# ON THE FRACTIONAL PARTS OF $\alpha n^2$ AND $\beta n$

*by* R. C. BAKER

(Received 29 January, 1980)

We denote by $\|\ldots\|$ the distance to the nearest integer. Let $\alpha$ and $\beta$ be real. W. M. Schmidt [5] proved that for $\varepsilon > 0$ and $N > c_1(\varepsilon)$ there is a natural number $n$ such that

$$n \leqq N, \qquad \|\alpha n^2 + \beta n\| < N^{-(1/2)+\varepsilon}.$$

This extends a theorem of H. Heilbronn [4] and also sharpens a theorem of H. Davenport [3].

In the present note I use the ideas of [5] to prove that for $N > c_2(\varepsilon)$ there is a natural number $n$ such that

$$n \leqq N, \qquad \|\alpha n^2\| < N^{-(1/4)+\varepsilon}, \qquad \|\beta n\| < N^{-(1/4)+\varepsilon}. \tag{1}$$

This sharpens a theorem of the author and J. Gajraj [2]. The other results of [2] can be improved; this is discussed in [1].

We require several lemmas. We write $e(x) = e^{2\pi i x}$ and $M = N^{(1/4)-\varepsilon}$. Let $\varepsilon_1 = \varepsilon/3$. Constants implied by '$\ll$' and '$\gg$' will depend at most on $\varepsilon$.

Lemma 1. *Let $N > c_2(\varepsilon)$. Suppose that there is no natural number n satisfying* (1). *Then either*

(i) *there is a natural number $r \leqq MN^{\varepsilon}$ such that*

$$\|r\beta\| < N^{-1+\varepsilon}, \tag{2}$$

*or*

(ii) *we have*

$$\sum_{|u| < MN^{\varepsilon_1}} \sum_{v=1}^{[MN^{\varepsilon_1}]} \left| \sum_{x=1}^{N} e(v\alpha x^2 + u\beta x) \right|^2 > N^{2-\varepsilon_1} M^{-2}. \tag{3}$$

*Moreover, in case* (ii) *there is a natural number $q \leqq M^4 N^{\varepsilon_1}$ such that*

$$|q\alpha - p| < M^3 N^{-2+\varepsilon_1}, \qquad (q, p) = 1. \tag{4}$$

*Proof.* See [2, pp. 329–331].

Lemma 2. *We have*

$$\left| \sum_{x=1}^{N} e(\alpha x^2 + \beta x) \right|^2 \ll \sum_{w=1}^{2N} \min(N, \|2(\alpha w + \beta)\|^{-1}).$$

*Proof.* See [5, p. 822].

Glasgow Math. J. **22** (1981) 181–183.

https://doi.org/10.1017/S0017089500004651 Published online by Cambridge University Press

LEMMA 3. *Suppose p, q are coprime, with* $1 \leqq q < N \leqq H$. *Suppose that*

$$\|\alpha q\| = |\alpha q - p| < (2H)^{-1}.$$

*Then for any real* $\gamma$,

$$\sum_{u=1}^{H} \min(N, \|\alpha u + \gamma\|^{-1}) \ll (\log H) \min\left(\frac{NH}{q}, \frac{H}{\|\gamma q\|}, \frac{1}{\|\alpha q\|}\right).$$

*Proof.* This is a straightforward extension of Lemma 5 of [5].

*Proof that* (1) *is soluble.* We suppose that there is no natural number $n \leqq N$ satisfying (1), where $N > c_2(\varepsilon)$. We shall obtain a contradiction.

Suppose first that alternative (i) takes place in Lemma 1. Let $r$ be the natural number defined there. By the theorem of Heilbronn [4] there is a natural number $s \leqq M^2 N^{\varepsilon}$ such that

$$\|s^2 r^2 \alpha\| < M^{-1}. \tag{5}$$

Let $n = sr$, then $n \leqq M^3 N^{2\varepsilon} < N$, and

$$\|n\beta\| \leqq s\, \|r\beta\| < M^2 N^{\varepsilon} \,.\, N^{-1+\varepsilon} < M^{-1}.$$

This together with (5) contradicts the insolubility of (1). Thus alternative (ii) must hold in Lemma 1. Let $q$ be the natural number defined there.

By combining (3) with Lemma 2 we find that

$$N^{2-\varepsilon_1} M^{-2} \ll \sum_{|u| < MN^{\varepsilon_1}} \sum_{v=1}^{[MN^{\varepsilon_1}]} \sum_{w=1}^{2N} \min(N, \|2(\alpha v w + \beta u)\|^{-1})$$

$$\ll N^{\varepsilon_1} \sum_{|y| < 2MN^{\varepsilon_1}} \sum_{x=1}^{[4MN^{1+\varepsilon_1}]} \min(N, \|\alpha x + \beta y\|^{-1}). \tag{6}$$

For the last inequality we write $x = 2vw$, $y = 2u$ and observe that for a given $x$ there are fewer than $N^{\varepsilon_1}$ possibilities for $v$, $w$. With an application of Lemma 3 to the sum over $x$, we obtain

$$N^{2-3\varepsilon_1} M^{-2} \ll \sum_{|y| < 2MN^{\varepsilon_1}} \min\left(\frac{MN^{2+\varepsilon_1}}{q}, \frac{MN^{1+\varepsilon_1}}{\|\beta q y\|}, \frac{1}{\|\alpha q\|}\right). \tag{7}$$

By Dirichlet's theorem there is a natural number $t \leqq 4MN^{\varepsilon_1}$ satisfying

$$|\beta q t - z| < (4MN^{\varepsilon_1})^{-1}, \qquad (t, z) = 1. \tag{8}$$

It is not difficult to see that

$$\|\beta q y\| \geqq (2t)^{-1}$$

whenever $|y| < 2MN^{\varepsilon_1}$ and $t \nmid y$. The contribution of these integers $y$ to the right hand side

https://doi.org/10.1017/S0017089500004651 Published online by Cambridge University Press

of (7) is thus

$$\leqq MN^{1+\varepsilon_1} \sum_{|y|<2MN^{\varepsilon_1}} \min(2t, \|\beta qy\|^{-1})$$

$$\ll MN^{1+\varepsilon_1}\left(\frac{MN^{\varepsilon_1}}{t}+1\right)(2t+t\log t) \ll M^2N^{1+3\varepsilon_1}$$

by a standard argument. Since $M^2N^{1+3\varepsilon_1}=o(N^{2-3\varepsilon_1}M^{-2})$, we must have

$$N^{2-3\varepsilon_1}M^{-2} \ll \sum_{\substack{|y|<2MN^{\varepsilon_1}\\ t|y}} \min\left(\frac{MN^{2+\varepsilon_1}}{q}, \frac{1}{\|\alpha q\|}\right).$$

As the number of terms in the last sum is $\ll MN^{\varepsilon_1}/t$, it is easy to see that

$$\max(qtM^{-1}N^{-2-\varepsilon_1}, t\|\alpha q\|) \ll M^3N^{-2+4\varepsilon_1}. \tag{9}$$

Now we get a contradiction by combining (8) and (9) to show that $n=qt$ solves (1). This completes the proof that (1) is soluble.

## REFERENCES

1. R. C. Baker, Recent results on fractional parts of polynomials, *Number theory, Carbondale 1979*, Lecture Notes in Mathematics No. 751 (Springer-Verlag, 1979), 10–18.

2. R. C. Baker and J. Gajraj, Some non-linear Diophantine approximations. *Acta Arith.* **31** (1976), 325–341.

3. H. Davenport, On a theorem of Heilbronn, *Quart. J. Math. Oxford Ser. 2*, **18** (1967), 339–344.

4. H. Heilbronn, On the distribution of the sequence $n^2\theta$ (mod 1), *Quart. J. Math. Oxford Ser. 1*, **19** (1948), 249–256.

5. W. M. Schmidt, On the distribution modulo 1 of the sequence $\alpha n^2+\beta n$, *Canad. J. Math.* **29** (1977), 819–826.

Royal Holloway College
Egham
Surrey