# NEW APPROACH TO THE PLANETARY THEORY

Manabu Yuasa and Gen'ichiro Hori
Department of Astronomy, University of Tokyo, Bunkyo-ku,
Tokyo 113, Japan

## 1. INTRODUCTION

A new approach to the planetary theory is examined under the following procedure: 1) we use a canonical perturbation method based on the averaging principle; 2) we adopt Charlier's canonical relative coordinates fixed to the Sun, and the equations of motion of planets can be written in the canonical form; 3) we adopt some devices concerning the development of the disturbing function. Our development can be applied formally in the case of nearly intersecting orbits as the Neptune-Pluto system. Procedure 1) has been adopted by Message (1976).

## 2. CANONICAL RELATIVE COORDINATES FIXED TO THE SUN

We consider n+1 celestial bodies. Let their masses be $m_i (i=0,\dots,n)$ and their coordinates referred to the center of mass be $\vec{\rho}_i (i=0,\dots,n)$. Then the Hamiltonian F of this system can be written as

$$F = -T + V = -\frac{1}{2}\sum_{i=0}^{n} m_i \dot{\vec{\rho}}_i^2 + \sum_{i>j\geq 0} \frac{k^2 m_i m_j}{\rho_{ij}} \quad , \tag{1}$$

where T, V, $k^2$, and $\rho_{ij}$ represent the kinetic energy, the potential energy, the gravitational constant of Gauss, and $|\vec{\rho}_i - \vec{\rho}_j|$ respectively. We regard $m_0$ as the Sun and $m_i (i=1,\dots,n)$ as the planets. The relative coordinates $\vec{r}_i (i=0,\dots,n)$ fixed to the Sun are introduced by putting $\vec{r}_i = \vec{\rho}_i - \vec{\rho}_0 (i=0,\dots,n)$. Next, we introduce the momenta $\vec{p}_i (i=1,\dots,n)$ which are conjugate to the coordinates $\vec{r}_i (i=1,\dots,n)$ as follows (Charlier 1902):

$$\vec{p}_i = \frac{\partial T}{\partial \dot{\vec{r}}_i} = \frac{\partial}{\partial \dot{\vec{r}}_i} \frac{1}{2} \sum_{i=0}^{n} m_i \dot{\vec{\rho}}_i^{\,2} = m_i \dot{\vec{\rho}}_i \quad . \tag{2}$$

Then the Hamiltonian of the system is given by



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$$F = \sum_{i=1}^{n} \{-\frac{1}{2}\frac{\vec{p}_i{}^2}{m_i{}'} + \frac{\mu_i m_i{}'}{r_{i0}}\} + \sum_{i>j\geq 1} (-\frac{\vec{p}_i \cdot \vec{p}_j}{m_0} + \frac{k^2 m_i m_j}{r_{ij}}) \quad , \tag{3}$$

where $m_i{}'=m_0 m_i/(m_0+m_i)$, $\mu_i=k^2(m_0+m_i)$, and $r_{ij}=|\vec{r}_i-\vec{r}_j|$.

Let the quantities $a_i$, $e_i$, $I_i$, $\ell_i$, $\omega_i$, and $\Omega_i$ be the semi-major axis, the eccentricity, the inclination, the mean anomaly, the argument of perihelion, and the longitude of the node of the motion of the i-th planet around the Sun. Then the canonical variables $L_i$, $G_i$, $H_i$, $\ell_i$, $g_i$, and $h_i$ can be defined as

$$\begin{aligned} L_i &= m_i{}'\sqrt{\mu_i a_i} \quad , \quad G_i = L_i\sqrt{1-e_i{}^2} \quad , \quad H_i = G_i \cos I_i \\ \ell_i &= \text{mean anomaly}, \quad g_i = \omega_i \quad , \quad h_i = \Omega_i \quad . \end{aligned} \tag{4}$$

The equations of motion are

$$\frac{d(L_i,G_i,H_i)}{dt} = \frac{\partial F}{\partial(\ell_i,g_i,h_i)} \quad , \quad \frac{d(\ell_i,g_i,h_i)}{dt} = -\frac{\partial F}{\partial(L_i,G_i,H_i)} \quad , \tag{5}$$

with

$$F = F_0 + F_1 \quad , \quad F_0 = \sum_{i=1}^{n} \frac{\mu_i{}^2 m_i{}'^3}{2L_i{}^2} \quad , \quad F_1 = \sum_{i>j\geq 1} (-\frac{\vec{p}_i \cdot \vec{p}_j}{m_0} + \frac{k^2 m_i m_j}{r_{ij}}) \; . \tag{6}$$

The function $F_1$ is the disturbing function and to be represented by $L_i$, $G_i$, $H_i$, $\ell_i$, $g_i$, and $h_i$.

## 3. DEVELOPMENT OF THE DISTURBING FUNCTION IN TERMS OF THE INCLINATIONS

We consider only two planets $m_1$ and $m_2$. If $v_1$ and $v_2$ are the true longitudes of the two planets, the mutual distance $r_{12}$ is given by

$$\begin{aligned} r_{12}^2 = r_1^2+r_2^2-2r_1r_2[&c_1^2c_2^2\cos(v_1-v_2)+c_1^2s_2^2\cos(v_1+v_2-2\Omega_2)+s_1^2c_2^2\cos(v_1+v_2-2\Omega_1) \\ &+s_1^2s_2^2\cos(v_1-v_2-2\Omega_1+2\Omega_2)+2c_1s_1c_2s_2\{\cos(v_1-v_2-\Omega_1+\Omega_2) \\ &-\cos(v_1+v_2-\Omega_1-\Omega_2)\}] \quad , \end{aligned} \tag{7}$$

where $c_i=\cos(I_i/2)$, $s_i=\sin(I_i/2)$, $(i=1,2)$. At this stage we define

$$q \equiv (r_1^2+r_2^2)/2r_1r_2(c_1c_2-s_1s_2)^2 \quad , \tag{8}$$

and the inverse of the mutual distance is expressed as

$$\frac{1}{r_{12}} = \frac{1}{\sqrt{2r_1r_2}(c_1c_2-s_1s_2)} [q - \cos(v_1-v_2) - \frac{\delta}{(c_1c_2-s_1s_2)^2}]^{-1/2} \quad , \tag{9}$$

with

$$\delta = s_1^2c_2^2\cos(v_1+v_2-2\Omega_1)+c_1^2s_2^2\cos(v_1+v_2-2\Omega_2)+s_1^2s_2^2\cos(v_1-v_2-2\Omega_1+2\Omega_2)$$
$$+ 2c_1s_1c_2s_2\{\cos(v_1-v_2-\Omega_1+\Omega_2)-\cos(v_1+v_2-\Omega_1-\Omega_2)\} \quad (10)$$
$$+ (2c_1s_1c_2s_2-s_1^2s_2^2)\cos(v_1-v_2) \quad .$$

By the binomial expansion of the equation (9), $1/r_{12}$ is written in the form

$$\frac{1}{r_{12}} = \frac{1}{\sqrt{2r_1r_2}(c_1c_2-s_1s_2)} \sum_{n=0}^{\infty}(-1)^n\binom{-1/2}{n}\left[\frac{\delta}{(c_1c_2-s_1s_2)^2}\right]^n \times$$
$$\times\ [q-\cos(v_1-v_2)]^{-(n+1/2)} \quad . \quad (11)$$

Furthermore, we expand $[q-\cos(v_1-v_2)]^{-(n+1/2)}$ by the 2-nd kind associated Legendre function $Q_\mu^\nu$. And we get

$$\frac{1}{r_{12}} = \sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty}\frac{2^n}{n!}(c_1c_2-s_1s_2)^{-2n}\delta^n\beta_{n+1/2}^{(j)}(q)\cos j(v_1-v_2) \quad , \quad (12)$$

where

$$\beta_{n+1/2}^{(j)} = \frac{(-1)^n}{2^n\pi}\frac{(q^2-1)^{-n/2}}{\sqrt{r_1r_2}(c_1c_2-s_1s_2)}Q_{j-1/2}^n(q) \quad . \quad (13)$$

These expansions converge regardless of the values of $r_1$ and $r_2$ except for the following two cases: 1) the case when two planets collide; 2) the case when $\Omega_1-\Omega_2=\pi$, $v_1=v_2$, $v_1+v_2-\Omega_1-\Omega_2=0$, and $r_1=r_2$. Consequently, above development can be applied formally even in the case of nearly intersecting orbits as the Neptune-Pluto system.

## 4. DEVELOPMENT OF THE DISTURBING FUNCTION IN TERMS OF THE ECCENTRICITIES

We use Newcomb's operator and $r_1$, $r_2$, $v_1$, $v_2$ can be expressed in terms of $a_1$, $a_2$, $e_1$, $e_2$, $\lambda_1$, $\lambda_2$, $\ell_1$, $\ell_2$, where $\lambda_1$, $\lambda_2$ are the mean longitudes. For the simplicity of notations, we put

$$\frac{2^n}{n!}(c_1c_2-s_1s_2)^{-2n}\delta^n\cos j(v_1-v_2)= \sum_y C_{n,y}(I_1,I_2)\cos[j(v_1-v_2)+y_1v_1$$
$$+y_2v_2+y_3\Omega_1+y_4\Omega_2] \quad , \quad (14)$$

where the summation is taken in all the combinations of $y_1,..,y_4$ appeared. Then the inverse of the mutual distance can be expanded as follows:

$$\frac{1}{r_{12}} = \sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty}\sum_y\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{s_1=|k_1|+0,2,..}\sum_{s_2=|k_2|+0,2,..}C_{n,y}(I_1,I_2)\times$$
$$\times\ \Pi_{k_1}^{s_1}(D_1|j+y_1)\Pi_{k_2}^{s_2}(D_2|-j+y_2)e_1^{s_1}e_2^{s_2}\beta_{n+1/2}^{(j)}(q_0)\times \quad (15)$$

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$$\times \cos[j(\lambda_1-\lambda_2)+y_1\lambda_1+y_2\lambda_2+y_3\Omega_1+y_4\Omega_2+k_1\ell_1+k_2\ell_2] \quad ,$$

where $D_1=a_1\cdot\partial/\partial a_1$, $D_2=a_2\cdot\partial/\partial a_2$, $q_0=(a_1^2+a_2^2)/2a_1a_2(c_1c_2-s_1s_2)^2$, and $\Pi_{k_1}^{s_1}(D_1|j+y_1)$, $\Pi_{k_2}^{s_2}(D_2|-j+y_2)$ are Newcomb's operators.

## 5. EVALUATIONS OF $\beta_{n+1/2}^{(j)}(q_0)$

From the equations (11) and (12) we get

$$\frac{(2n-1)!!}{2^{2n}\sqrt{2a_1a_2}(c_1c_2-s_1s_2)}[q_0-\cos(v_1-v_2)]^{-(n+1/2)}$$
$$= \beta_{n+1/2}^{(0)} + 2\sum_{j=1}^{\infty}\beta_{n+1/2}^{(j)}\cos j(v_1-v_2) \quad , \tag{16}$$

and we can determine the values of $\beta_{n+1/2}^{(j)}$ by the numerical Fourier analysis if $a_1$, $a_2$, $c_1$, $c_2$, $s_1$, $s_2$ are given. On the other hand, the equation (13) and the recurrence formulas of $Q_\mu^\nu$ give rise to the recurrence formulas of $\beta$, $D_1{}^\nu\beta$, and $D_2{}^\nu\beta$. These recurrence formulas are of much help for the evaluation of $\beta$.

The practical development of the disturbing function has been performed to the fourth order of the eccentricity and the inclination. As an application, we are trying to study the Neptune-Pluto system by a canonical perturbation method.

## REFERENCES

Charlier, C. L. : 1902, "Die Mechanik des Himmels", Erster Band, pp. 234-237.

Message, P. J. : 1976, in "Long-Time Predictions in Dynamics", ed. V. Szebehely and B. D. Tapley (D. Reidel Publishing Company), pp. 279-293.