular Conway/Sloane [7]).

Many problems in the classical (Euclidean) geometry of numbers have their analogues in hyperbolic geometry. Let T_g, $g \geq 2$, be the Teichmüller space of closed Riemann surfaces of genus g, the surfaces being equipped with a complete hyperbolic metric (a metric of constant curvature -1). Let syst be the function on T_g which associates to $M \in T_g$ the length of a systole of M (a systole is a shortest closed geodesic). Then $M_0 \in T_g$ is an *extremal surface* if syst has a local maximum in M_0. This is a non-Euclidean analogue of extremal lattice sphere packings (for closed Riemann surfaces of genus 1 the two concepts coincide). It has been introduced in Schmutz ([18], [19], see also the recent survey paper [21]). Extremal surfaces are also the subject of [15], see further Bavard [4], Quine/Zhang [16].

(b) In order that a positive definite quadratic form f is extremal, it is not sufficient that f is perfect. Voronoï [22] had already announced the example of a perfect non-extremal form of dimension 6. Such an example has eventually been described by Barnes [2], [3], answering a question of Coxeter [8]. Also, of the 33 different perfect forms of dimension 7, only 30 are extremal, see Conway/Sloane [6].

The aim of this paper is to provide non-Euclidean examples, namely to construct perfect non-extremal Riemann surfaces. *Perfect surfaces* are those which are determined by their set of systoles, see Section 2 for the precise definition.

In Section 3, I construct a family of infinitely many perfect non-extremal Riemann surfaces. The construction is based on normal subgroups of the modular group. More precisely, let G be a torsion-free normal subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ of finite index and of level 6. The corresponding surface $\mathbb{H}/G$ ($\mathbb{H}$ is the hyperbolic plane) has genus 1 and a number $n = n(G)$ of cusps. Replace the n cusps in $\mathbb{H}/G$ by n simple closed geodesics of the same length $12z$ such that the automorphism group of $\mathbb{H}/G$ is preserved; denote by $R_G(z)$ this

Received by the editors April 8, 1998; revised October 1, 1998.
AMS subject classification: Primary: 11H99; secondary: 11F06, 30F45.

surface with boundary. Let $D_G(z)$ be the double of $R_G(z)$, $D_G(z)$ is then a closed Riemann surface. $D_G(z)$ is contained in a 2-parameter family A_G of closed surfaces (A_G corresponds to the 2-dimensional family of $(2, 2, 2, 3)$-quadrilateral groups); one parameter is the length z, the second parameter is provided by a simultaneous twist deformation along the n simple closed geodesics of length $12z$. In A_G one finds an, up to isometry, unique closed surface M_G which has $10n$ systoles of length $12z$. We shall see that M_G is perfect, but not extremal.

At the end of the paper I add some remarks concerning possible generalizations of this construction.

(c) The term "eutactic" (introduced by Coxeter [8]) for positive definite quadratic forms (or for lattice sphere packings) is not easy to define. However, it has been shown by Ash [1] that eutactic forms correspond to critical points of the packing function, which is a topological Morse function. I have shown in [20] (see also [21]) that also syst is a topological Morse function for Riemann surfaces. Therefore, we can say that the above described examples are perfect Riemann surfaces which are not critical points of syst.

(d) Let $\mathcal{G}$ be the set of torsion-free normal subgroups of $\mathrm{PSL}(2, \mathbb{Z})$ of finite index and of level 6. $\mathcal{G}$ can be interpreted in a Euclidean context. The elements of $\mathcal{G}$ correspond to normal subgroups of the $(2, 3, 6)$ triangle group (which uniformize Euclidean tori). Coxeter/Moser [9]) have classified all Euclidean tori which can be obtained by these subgroups and I shall use their classification in the present context.

An element $G \in \mathcal{G}$ will be characterized by the number n of cusps of $\mathbb{H}/G$ (this is however not a $1 - 1$ classification). For example, $\mathcal{G}$ contains the principal congruence subgroup $\Gamma(6)$ (which has index 72 in $\mathrm{PSL}(2, \mathbb{Z})$) and $\mathbb{H}/\Gamma(6)$ has 12 cusps. $\mathcal{G}$ contains exactly five elements with a smaller index in $\mathrm{PSL}(2, \mathbb{Z})$ than $\Gamma(6)$, the corresponding surfaces have $1, 3, 4, 7$, and 9 cusps, respectively. I further note that it follows by a result of Zograf [23] and by the main result of Luo/Rudnick/Sarnak [12] that $G \in \mathcal{G}$ is not a congruence subgroup (of $\mathrm{PSL}(2, \mathbb{Z})$) for $n \geq 37$ (this does not mean, of course, that all elements of $\mathcal{G}$ with $n \leq 36$ are congruence subgroups).

Acknowledgment I thank Jack Quine for helpful discussions.

2 Definition of Perfect Non-Extremal Surfaces

Definitions

(i) A *surface M* is a Riemann surface of constant curvature -1. If M is a surface with boundary, then, by definition, the boundary components are simple closed geodesics, called *boundary geodesics*. If M is compact without boundary, then M is called a *closed* surface.

(ii) Denote by $\mathbb{H}$ the upper halfplane. For a complete surface M I also write $M = \mathbb{H}/\Gamma$ where Γ is a Fuchsian group which uniformizes M.

(iii) A *systole* of a surface M is a shortest closed geodesic of M. $S(M)$ denotes the set of systoles of M.

(iv) Let $\mathcal{M}$ be a closed surface. Denote by $T(\mathcal{M})$ the Teichmüller space of $\mathcal{M}$. Denote by syst the function

$$\mathrm{syst}\colon T(\mathcal{M}) \longrightarrow \mathbb{R}$$

https://doi.org/10.4153/CMB-2000-018-4 Published online by Cambridge University Press

which associates to every $M \in T(\mathcal{M})$ the length of a systole of M. Let $M_0 \in T(\mathcal{M})$. Then M_0 is called an *extremal* surface if syst has a local maximum in M_0.

(v) Let $\mathcal{M}$ be a closed surface. Let $M_0 \in T(\mathcal{M})$. Then M_0 is called a *perfect* surface if the following condition holds.

There exists an open neighborhood U of M_0 in $T(\mathcal{M})$ such that

a) the length functions of the elements of $S(M_0)$ parameterize U and
b) $S(M_0) \neq S(M)$ for every $M \in U \setminus \{M_0\}$ ($S(M)$ and $S(M_0)$ are understood as sets of *marked* geodesics).

3 Construction of the Examples

Definitions

(i) Let $\mathcal{G}$ be the set of torsion-free normal subgroups of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ of finite index and of level 6 (the level is defined in the sense of Wohlfahrt, see for example [14, p. 147]; see also the proof of Theorem 1 below for a geometric characterization). Let $G \in \mathcal{G}$. Then I write $G = G(n)$ if $\mathbb{H}/G$ has n cusps.
(ii) Let $\mathcal{N} = \{b^2 + bc + c^2 : b, c \in \mathbb{Z}, b \geq c \geq 0, b + c > 0\} = \{1, 3, 4, 7, 9, 12, 13, \dots\}$.

Remark Compare the introduction for some comments on $\mathcal{G}$.

Theorem 1

(i) Let $G = G(n) \in \mathcal{G}$. Then $\mathbb{H}/G$ has genus 1 *and $n \in \mathcal{N}$.*
(ii) Let $n \in \mathcal{N}$. Then there exists $G(n) \in \mathcal{G}$.

Proof Since $G(n)$ has level 6, around each cusp of $\mathbb{H}/G$, there are exactly 12 different triangles of angles $\pi/2, \pi/3, 0$. It follows that the area of $\mathbb{H}/G$ is $2n\pi$ which implies that $\mathbb{H}/G$ has genus 1. Replace the triangles of angles $\pi/2, \pi/3, 0$ by triangles of angles $\pi/2, \pi/3, \pi/6$. Thereby, $\mathbb{H}/G$ is transformed into an Euclidean torus. These Euclidean tori have been classified in Coxeter/Moser [9] and the theorem follows by their classification. ■

Remark Let $n \in \mathcal{N}$. Then $G(n)$ is not unique if the representation $n = b^2 + bc + c^2$, $b \geq c \geq 0$, is not unique. $G(1)$ is however unique and $\mathbb{H}/G(1)$ is the so-called *modular torus*.

Lemma 2 *Let $G \in \mathcal{G}$. Then G is a normal subgroup of $G(1)$.*

Proof Obvious since the automorphism group of $\mathbb{H}/G$ acts transitively on the cusps. ■

Definitions

(i) Let $z \in \mathbb{R}$, $z \geq 0$. Let $Q(z)$ be a hyperbolic quadrilateral with three angles $\pi/2$ and one angle $\pi/3$ such that one of the sides of the quadrilateral, which joins two angles $\pi/2$, has length z.

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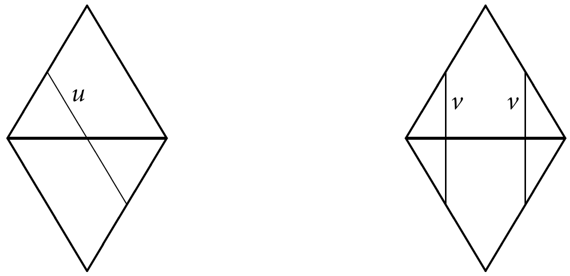

Figure 1: The modular torus with the canonical triangulation (thick lines), opposite sides have to be identified. On the left hand side, a closed geodesic u with $N(u) = 2$, on the right hand side a closed geodesic v with $N(v) = 4$.

(ii) Take six copies of $Q(z)$ such that they form a right-angled hexagon, the latter is denoted by $H(z)$.

Remarks

(i) $Q(z)$ is, up to isometry, uniquely determined for every non-negative real z. Note that for $z = 0$ we obtain a hyperbolic triangle with angles $\pi/2, \pi/3, 0$, which is denoted by $Q(0)$.
(ii) $H(0)$ is a triangle, its three angles being zero.

Definitions

(i) Let $G \in \mathcal{G}$. Then $\mathbb{H}/G$ is triangulated into triangles of angles $\pi/2, \pi/3, 0$. This is called the $(2,3,6)$-*triangulation*. This triangulation induces a new triangulation by triangles of type $H(0)$. It will be called the *canonical triangulation*. (The $(2,3,6)$-triangulation is the (first) barycentric subdivision of the canonical triangulation.)
(ii) Let u be a closed geodesic of $\mathbb{H}/G$. Then the canonical triangulation of $\mathbb{H}/G$ separates u into a number of geodesic segments, denote by $N(u)$ this number (see Figure 1 and Figure 2 for examples).
(iii) The length of a closed geodesic u will be denoted by $L(u)$.

Lemma 3 *Let u be a closed geodesic in the modular torus $\mathbb{H}/G(1)$. If $N(u) < 6$, then*

$$2\cosh\bigl(L(u)/2\bigr) \in \{3, 6, 7\}.$$

Proof If u has self-intersections, then the component of $\mathbb{H}/G(1) \setminus u$ which contains the cusp, is homeomorphic to a horodisc so that $N(u) \geq 6$. Therefore, we can assume that u is simple and passes through two fixed points of the hyperelliptic involution of $\mathbb{H}/G(1)$; these fixed points are the centres of the sides of the triangles of the canonical triangulation. It follows (compare Figure 1) that, up to isometry, there is a unique possibility for $N(u) = 2$ and there are two possibilities for $N(u) = 4$ (one of them is the geodesic u with $N(u) = 2$ passed twice). The lemma follows by an easy calculation (for formula of hyperbolic trigonometry see for example [5]). ■

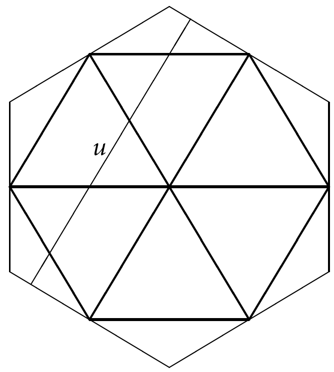

Figure 2: The surface $\mathbb{H}/G(4)$ with the canonical triangulation (thick lines), opposite sides have to be identified; u is a closed geodesic with $N(u) = 4$.

Corollary 4 *Let $G(n) \in \mathcal{G}$. Let u be a closed geodesic of $\mathbb{H}/G$. If $N(u) < 6$, then $n \in \{1, 3, 4\}$.*

Proof By Lemma 2, u is induced by a closed geodesic u_0 of the modular torus and $N(u_0) \leq N(u) < 6$. We therefore can apply Lemma 3.

Assume that $n > 4$. Then $\mathbb{H}/G(n)$ contains an embedded part as in Figure 2 such that the opposite sides (in Figure 2) are disjoint. It follows by Lemma 3 that $N(u) < 6$ is impossible. ■

Definition (i) Let $G \in \mathcal{G}$. Let $z \in \mathbb{R}$, $z > 0$. Replace each triangle $Q(0)$ of the $(2, 3, 6)$-triangulation of $\mathbb{H}/G$ by a quadrilateral $Q(z)$. Denote by $R_G(z)$ the corresponding surface with boundary.

(ii) Denote by $2x$ the length of a boundary geodesic of $R_G(z)$ (note that $x = 6z$).

(iii) A *common orthogonal* of $R_G(z)$ is a simple geodesic which is a common orthogonal of two boundary geodesics of $R_G(z)$.

(iv) Denote by τ the sum of the lengths of the two sides of $Q(z)$ which enclose the angle $\pi/3$ of $Q(z)$. Let C be the sum of the lengths of the four sides of $Q(z)$. Put $t/2 = C - \tau - z$ ($t/2$ is the length of a side of $Q(z)$).

(v) Denote by $\mathcal{T}$ the set of the common orthogonals (of $R_G(z)$) of length t, corresponding to the sides of the hexagons $H(z)$ (induced by the triangles $H(0)$ of the canonical triangulation of $\mathbb{H}/G$).

Lemma 5 *Let $G(n) \in \mathcal{G}$ and let $R_{G(n)}(z)$, $z > 0$, be the corresponding surface with boundary.*

- *(i) The shortest common orthogonals in $R_{G(n)}(z)$ are the $3n$ elements of $\mathcal{T}$.*
- *(ii) Among all common orthogonals of $R_{G(n)}(z)$ which are not elements of $\mathcal{T}$, let v be one of shortest length. Then the length of v is 2τ.*
- *(iii) Let w be a common orthogonal of $R_{G(n)}(z)$ which is strictly longer than 2τ. Then the length of w is at least θ with $\cosh\theta = \cosh x \sinh^2 t - \cosh^2 t$.*

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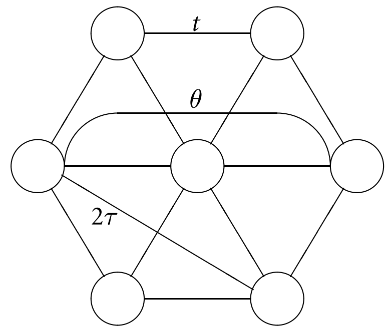

Figure 3: A part of $R_G(z)$ with right-angled hexagons and common orthogonals of length t, 2τ and θ, respectively. (The circles correspond to boundary geodesics of $R_G(z)$.)

Proof The canonical triangulation of $\mathbb{H}/G(n)$ induces a partition of $R_{G(n)}(z)$ into right-angled hexagons. The lemma follows, compare Figure 3. ■

Definition Let $G \in \mathcal{G}$, let $R_G(z)$ be the corresponding surface with boundary. Let $D_G(z)$ be the double of $R_G(z)$. Denote by Z the set of simple closed geodesics of $D_G(z)$ which were boundary geodesics of $R_G(z)$. (The double $D_G(z)$ of $R_G(z)$ is defined such that $D_G(z)$ has an orientation reversing involution fixing pointwise the elements of Z.)

Remark By construction, $D_G(z)$ is a closed surface and the automorphism group of $D_G(z)$ contains a subgroup Σ of a $(2, 2, 2, 3)$-quadrilateral group. Let T_G be the Teichmüller space of $D_G(z)$. Then T_G contains a family, of two real parameters, of surfaces such that their automorphism group contains Σ; one parameter of this family is z, the other parameter is provided by a simultaneous twist deformation of the same amount (and in the same direction) along all elements of Z.

Definition (i) Denote by A_G the family of two real parameters in T_G which is described in the previous remark. Denote by Σ the subgroup of a $(2, 2, 2, 3)$-quadrilateral group which characterizes the family $A(G)$.

(ii) I now define a particular element M_G in A_G. Let $q \in \mathcal{T}$; q joins two boundary geodesics of $R_G(z)$ which are denoted by z_1 and z_2. Denote by $Y(q)$ the unique embedded pair of pants (a surface of genus zero with three boundary geodesics) of $R_G(z)$ which contains z_1, z_2 and q. Let $X(q)$ be the double of $Y(q)$, embedded in $D_G(z)$; $X(q)$ has genus 1 and two boundary geodesics. Then M_G is defined by the property that $X(q)$ has five geodesics of length $2x$ which are systoles of $X(q)$.

Define $X(\mathcal{T}) = \{X(q) : q \in \mathcal{T}\}$.

Remark Up to isometry, M_G is uniquely defined in A_G, compare [18]. Note however that the (common) sign of the twist deformations along the elements of Z is not well-defined.

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Lemma 6 *Let $G \in \mathcal{G}$ and let M_G be the corresponding closed surface. Then M_G has three sets of simple closed geodesics of length $2x$ which are invariant with respect to Σ, one of these sets is Z. The two other sets have $3n$ and $6n$ elements, respectively.*

Proof Σ acts transitively on the elements of $X(\mathcal{T})$ (since Σ acts transitively on the elements of $\mathcal{T}$); by Lemma 5, $X(\mathcal{T})$ has $3n$ elements. Let $X(q) \in X(\mathcal{T})$. Then $X(q)$ has five simple closed geodesics of length $2x$. One of them intersects all other four, this induces a set V of $3n$ simple closed geodesics of length $2x$, invariant with respect to Σ. $X(q)$ contains further two elements of Z. The last two closed geodesics of length $2x$ of every $X(q) \in X(\mathcal{T})$ are contained in a set W of $6n$ elements, invariant with respect to Σ. ■

Definition Define the sets V and W of simple closed geodesics in M_G of length $2x$ as in the proof of Lemma 6.

Lemma 7 *Let $X(q) \in X(\mathcal{T})$. Let $v \in V$, $w \in W$, $z \in Z$ be three simple closed geodesics of length $2x$ in $X(q)$. Let γ be defined by*

$$\cos\gamma = \frac{\cosh x}{1 + \cosh x}.$$

Then both w and z intersect v in the angle γ.

Proof This follows by a calculation (compare [18]). ■

Lemma 8 *Let $G \in \mathcal{G}$ and let M_G be the corresponding closed surface. Let θ be defined as in Lemma 5. Then*

(i) $x \sim 3.17575$
(ii) $t \sim 1.6206$
(iii) $2\tau \sim 3.5809$
(iv) $\theta \sim 4.8499$.

Proof The lengths t, τ, and θ depend on x. The length x can be calculated using the identities

$$\cosh t = \frac{\cosh(x/3)}{\cosh(x/3) - 1}, \tag{1}$$

provided by the hexagons $H(z)$, and

$$\sinh(t/2) = \sinh(x/2)\sin\gamma,$$

provided by an analysis of an element of $X(\mathcal{T})$; the latter identity, together with Lemma 7, implies

$$\cosh t = \frac{\cosh x(1 + 3\cosh x)}{(1 + \cosh x)^2} \tag{2}$$

and x is determined by (1) and (2). ■

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Theorem 9 *Let $G(n) \in \mathcal{G}$, $n > 7$, and let $M = M_{G(n)}$ be the corresponding closed surface. Then M has exactly $10n$ systoles of length $2x$.*

Proof Let u be a systole of M. Assume that u does not intersect an element of Z. Then $u \in Z$ or u has the same length as a systole of $R_{G(n)}(z)$. Since $n > 7$, a systole of $R_{G(n)}(z)$ intersects at least six right-angled hexagons $H(z)$ (by Corollary 4) so that it is longer than $2x$. Therefore, $u \in Z$.

Assume now that u intersects $2m > 0$ elements of Z. Then $L(u) \geq 2mt$ by Lemma 5 which implies by Lemma 8 that $m = 1$ (since $2t > x$). Therefore, u is separated by Z into two parts u_1 and u_2, each one is homotopic to a common orthogonal in a copy of $R_{G(n)}(z)$. By Lemma 5 and Lemma 8, one of them, u_1 say, is homotopic to an element of $\mathcal{T}$ (since $2\tau > x$), and u_2 is then homotopic to a common orthogonal of length t or of length 2τ (since $t + \theta > 2x$). The latter case is only possible for $n \leq 7$ (which is excluded by hypothesis). Therefore, u_1 and u_2 are both homotopic to an element of $\mathcal{T}$. Since $n > 7$ it follows (compare the proof of Corollary 4) that in M, there exists a subsurface $X(q)$ containing u. This implies $L(u) = 2x$. We have therefore proved that the systoles of M have length $2x$. The number $10n$ follows by Lemma 6. ■

Theorem 10 *Let $G(n) \in \mathcal{G}$, $n > 7$, and let $M = M_{G(n)}$ be the corresponding closed surface. Then M is a perfect closed surface which is not extremal.*

Proof By Theorem 9, the systoles of M are identified. Let $z \in Z$. Then there are exactly six elements of $X(\mathcal{T})$ which contain z. The six elements of V in these six elements of $X(\mathcal{T})$ are the boundary geodesics of an embedded subsurface S of M of genus 1. In the interior, S has 13 systoles of M, namely z, six elements of V and six elements of W. The corresponding length functions determine S, compare [17], [18]. It follows that M is perfect.

To prove that M is not extremal, requires some calculation. Let ξ be a vector in the tangent space of M which is induced by a twist deformation of the same amount (and in the same direction) along all elements of V. Denote by $\xi(u)$ the real number obtained by applying ξ to the length function of a simple closed geodesic u. Then, see for example [11],

$$\xi(u) = \sum_i \cos \gamma_i$$

where the sum is over all directed angles γ_i in the intersection points of u with the elements of V. Since the elements of V are mutually disjoint, we have

$$\xi(v) = 0, \quad \forall v \in V.$$

By Lemma 7 it follows that

$$\xi(z) = 6 \cos \gamma > 0, \quad \forall z \in Z$$

(we measure the angles clockwise (in Figure 4) from the elements of V to z). Let $w \in W$. Then w is intersected by exactly six elements of V. It follows by Figure 4 (and by the symmetry of M) that the six angles of the corresponding intersections are (recall that we measure the angles clockwise) $2\gamma, -\gamma_1, -\gamma_2, -\gamma, -\gamma_2, -\gamma_1$.

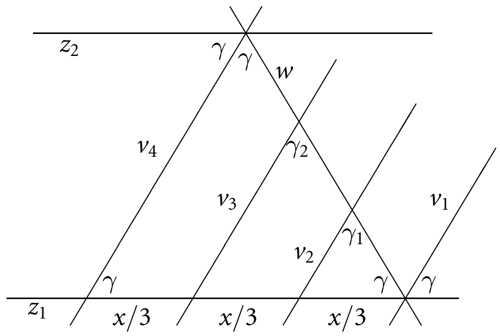

Figure 4: An element $w \in W$ intersects $z_i \in Z$, $i = 1, 2$, and six elements of V, four of them (v_k, $k = 1, \dots, 4$) are drawn.

Again by Figure 4, we have, for $j = 1, 2$,

$$\cos \gamma_j = \sin^2 \gamma \cosh(jx/3) - \cos^2 \gamma.$$

Since we know x and γ (by Lemma 8 and Lemma 7), we can calculate $\cos \gamma_j$, $j = 1, 2$. A calculation then gives

$$\xi(w) = -\cos \gamma - 2\cos \gamma_2 - 2\cos \gamma_1 + \cos(2\gamma) \sim 1.46, \quad \forall w \in W.$$

We have therefore shown that $\xi(u) > 0, \forall u \in Z \cup W$.

Let ζ be the vector in the tangent space of M which is induced by a twist deformation of the same amount (and in the same direction) along all elements of Z such that

$$\zeta(v) = 2\cos \gamma > 0, \quad \forall v \in V$$

(this is possible by Lemma 7). It follows that the span of ξ and ζ (in the tangent space of M) contains a vector η such that

$$\eta(u) > 0, \quad \forall u \in Z \cup V \cup W.$$

This implies that M is not a local maximum for syst. ■

Corollary 11 *Let m be any positive integer. Then there exists an integer $g \geq 2$ such that in the Teichmüller space T_g of closed surfaces of genus g, there exist more than m mutually non-isometric surfaces which all are perfect, but not extremal. Moreover, these surfaces can be*

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chosen such that the length of their systoles is the same and the number of their systoles is the same.

Proof Take $G(n) \in \mathcal{G}$ such that n has more than m different representations $n = b^2+bc+c^2$, $b \geq c \geq 0$, b, c integers. Each different representation gives a different surface $M_{G(n)}$. By Theorem 10 they are all perfect, but not extremal. ■

Remark I add some remarks concerning possible generalizations of the construction of perfect non-extremal surfaces given in this paper.

(i) Here is another idea for a possible proof of Corollary 11. Let $G(n) \in \mathcal{G}$. In the definition of $M = M_{G(n)}$ we have seen that the (common) direction of the twist deformations along the elements of Z is not well-defined, they can be all positive or all negative, say (but the amount of the twist deformations is well-defined). Choose a subset $Z' \subset Z$. Construct a new surface $M(Z')$ by inverting the twist deformation along the elements of Z'. Then $M(Z')$ has still $10n$ closed geodesics of length $2x$ and they are still the systoles since the proof of Theorem 9 goes through. Also, $M(Z')$ is still perfect by the argument in the proof of Theorem 10. But $M(Z')$ may be extremal.

(ii) Let $\mathcal{G}(N)$ be the set of the torsion-free normal subgroups of the modular group of finite index and of level N (for a fixed integer $N \geq 7$). As in the case $N = 6$, we can also construct a closed surface M_G for every $G \in \mathcal{G}(N)$. It is very probable that the proofs of Theorem 9 and Theorem 10 also work in this case if we exclude those $G \in \mathcal{G}(N)$ which produce only a few cusps; for example for big N, we have to exclude the principal congruence subgroup $\Gamma(N)$ (remember that in the case $N = 6$ we had to exclude $n \leq 7$).

(iii) I finally note that we could also work with a torsion-free subgroup Γ of the modular group of finite index which is not normal, but shares the property of normal subgroups that around each cusp in $\mathbb{H}/\Gamma$, there is the same number of triangles with angles $\pi/2, \pi/3, 0$.

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