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EXISTENCE AND MULTIPLICITY RESULTS FOR QUASILINEAR ELLIPTIC EQUATIONS

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The goal of this paper is to study the multiplicity of positive solutions of a class of quasilinear elliptic equations. Based on the mountain pass theorems and sub- and supersolutions argument for p-Laplacian operators, under suitable conditions on nonlinearity $f(x,s)$, we show the follwing problem:

$$-\Delta_p u = \lambda f(x,u) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,$$

where Ω is a bounded open subset of R^N, $N \geqslant 2$, with smooth boundary, λ is a positive parameter and Δ_p is the p-Laplacian operator with $p > 1$, possesses at least two positive solutions for large λ.

1. Introduction

Let Ω be a bounded open subset of R^N, $N \geqslant 2$, with smooth boundary $\partial\Omega$. We are interested in the existence and multiplicity of positive solutions for the problem

$$-\Delta_p u = \lambda f(x,u) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.1}$$

where λ is a positive parameter, Δ_p is the p-Laplacian operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2} Du\right)$$

with $p > 1$, and $f(x,s)$ is a real function defined on $\overline{\Omega} \times R^+$.

Problem (1.1) with $p = 2$ has been extensively studied. Rabinowitz [8] proved that problem (1.1) has two distinct positive solutions for large λ provided that $f(x,s)$ satisfies

- (f_1) f is locally Lipschitz continuous on $\overline{\Omega} \times R^+$ with $f(x,0) = 0$;
- (f_2) there exists a positive number k such that $f(x,s) < 0$ if $s \geqslant k$;
- (f_3) there exists $(a,y) \in \Omega \times [0,k)$ such that $F(a,y) > 0$, where $F(x,z) = \displaystyle\int_0^z f(x,s)\,ds$; and
- (f_4) $f(x,s) = o(s)$ at $s = 0$ uniformly in $x \in \overline{\Omega}$.

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This result has been generalised by various authors in several directions. Recently, Rabinowitz's result was extended to the case of $1 < p < 2$ in [6] and $p > 2$ in [10] and special nonlinearity $f(s)$.

In this paper, we show that the main results in [8] continue to hold for all $p > 1$ and for a more general nonlinearity. Some techniques are adapted from [2] and [1]. In particular, we prove the existence of a local minimiser in $W_0^{1,p}(\Omega)$, and use this fact to prove the multiplicity result under conditions on $f(x,s)$ similar to those in [8]. Our results improve the previous results.

Throughout this paper, we assume that the function $f(x,s)$ satisfies:

- (H_1) $f \in C(\overline{\Omega} \times R^+)$ with $f(x,0) = 0$ and $\lim\limits_{s\to 0}(f(x,s))/s^{p-1}$ uniformly in $x \in \overline{\Omega}$;
- (H_2) there exists a positive number k such that $f(x,s) < 0$ if $s \geqslant k$;
- (H_3) there exists $(a,y) \in \Omega \times [0,k)$ such that $F(a,y) > 0$, where $F(x,z) = \displaystyle\int_0^z f(x,s)\,ds$; and
- (H_4) there exists a nondecreasing locally Lipschitz continuous function $L(s)$ such that $L(0) = 0$, $\displaystyle\int_0^1 \left(\int_0^s L(t)\,dt\right)^{-(1/p)} ds = \infty$, and the map $s \longrightarrow f(x,s) + L(s)$ is nondecreasing in $s \in [0,k]$ for almost all $x \in \Omega$, where the constant k is the same as in (H_2).

Clearly our case includes the cases considered in [6, 10]. We only consider the positive solutions of (1.1). By a positive solution of (1.1), we mean that $u \in C^1(\overline{\Omega})$ satisfies (1.1) in the weak sense with $u > 0$ in Ω. Standard regularity theory for p-Laplacian operators (see [12, 13]) assures that any weak solution of (1.1) belongs to $C^1(\overline{\Omega})$.

Our main results are the following.

Theorem 1.1. *There exists a positive number λ^* such that for $\lambda \geqslant \lambda^*$, problem* (1.1) *possesses at least one positive solution, and* (1.1) *has no positive solutions if $\lambda < \lambda^*$.*

Theorem 1.2. *For all $\lambda > \lambda^*$, problem* (1.1) *possesses at least two distinct positive solutions, where λ^* is the same as in Theorem 1.1.*

The rest of this paper is organised as follows. In Section 2, we use variational techniques and standard sub- and supersolutions arguments to obtain the existence result Theorem 1.1. In section 3, we use the idea in [1] and the Mountain Pass Theorems in [9] and [7] to obtain our multiplicity result, Theorem 1.2.

2. The existence result

In this section, we use directly methods in the calculus of variations in [2] and standard sub- and supersolutions arguments in [3, 4] to prove Theorem 1.1. We first recall the comparison principle for p-Laplacian (see, for example, [11, Lemma 2.2]).

Lemma 2.1. (Comparison Principle). *Let u and v be continuous functions in the Sobolev space $W^{1,p}_{loc}(\Omega)$ which satisfy the distributional inequality*

$$\Delta_p u - \Delta_p v \leqslant 0$$

in a domain Ω of R^N. Suppose that $u \geqslant v$ on $\partial\Omega$, in the sense that the set $\{u - v + \varepsilon \leqslant 0\}$ has compact support in Ω for every $\varepsilon > 0$. Then $u \geqslant v$ in Ω.

We also need the following lemmas.

Lemma 2.2. *Suppose that $u(x) \in C^1(\overline{\Omega})$ is a positive solution of* (1.1). *Then*

$$\max_{x\in\overline{\Omega}} u < k,$$

where k is the positive number in condition (H_2).

Proof: Otherwise, by continuity there exists a connected domain $\Omega_1 \subset \Omega$ such that $u \geqslant k$ in Ω_1 and $u = k$ on $\partial\Omega_1$. Hence u and $v = k$ satisfy

$$-\Delta_p u < 0 = -\Delta_p v \ \text{ in } \ \Omega_1, \ \ u|_{\partial\Omega_1} = v|_{\partial\Omega_1} \tag{2.1}$$

due to condition (H_2). Thus, the comparison principle, Lemma 2.1, shows that $u \equiv k$ in Ω_1. It contradicts (2.1). Therefore $u < k$ in Ω, and the proof is finished. ∎

Lemma 2.3. *For any $\lambda_0 > 0$, there exists a $\rho = \rho(\lambda_0)$ such that for $\lambda \in (0, \lambda_0]$, $u \equiv 0$ is the unique nonnegative solution of* (1.1) *in $\mathcal{B}_\rho(0) \subset C(\overline{\Omega})$, where $\mathcal{B}_\rho(0)$ is a ball with centre 0 and radius ρ.*

Proof: Suppose not; then (1.1) possesses solutions (λ_m, u_m) such that $u_m \geqslant 0$ and $u_m \not\equiv 0$, and $\lim\limits_{m\to\infty} u_m = 0$ in $C(\overline{\Omega})$; $\lambda_m > 0$ and $\lambda_m \in (0, \lambda_0]$. Since $v_m = u_m/\left(\|u_m\|_{C(\overline{\Omega})}\right)$ is bounded in $C(\overline{\Omega})$ and satisfies

$$-\Delta_p v_m = \lambda_m \frac{f(x, u_m)}{\|u_m\|^{p-1}_{C(\overline{\Omega})}} \to 0$$

and

$$v_m = 0 \ \text{ on } \ \partial\Omega,$$

the standard regularity theory implies that $v_m \to 0$ in $C(\overline{\Omega})$, which contradicts $\|v_m\|_{C(\overline{\Omega})} = 1$ and $v_m \to v$ in $C(\overline{\Omega})$. This completes the proof. ∎

Next a variational argument will be employed to show that problem (1.1) possesses a positive solution for large λ. For technical reason, the function $f(x,s)$ has to be modified to $\overline{f}(x,s)$, where $\overline{f}(x,s) := f(x,s)$ for $s \in [0,k]$; $\overline{f}(x,s) := f(x,k) < 0$ for $s \geqslant k$; $\overline{f}(x,s) := f(x,0)$ for $s \leqslant 0$. Obviously, the maximum principle implies that any solutions of the problem

$$-\Delta_p u = \overline{f}(x,u), \ \ x \in \Omega, \ \ u|_{\partial\Omega} = 0 \tag{2.2}$$

are positive and satisfy $u \leqslant k$. Then problem (2.1) is equivalent to problem (1.1) when only nonnegative solutions are considered. From now on, we treat problem (2.2) instead of (1.1). Now we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1: Define the functional $J(u)$ on $W_0^{1,p}(\Omega)$ for problem (2.2) by

$$J(u) = \frac{1}{p}\int_\Omega |Du|^p\,dx - \lambda\int_\Omega \overline{F}(x,u)\,dx, \tag{2.3}$$

where $\overline{F}(x,u) = \displaystyle\int_0^u \overline{f}(x,s)\,ds$.

It follows from (H_1) and the boundedness of $\overline{f}(x,s)$ that $J(u)$ is well defined on $W_0^{1,p}(\Omega)$. Standard arguments show that J is weakly lower semicontinuous. Clearly,

$$\left|\int_\Omega \overline{F}(x,u)\,dx\right| \leqslant M\int_\Omega |u|\,dx.$$

Since $p > 1$, it is easily seen that

$$\int_\Omega |u|\,dx \leqslant C_1\left(\int_\Omega |u|^p\,dx\right)^{1/p}$$

for some constant C_1 by using the Hölder inequality. It follows from Sobolev's inequality that

$$J(u) \geqslant \left(\frac{1}{p} - \lambda C\|u\|_{W_0^{1,p}}^{(1-p)/p}\right)\|u\|_{W_0^{1,p}}^p.$$

Therefore, $J(u)$ is coercive, that is

$$J(u) \to \infty \text{ as } \|u\|_{W_0^{1,p}(\Omega)} \to \infty.$$

Therefore, we conclude that J possesses a minimiser in $W_0^{1,p}(\Omega)$. Since $\overline{f}(x,s) = 0$, $s \leqslant 0$ and $J(|u|) \leqslant J(u)$, $J(u)$ posseses a nonnegative minimiser, which we denote by u_λ.

By (H_3) there exists a $\delta > 0$ such that

$$\overline{F}(x,s) \geqslant \gamma > 0 \text{ if } |x-a| \leqslant \delta,\ \ |y-s| \leqslant \delta.$$

Define $\phi(x) = y$ if $|x-a| \leqslant \delta/2$ and let ϕ go to 0 linearly along rays in the shell $\delta/2 \leqslant |x-a| \leqslant (\delta/2)+\eta$, where η is so small that

$$\int_{\delta/2\leqslant|x-a|\leqslant(\delta/2)+\eta} \overline{F}\big(x,\phi(x)\big)\,dx < \frac{1}{2}\int_{|x-a|\leqslant\delta/2} \overline{F}\big(x.\phi(x)\big)\,ds.$$

Therefore,

$$\int_\Omega \overline{F}\big(x,\phi(x)\big)\,dx > 0.$$

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Thus, there exists a λ_0 sufficiently large such that $J(\phi) < 0$ for all $\lambda \geqslant \lambda_0$. So $u_\lambda \geqslant 0$, $u_\lambda \not\equiv 0$ for all $\lambda \geqslant \lambda_0$, and u_λ satisfies

$$-\Delta_p u_\lambda + \lambda L(u_\lambda) = \lambda\big(f(x,u_\lambda) + L(u_\lambda)\big) \geqslant 0 \text{ in } \Omega, \quad u_\lambda|_{\partial\Omega} = 0,$$

where $L(s) = \beta(s) + s^p$ and $\beta(s)$ is the same as in condition (H_4). Clearly, $L(s)$ satisfies $\displaystyle\int_0^1 \left(\int_0^s L(t)\,dt\right)^{-(1/p)} ds = \infty$. The strong maximum principle in **[14]** implies that u_λ is a positive solution of (2.2), and hence it is a positive solution of (1.1).

Let

$$\lambda^* = \inf\{\lambda > 0,\ (1.1) \text{ has a positive solution}\}.$$

Then, $0 \leqslant \lambda^* \leqslant \lambda_0$ and for all $\lambda > \lambda^*$ there exists λ_1 such that $\lambda^* < \lambda_1 < \lambda$. Moreover, problem (1.1) with $\lambda = \lambda_1$ possesses at least one positive solution, denoted by u_{λ_1} such that $u_{\lambda_1} < k$ because of Lemma 2.2. Clearly u_{λ_1} is a subsolution of (1.1) and it is easy to check that the constant k in (H_2) is a supersolution of (1.1) such that $u_{\lambda_1} < k$. A standard sub- and supersolution argument (see **[3, 4]**) implies that problem (1.1) has a positive solution u_λ for all $\lambda > \lambda^*$ such that $u_{\lambda_1} \leqslant u_\lambda < k$ due to Lemma 2.2. Standard regularity theory implies that $u_\lambda \in C^1(\overline{\omega})$.

Next we shall show that problem (1.1) possesses a positive solution when $\lambda = \lambda^*$. In fact, let $\{\lambda_n\}$ be a sequence which decreases to λ^*. By the discussion above, there exists a positive solution $u_n \in C_0^1(\overline{\Omega})$ of (1.1) with $\lambda = \lambda_n$ such that $u_n \leqslant k$. The standard regularity theorey implies that $\{u_n\}$ is bounded in $C^{1,\mu}(\overline{\Omega})$, $\mu \in (0,1)$. Hence there exists a subsequence $\{u_{n_j}\} \subset \{u_n\}$ and $u \in C^1(\overline{\Omega})$ such that $u_{n_j} \to u$ in $C^1(\overline{\Omega})$ and u is a nonnegative solution of (1.1) with $\lambda = \lambda^*$. By using the same kind of arguments as in Lemma 2.3, we obtain $u > 0$ for $x \in \Omega$. The strong maximum principle implies $\lambda^* > 0$. This finishes the proof. ∎

3. The multiplicity result

In this section, we first show that the existence of local minimisers of the functional for (1.1) in $W_0^{1,p}(\Omega)$. Using this fact and the mountain pass theorem, we prove the multiplicity result for positive solutions to problem (1.1)

Lemma 3.1. *For all $\lambda > \lambda^*$, the functional $J(u)$ defined by (2.3) has a local minimiser in $W_0^{1,p}(\Omega)$, where λ^* is the same as in Theorem 1.1.*

Proof: By (H_4), $g(x,s) = \overline{f}(x,s) + L(s)$ is nondecreasing in $s \in [0,k]$ for almost all $x \in \Omega$. For any fixed $\lambda > \lambda^*$, there exists positive constant λ_1 such that $\lambda^* < \lambda_1 < \lambda$. We shall treat (1.1) in the form

$$-\Delta_p u + \lambda L(u) = \lambda g(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{3.1}$$

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Let u_{λ_1} be a positive solution of (1.1) with $\lambda = \lambda_1$. [**2**, Theorem 1.1] shows that the following problems:

$$-\Delta_p u + \lambda L(u) = \lambda_1 g(x, u_{\lambda_1}) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{3.2}$$

and

$$-\Delta_p u + \lambda L(u) = \lambda g(x, k) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{3.3}$$

have positive solutions u_1, u_2, respectively. By the monotonicity of $L(s)$, we can easily show

$$-\Delta_p u_1 + \lambda_1 L(u_1) < \lambda_1 g(x, u_{\lambda_1}) \text{ in } \Omega.$$

[**2**, Theorem 1.1] shows that $u_1 \leqslant u_{\lambda_1} < k$. By a similar argument to that used in Lemma 2.2, we obtain $u_2 < k$ in Ω. It follows from the monotonicity of $g(x, s)$ in $s \in [0, k]$ that

$$-\Delta_p u_2 + \lambda L(u_2) > \lambda g(x, u_{\lambda_1}) > -\Delta_p u_1 + \lambda L(u_1), \text{ in } \Omega \tag{3.4}$$

and $u_1 = u_2 = 0$ on $\partial\Omega$. By [**2**, Theorem 1.1], we obtain $u_1 \leqslant u_2$, $u_1 \not\equiv u_2$ in Ω. Moreover, by (H_4), $\int_0^1 \left(\int_0^s L(t)\,dt\right)^{-(1/p)} ds = \infty$, from [**14**, Theorem 5], we obtain

$$\frac{\partial u_1}{\partial n} < 0 \text{ and } \frac{\partial u_2}{\partial n} < 0 \text{ on } \partial\Omega, \tag{3.5}$$

where n is the outward normal vector on $\partial\Omega$.

Next we shall show that $u_1 < u_2$ in Ω and $(\partial(u_2 - u_1)/\partial n) < 0$ on $\partial\Omega$. Since $u_1 < k$ in Ω and $g(x, s)$ is strictly increasing in $s \in [0, k]$, there exists a constant $\nu > 0$ such that

$$\lambda g(x, k) \geqslant \lambda_1 g(x, u_1) + \nu \text{ in } \Omega.$$

Because $\dfrac{\partial u_1}{\partial n}$ and $\dfrac{\partial u_2}{\partial n}$ are continuous, it follows from (3.4) that

$$\frac{\partial u_1}{\partial n} < -\delta \text{ and } \frac{\partial u_2}{\partial n} < -\delta \text{ on } \overline{\Omega_1}, \tag{3.6}$$

where δ is a positive constant and Ω_1 is an open connected neighbourhood of $\partial\Omega$ in Ω and

$$0 < -\Delta_p u_2 + \lambda L(u_2) - \left\{-\Delta_p u_1 + \lambda L(u_1)\right\} \leqslant -\mathcal{L}(u_2 - u_1) + C(u_2 - u_1) \text{ in } \Omega_1 \tag{3.7}$$

due to (H_4) and $p > 1$, where C is a positive constant and

$$\mathcal{L} = \sum_{i,j} \left[a_{ij}(x)\left(\frac{\partial}{\partial x_j}\right)\right]_{x_i},$$
$$a_{ij} = \int_0^1 \left[a_i\big(tDu_2 + (1-t)\big)Du_1\right]_{y_j} dt,$$
$$a_i(y) = |y|^{p-2} y_i \ (i = 1, 2, \ldots, N) = (y_1, y_2, \ldots, y_N) \in R^N.$$

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It is clear that $\mathcal{L}$ is a uniformly elliptic operator in Ω_1 because of (3.6). Suppose that there exists an $x_0 \in \Omega$ such that $u_1(x_0) = u_2(x_0)$. Then there exists an $x_1 \in \Omega_1$ such that $u_1(x_1) = u_2(x_1)$. Indeed, choose a subdomain Ω_2 of Ω with smooth boundary such that

$$\overline{\Omega}_2 \subset \Omega, \partial\Omega_2 \subset \Omega_1 \text{ and } x_0 \in \Omega_2.$$

Then we shall show there exists a point $x_1 \in \partial\Omega_2$ such that $u_1(x_1) = u_2(x_1)$. Suppose not. By continuity, we obtain $u_2 - u_1 \geqslant \tau$ on $\partial\Omega_2$, where the positive constant $\tau \leqslant 1$ will be specified below. Obviously, by condition (H_4) $v = u_1 + \tau$ satisfies

$$-\Delta_p v + \lambda L(v) = \lambda_1 g(x, u_1) + \lambda\big(L(v) - L(u_1)\big) \leqslant \lambda_1 g(x, u_1) + M\tau, \quad x \in \Omega_2$$

for some constant $M > 0$. Let τ be small enough so that $M\tau < \nu$. Then

$$-\Delta_p v + \lambda L(v) \leqslant -\Delta_p u_2 + \lambda L(u_2) \text{ in } \Omega_2, \quad u_2 \geqslant v \text{ on } \partial\Omega_2,$$

and [**2**, Theorem 1.1] shows that $u_2 \geqslant v$ in Ω_2. This contradicts $u_2(x_0) = u_1(x_0)$. Since $\mathcal{L}$ is uniformly elliptic in Ω_1 and $u_2(x_1) = u_1(x_1)$, $x_1 \in \Omega_1$, it follows from identity (3.7) and [**14**, Theorem 5] that $u_1 \equiv u_2$ in Ω_1. This contradicts (3.7). Thus, we obtain $u_2 > u_1$ in Ω. On the other hand, from (3.7) we have

$$-\mathcal{L}(u_2 - u_1) + C(u_2 - u_1) > 0 \text{ in } \Omega_1 \text{ with } u_2 = u_1 = 0 \text{ on } \partial\Omega.$$

By [**14**, Theorem 5], we obtain $(\partial(u_2 - u_1)/\partial n) < 0$ on $\partial\Omega$.

Now we define

$$g_1(x, s) := \begin{cases} g(x, u_1), & s \leqslant u_1; \\ g(x, s), & s \in (u_1, u_2); \\ g(x, u_2), & s \geqslant u_2, \end{cases}$$
$$G(x, u) = \int_0^u g_1(x, s)\, ds,$$

and

$$J_1 = \frac{1}{p}\int_\Omega |Du|^p\, dx + \lambda \int_\Omega L(u)\, dx - \lambda \int_\Omega G(x, u)\, dx.$$

Using a similar argument to the proof Theorem 1.1, we can prove that the functional $J_1(u)$ achieves a minimiser $u_\lambda \in W_0^{1,p}(\Omega)$. Standard regularity theory implies $u_\lambda \in C^1(\overline{\Omega})$ and satisfies

$$-\Delta_p u_\lambda + \lambda L(u_\lambda) = \lambda g_1(x, u_\lambda) \text{ in } \Omega, \quad u_\lambda = 0 \text{ on } \partial\Omega.$$

It is obvious that

$$g(x, u_1) \leqslant g_1(x, u_\lambda) \leqslant g(x, u_2) \text{ in } \Omega.$$

The same arguments used above, show

$$u_1 < u_\lambda < u_2 \text{ in } \Omega, \tag{3.8}$$

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and

$$\frac{\partial(u_2 - u_\lambda)}{\partial n} < 0, \quad \frac{\partial(u_\lambda - u_1)}{\partial n} < 0. \tag{3.9}$$

Hence $g_1(x, u_\lambda) = g(x, u_\lambda)$ and u_λ is a positive solution of (1.1). It follows from (3.8)–(3.9) that if

$$\|v - u_\lambda\|_{C^1(\overline{\Omega})} = \theta$$

with θ small, then $u_1 \leqslant v \leqslant u_2$. Moreover $J(v) - J_1(v)$ is constant for $u_1 \leqslant v \leqslant u_2$ and therefore u_λ is also a local minimiser of J in $C^1(\overline{\Omega})$ such that $u_\lambda > 0$ in Ω. By [**1**, Theorem 1.1], we conclude that u_λ is a local minimiser of J in $W_0^{1,p}(\Omega)$. The proof is complete. ∎

Now we are ready to prove the multiplicity result, Theorem 1.2.

PROOF OF THEOREM 1.2: We still treat problem (1.1) in the form (2.2), with the functional $J(u)$ on $W_0^{1,p}(\Omega)$ for (2.2) defined by (2.3):

$$J(u) = \frac{1}{p}\int_\Omega |Du|^p\,dx - \lambda\int_\Omega \overline{F}(x,u)\,dx,$$

where $\overline{F}(x,u) = \displaystyle\int_0^u \overline{f}\big(x, u(s)\big)\,ds$. Now for any fixed $\lambda > \lambda^*$, we shall use the Mountain Pass Theorem in [**8**, **9**] to find another solution $\overline{u}_\lambda \not\equiv u_\lambda$, where u_λ is a local minimiser for J obtained in Lemma 3.1. We now check the conditions of the Mountain Pass Theorem. Take any fixed $q \in (p, p^*)$, where $p^* := Np/(N-p)$ if $p < N$; $p^* := +\infty$ if $p \geqslant N$, and fix it. By the construction of $\overline{f}$ and condition $(H_1) - (H_2)$, for any $\delta > 0$ there exists $C_\delta > 0$ such that $\left|\overline{f}(x,s)\right| \leqslant \delta s^{p-1} + C_\delta s^{q-1}$. Thus, J is of class C^1. Next, it is easy to see that the functional J satisfies the Palais-Smale condition. Indeed, let $\{u_n\}$ be any sequence in $W_0^{1,p}(\Omega)$ such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$. Then, it follows from the boundedness of $\overline{F}$ that $\left\{\|Du_n\|_{L^p(\Omega)}\right\}$ is bounded; that is $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, [**5**, Lemma 2.1] yields the assertion. In addition, the Sobolev inequality assures there exist positive constants γ and ρ such that $\rho < \|u_\lambda\|_{W_0^{1,p}(\Omega)}$ and $J(u) \geqslant \gamma$ if $\|Du\|_{L^p(\Omega)} = \rho$, because

$$\begin{aligned} J(u) &\geqslant \frac{1}{p}\int_\Omega |Du|^p\,dx - \frac{\lambda}{p}\int_\Omega |u|^p\,dx - \frac{\lambda C_\delta}{q}\int_\Omega |u|^q\,dx \\ &\geqslant \Big(\frac{1-\lambda C_1\delta}{p} - \frac{\lambda C_2 C_\delta}{q}\|u\|^{q-p}_{W_0^{1,p}(\Omega)}\Big)\|u\|^p_{W_0^{1,p}(\Omega)} \geqslant \gamma, \end{aligned}$$

where C_1 and C_2 are positive constants, provided that $\delta \in \Big(0, \big(1/(\lambda C_1)\big)\Big)$ and ρ is sufficiently small. If $J(u_\lambda) \leqslant 0$, then since $J(0) = 0$, the Mountain Pass Theorem in [**9**], shows problem (2.2) possesses a nontrivial solution $\overline{u}_\lambda$ such that $J(\overline{u}_\lambda) > 0$, and which is distinct from u_λ. If $J(u_\lambda) > 0$, there exists $\delta > 0$ such that if $v \in B_\delta(u_\lambda)$, then $J(v) \geqslant J(u_\lambda)$, where

$$B_\delta(u_\lambda) := \left\{v \in W_0^{1,p}(\Omega) \;\middle|\; \|v - u_\lambda\|_{W_0^{1,p}(\Omega)} \leqslant \delta\right\}.$$

Hence

$$J|_{\partial B_\delta(u_\lambda)} \geqslant \max\{J(0), J(u_\lambda)\}.$$

Then by the extensions of the Mountain Pass Lemma in [7], problem (2.2) has a nontrivial solution which is distinct from u_λ. Therefore, for any fixed $\lambda > \lambda^*$, problem (2.2) possesses at least two nontrivial nonnegative distinct solutions. The standard regularity theory and the maximum principle imply that these two nontrivial nonnegative solutions are positive and belong to $C^1(\overline{\Omega})$. Hence for all $\lambda > \lambda^*$, problem (1.1) possesses two distinct positive solutions. This finishes the proof. ∎

References

[1] J.P Garcia Azorero, J.J. Manfredi and I. Peral Alonso, Sobolev versus Hölder local minimisers and global multiplicity for some quasilinear elliptic equations, *Comm.um Contemp. Math.* **2** (2000), 385–404.

[2] J. Diaz, *Nonlinear partial differential equation and free boundary problems, Elliptic equations Vol.1*, Pitman Research Notes in Math. **106** (Pitman, Boston, M.A., 1985).

[3] Y. Du and Z.M. Guo, 'Liouville type results and eventual flatness of positive solutions for p-Laplacian equations', *Adv. Differential Equations* **7** (2002), 1479–1512.

[4] Y. Du and Z.M. Guo, 'Boundary blow-up solutions and their applications in quasilinear elliptic equations', *J. Analyse Math.* **89** (2003), 277–302.

[5] G. Dinca, P. Jebelean and J. Mawhin, 'A result of Ambrosetti-Rabinowitz type for p-Laplacian', in *Qualitative problems for differential equations and control theory* (World Sci. Punlishing, Rever Edge, 1995), pp. 231–242.

[6] Z.M. Guo, 'Some existence and multiplicity results for a class of quasilinear elliptic eigenvalue problem', *Nonlinear Anal.* **18** (1992), 957–971.

[7] D. Guo, J. Sun and G. Qi, 'Some extensions of the mountain pass lemma', *Diffential Integral Equations* **1** (1988), 351–358.

[8] P.H. Rabinowitz, 'Pairs of positive solutions of nonlinear elliptic partial differential equations', *Indiana Univ. Math. J.* **23** (1973), 173–186.

[9] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math. **65** (Amer. Math. Soc., Province, R.I., 1986).

[10] S. Takeuchi, 'Positive solutions of a degenerate elliptic equation with logistic reaction', *Proc. Amer. Math. Soc.* **129** (2000), 433–441.

[11] J. Serrin and H. Zou, 'Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities', *Acta. Math.* **189** (2002), 79–142.

[12] P. Tolksdorf, 'On the Dirichlet problem for quasilinear equations in domains with conial boundary points,', *Comm. Partial Differential Equations* **8** (1983), 773–817.

[13] P. Tolksdorf, 'Regularity for more general class of quasilinear elliptic equations', *J. Differential Equations* **51** (1984), 126–150.

[14] J.L. Vazquez, 'A strong maximum principle for some quasilinear elliptic equations', *Appl. Math. Optim.* **12** (1984), 191–202.

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