Canad. J. Math. Vol. **55** (3), 2003 pp. 576–608

# Automorphic Orthogonal and Extremal Polynomials

A. L. Lukashov and F. Peherstorfer

*Abstract.* It is well known that many polynomials which solve extremal problems on a single interval as the Chebyshev or the Bernstein-Szegö polynomials can be represented by trigonometric functions and their inverses. On two intervals one has elliptic instead of trigonometric functions. In this paper we show that the counterparts of the Chebyshev and Bernstein-Szegö polynomials for several intervals can be represented with the help of automorphic functions, so-called Schottky-Burnside functions. Based on this representation and using the Schottky-Burnside automorphic functions as a tool several extremal properties of such polynomials as orthogonality properties, extremal properties with respect to the maximum norm, behaviour of zeros and recurrence coefficients *etc.* are derived.

## 1 Introduction and Notation

Let $l \in \mathbb{N}$, $a_k \in \mathbb{R}$ for $k = 1, \dots, 2l$, $a_1 < a_2 < \cdots < a_{2l}$, and put

$$E = \bigcup_{k=1}^{l} [a_{2k-1}, a_{2k}],\ H(x) = \prod_{k=1}^{2l} (x - a_k),$$

and set

$$\frac{1}{h(x)} = \begin{cases} \frac{1}{\pi} \operatorname{sgn}\left(\prod_{k=1}^{l} (x - a_{2k-1})\right) / \sqrt{-H(x)} & \text{for } x \in E, \\ 0 & \text{elsewhere}. \end{cases}$$

The symbols $R$ and $S$ denote monic polynomials of degree $r$, respectively, $s$, that satisfy the relation

$$R(x)S(x) = H(x)$$

and $\rho_\nu$ denotes a real polynomial of degree $\nu$ that has no zero in $E$, *i.e.*,

$$\rho_\nu(x) = c \prod_{k=1}^{\nu^*} (x - w_k)^{\nu_k}$$

where $c \in \mathbb{R} \setminus \{0\}$, $\nu^* \in \mathbb{N}_0$, $\nu_k \in \mathbb{N}$ for $k = 1, \dots, \nu^*$, $\nu = \sum_{k=1}^{\nu^*} \nu_k$, $w_k \in \mathbb{C} \setminus E$, for $k = 1, \dots, \nu^*$, and the $w_k$'s are real or appear in pairs of conjugate complex numbers.

Received by the editors December 7, 2001; revised October 10, 2002.

Research of the first author was supported by the Austrian Science Fund FWF, project-number P12985-TEC, as well as by the J. Kepler University Linz and also in part by the Russian Foundation for Basic Research under grants No. 99-01-01120 and No. 00-15-96123 and by the Ministry of Education of Russian Federation, grant E00-1.0-192.

AMS subject classification: Primary: 42C05; secondary: 30F35, 31A15, 41A21, 41A50.

Furthermore we set

$$\rho_{\nu,k}(x) = \rho_\nu(x)/(x - w_k)^{\nu_k} \quad \text{for } k = 1, \dots, \nu^*,$$

and

$$\Xi_{\nu^*} = \{(\varepsilon_1, \dots, \varepsilon_{\nu^*}) : \varepsilon_k \in \{-1, +1\} \text{ for } k = 1, \dots, \nu^*\}.$$

For given $R$, $\rho_\nu$, and $\varepsilon \in \Xi_{\nu^*}$ we define the following linear functionals on the space of real polynomials $\mathbb{P}$:

$$L_{R,\rho_\nu,\varepsilon}(p) = \sum_{k=1}^{\nu^*} \frac{1 - \varepsilon_k}{(\nu_k - 1)!} \Big( \frac{pR}{\rho_{\nu,k}} \sqrt{H} \Big)^{(\nu_k - 1)}(w_k) \quad \text{for } p \in \mathbb{P}, \tag{1}$$

$$\Psi_{R,\rho_\nu,\varepsilon}(p) = \int p \frac{R}{\rho_\nu h}\, dx + L_{R,\rho_\nu,\varepsilon}(p) \quad \text{for } p \in \mathbb{P} \tag{2}$$

where we make the additional assumption that $\varepsilon_{k+1} = \varepsilon_k$ if $w_k$ and $w_{k+1}$ are complex conjugate and where that branch of $\sqrt{H}$ is chosen that is analytic on $\mathbb{C} \setminus E$ and satisfies

$$\operatorname{sgn} \sqrt{H(y)} = \operatorname{sgn} \prod_{k=1}^{l} (y - a_{2k-1}) \quad \text{for } y \in \mathbb{R} \setminus E.$$

We use $g^{(j)}$ to denote the $j$th derivative of $g$. If there is no confusion possible, we omit the index $\nu$ and we write $\Psi_{\rho,\varepsilon}$ instead of $\Psi_{1,\rho,\varepsilon}$.

In this paper we give, in terms of automorphic Schottky-Burnside functions, an explicit representation of the polynomials $p_n = x^n + \cdots$, $n \in \mathbb{N}$, which are orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$, that is, which satisfy

$$\Psi_{R,\rho,\varepsilon}(x^j p_n) = 0 \quad \text{for } j = 0, \dots, \tilde{n},$$

where $\tilde{n} \geq n - 1$. Let us note that $\tilde{n} > n - 1$ is possible since the linear functional need not be definite. It is known (see, for example, [34]) that for given $R$, $\rho_\nu$, $\varepsilon$ there exist a unique sequence of so-called basic integers $(i_n)$, with $i_0 := 0 < i_1 < i_2 < \cdots$, and a unique sequence of polynomials $p_{i_n}$ with $p_{i_n} = x^{i_n} + \cdots$, $n \in \mathbb{N}_0$, such that

$$\Psi_{R,\rho,\varepsilon}(x^j p_{i_n}) = 0, \quad j = 0, \dots, i_{n+1} - 2, \tag{3}$$

and

$$\Psi_{R,\rho,\varepsilon}(x^{i_{n+1}-1} p_{i_n}) \neq 0,$$

and the $p_{i_n}$ satisfy a recurrence relation of the type

$$p_{i_n}(x) = d_{i_n}(x) p_{i_{n+1}}(x) - \lambda_{i_n} p_{i_{n-1}}(x) \quad \text{for } n \in \mathbb{N}, \tag{4}$$

where $\lambda_{i_n} \in \mathbb{R} \setminus \{0\}$, $d_{i_n} \in \mathbb{P}_{i_n - i_{n-1}}$ ($\mathbb{P}_n$ denotes the set of polynomials of degree at most $n$), $p_{i_0} = 1$ and $p_{i_{-1}} = 0$. From (4) it follows immediately that

$$\lambda_{i_{n+1}} = \Psi_{R,\rho,\varepsilon}(x^{i_{n+1}-1} p_{i_n}) / \Psi_{R,\rho,\varepsilon}(x^{i_n - 1} p_{i_{n-1}}). \tag{5}$$

Furthermore the sequence of basic integers with respect to $\Psi_{R,\rho,\varepsilon}$ satisfies

$$i_n + 1 \le i_{n+1} \le i_n + l \tag{6}$$

(which is more exact than the estimate before relation (1.3) in [34, p. 462]). This follows from the relation following relation (4.10) in [34], because otherwise the expression on the left-hand side there cannot vanish at all zeros of $\rho_\nu$, which is a contradiction.

Polynomials orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$ have been characterised by the second author in [34, 35], see also [32, 33]. Note if $\varepsilon_k = 1$ for $k = 1, \dots, \nu^*$, that is, $L_{R,\rho,\varepsilon} \equiv 0$, $R$ and $\rho_\nu$ are such that $R/h\rho_\nu > 0$ on $\operatorname{inte}(E)$, and thus $i_n = n$ for $n \in \mathbb{N}$, then the orthogonality condition (3) becomes

$$\int_E x^j p_n(x) \sqrt{\left|\frac{R(x)}{S(x)}\right|} \frac{dx}{|\rho_\nu(x)|} = 0 \quad \text{for } j = 0, \dots, n-1. \tag{7}$$

The orthogonal polynomials from (7) are sometimes called *Akhieser polynomials*, since they have been studied in the sixties in ([4, 5, 51]) for the case $\partial R \equiv l - 1$. Let us note that already Abel [1], Jacobi [23], Halphen [21] and I. L. Ptashitzkiĭ [42] studied certain types of continued fraction expansions of the functions $\sqrt{H(z)}$, $\left(\sqrt{H(z)} - \sqrt{H(y)}\right)/(z - y)$, which are closely related to the functions $\Psi_{H,\rho,1}\left(\frac{1}{z-x}\right)$.

Polynomials orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$ are important from different points of view. Taking a look at the orthogonality condition (7) we see that they can be considered as the counterparts of the so-called Bernstein-Szegö polynomials which are orthogonal on $[-1, 1]$ with respect to weight functions of the form $(1 - x)^\alpha \cdot (1 + x)^\beta/\rho(x)$, where $\alpha, \beta \in \{-1/2, 1/2\}$ and $\rho \in \mathbb{P}$ is positive on $[-1, 1]$. Let us recall that the whole theory of Szegö and Bernstein (see [10, 50]) concerning the asymptotic behaviour of orthogonal polynomials whose support consists of one interval is based on the Bernstein-Szegö polynomials.

But Bernstein-Szegö polynomials have also the remarkable property, due to Markov and Bernstein [3, 12, 43], that they are minimal polynomials on $[-1, 1]$ with respect to the maximum-norm and weight function $1/\sqrt{1 - x^2}\rho$, where $\rho \in \mathbb{P}$ is positive on $[-1, 1]$. The minimal polynomials are also called *Chebyshev-Markov polynomials*.

Now, in the case of several intervals, the polynomials orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$ share many properties with the Bernstein-Szegö polynomials; see, for instance, [35], where an overview is given also. However things become more involved; for instance the minimal property with respect to the max-norm holds under additional conditions only, see [35] and below. More precisely, if the polynomial $p_n$, orthogonal with respect to $1/\rho h$, with $\rho > 0$ on $E$, has maximum orthogonality, that is, is orthogonal to $\mathbb{P}_{n+l-1}$ and not only to $\mathbb{P}_{n-1}$, then $p_n$ is also that monic polynomial for which $p_n/\sqrt{\rho}$ has minimal max-norm on $E$ among all functions of the form $(x^n + a_{n-1}x^{n-1} + \cdots + a_0)/\sqrt{\rho}$, and in addition $p_n/\sqrt{\rho}$ has maximal number of extremal points on $E$, namely $n + l$. As usual a point $y \in E$ is called an *extremal point* of $f \in C(E)$ on $E$ if $|f(y)| = \|f\|_{C(E)}$. The converse statement is valid too. The corresponding minimal polynomials with the maximal number of extremal points were investigated by the authors in [30, 31] and [34, 35, 40] respectively. Related results and applications may be found in [6, 11, 18, 19, 24, 26, 32, 33, 37, 41, 47, 51, 53, 54] and in surveys [25, 27, 29, 36, 39, 46].

In this paper we use so called *automorphic Schottky-Burnside functions*. Probably, N. I. Akhieser was the first one who used such functions for the description of minimal polynomials with respect to the max-norm, that is, for the description of Zolotarev polynomials with three given leading coefficients [2]. Recently such automorphic functions have been used by the first author ([30, 31]) for the description of the minimal polynomials with respect to the max-norm and weight function $1/\sqrt{\rho}$ just mentioned above.

Note also that automorphic functions were used for the representation and qualitative analysis of the behaviour of the solutions of integrable equations for small gaps and small intervals [9, p. 174–177], respectively. Here we give not only the representations of orthogonal polynomials in terms of automorphic functions but use them also heavily as a tool for proving the location of the zeros of the orthogonal polynomials and limit properties of the recurrence coefficients. Furthermore we show how notions from potential theory as Green's functions and capacity, can be expressed with the help of Schottky-Burnside functions.

In the case of two intervals Schottky-Burnside functions degenerate into elliptic functions. From this point of view several statements of this paper about the polynomials orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$ can be considered as counterparts of results given in [38] by the second author with the help of elliptic functions.

This paper is organized as follows. First we derive an explicit representation of polynomials orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$ in terms of Schottky-Burnside automorphic functions. This enables us to determine the exact number of zeros of the orthogonal polynomials in each interval $E_j = [a_{2j-1}, a_{2j}]$, $j = 1, \dots, l$. As another consequence we obtain a criterion on the zeros of an arbitrary polynomial $\rho$, $\rho > 0$ on $\operatorname{int}(E)$ such that the polynomial orthogonal with respect to $\Psi_{\rho,\varepsilon}$ is a minimal polynomial with respect to the max-norm and weight function $1/\sqrt{\rho}$. In the third section we study the question when there are zeros in the gaps $[a_{2j}, a_{2j+1}]$, $j \in \{1, \dots, l-1\}$. We prove that for a given $\rho$ with $1/h\rho > 0$ on $\operatorname{int}(E)$ and arbitrary given $m \in \{0, \dots, l-1\}$, there is a subsequence $(n_k)$ of the natural numbers such that each polynomial $p_{n_k}$, $k = 1, 2, \dots$, orthogonal with respect to $\Psi_{\rho,\varepsilon}$ has $m$ zeros in $[a_1, a_{2l}] \setminus E_l$ which do not accumulate to $E$ if the harmonic measures of the intervals $E_j$ at $z = \infty$ are independent over the rationals. In this connection let us mention a recent related result by S. P. Suetin [49], who has shown that under the same assumption there exists a subsequence $(n_k)$ such that all zeros of $(p_{n_k})$ accumulate to $E$. Finally, we give explicit expressions for the recurrence coefficients and prove that they have an innumerable set of limit points.

Let us remember first the main points of the Schottky-Burnside automorphic functions theory. There are expositions of that theory in [7, Chapter 14], [9, Chapter 5], [11]. In [7] they are called Schottky functions and in [8] Burnside functions. Because of the many important contributions by W. Burnside [13, 14] to that theory, we call them *Schottky-Burnside functions*.

Denote by $G(K_1, \dots, K_{l-1}) \subset \mathbb{C}$ any domain which is the upper half of the complex plane without disjoint circles $K_1, \dots, K_{l-1}$, lying inside it with its centers on the imaginary axis. The domain $G(K_1,\dots,K_{l-1})$ together with the domain symmetric to it with respect to the real axis and together with the real axis and $\bigcup_{j=1}^{l-1}\partial K_j$ is called the *fundamental domain* $\mathfrak{T}$ of a Schottky group $\Gamma$, see [17]. The generators of the group $\Gamma$ are the Möbius maps $T_i(z) = \big(R_i^2/(z-\bar{o}_i)\big) + o_i$, $i = 1,\dots,l-1$, where $o_i$ denotes the center and $R_i$ the radius of the circle $K_i$, $i = 1,\dots,l-1$. The group $\Gamma$ consists of the mappings $\Gamma = \{T_i\}_{i=0}^{\infty}$, $T_0(z) \equiv z$. Recall that a function $f$ is called *automorphic* if it is a single-valued meromorphic function on the complex sphere $\bar{\mathbb{C}}$ without the singular points of the group $\Gamma$ and such that for any $T \in \Gamma$ the identity $f\big(T(z)\big) = f(z)$ holds for $z \in \mathfrak{T}$.

Now we introduce the following W. Burnside's functions :

$$\Omega(z,y) = (z-y)\prod_i{}' \frac{\big(T_i(z)-y\big)\big(T_i(y)-z\big)}{\big(T_i(z)-z\big)\big(T_i(y)-y\big)}, \quad [14,\text{ Section } 2], \tag{8}$$

$$\exp\Phi_i(z) = \frac{z-c_i}{z-c_{i^{-1}}}\prod_{\substack{j=1\\ j\neq i}}^{\infty}\frac{z-c_{j^{-1}i}}{z-c_{j^{-1}}}, \quad [13,\text{ Section } 4]. \tag{9}$$

Here and everywhere later, $c_j = T_j^{-1}(\infty)$, $c_{i^{-1}} = T_i(\infty)$, $c_{i^{-1}j} = T_i\big(T_j^{-1}(\infty)\big)$, and prime near signs of products means that of each pair of inverse substitutions $T$ and $T^{-1}$, only one is to be taken in the infinite product and $i > 0$.

According to [31, Lemma 2] it is possible to find for a given system of intervals $E$ a unique domain $G(K_1,\dots,K_{l-1})$ with $K_1 = \{z : |z-i| \le R_1\}$ such that the region $G(K_1,\dots,K_{l-1})$ is mapped conformally onto $\mathbb{C}\setminus E$ by the automorphic function

$$x = \phi(u) = (a_1-a_2)\prod_{i=0}^{\infty}\frac{\big(u-T_i(0)\big)^2}{\big(u-T_i(\xi)\big)\big(u-T_i(\bar{\xi})\big)} + a_2 \tag{10}$$

where $\xi$ is such that $\phi(\xi) = \infty$. Let us observe now that the images of the parts of the imaginary axis which are in $G(K_1,\dots,K_{l-1})$ are mapped by $\phi$ onto the parts of the real axis from $\mathbb{C}\setminus E$, in particular, $\xi$ is purely imaginary.

Indeed, from [52, Theorem IX.36] it follows that the conformal mapping of $G(K_1,\dots,K_{l-1})$ onto $\mathbb{C}\setminus E$ is unique. But it is obvious that the mapping $\overline{\phi(-\bar{u})}$ will be a conformal mapping of $G(K_1,\dots,K_{l-1})$ onto $\mathbb{C}\setminus E$, too. Hence $\phi(u) = \overline{\phi(-\bar{u})}$, and for $\Re u = 0$ we have $\Im\phi(u) = 0$, proving the claim.

Now let us extend $\phi$ into the lower halfplane by the Riemann-Schwartz symmetry principle as usual by $\phi(\bar{u}) = \overline{\phi(u)}$. The mapping $u \to -u$ may be considered as a composition of two symmetries in $\mathbb{C}$: one with respect to the imaginary axis and the other one with respect to the real axis. But to both symmetries in the plane $z = \phi(u)$ there corresponds a symmetry with respect to the real axis, so $\phi(-u) = \phi(u)$, *i.e.* $\phi$ is even.

For the following it is more convenient to use another normalization of the fundamental domain $\mathfrak{T}$, namely instead of $o_1 = i$ we require

$$\lim_{u\to\infty}\frac{\Omega^2(u,0)}{\Omega(u,\xi)\Omega(u,-\xi)} = 1.$$

Then $\phi$ from (10) becomes the following form:

$$\phi(u) = (a_1 - a_2)\frac{\Omega^2(u,0)}{\Omega(u,\xi)\Omega(u,-\xi)} + a_2. \tag{11}$$

The points $u = o_i + (-1)^j R_i$, $i = 1,\dots,l-1$, $j = 1,2$ correspond under the map (11) to the points $x = a_{2i+j}$, and the left and right semi-circumferences of $\partial K_i$ correspond to the upper and lower edges of the intervals $[a_{2i+1}, a_{2i}]$, $i = 1,\dots,l-1$, $a_1 = \phi(\infty)$, $a_2 = \phi(0)$, and the left and right half of the real axis correspond to the upper and lower edge of the interval $[a_1, a_2]$. Let us observe that the mapping $u \to -u$ corresponds to the hyperelliptic involution on the Riemann surface of the function $w = \sqrt{H(z)}$ (*i.e.* the change of the branch of the square root).

We will need also the following properties of the functions (8), (9) under the substitutions of the group $\Gamma$:

$$\frac{\Omega\big(T_p(z), y\big)}{\Omega(z,y)} \equiv (\gamma_p z + \delta_p)^{-1} \exp\Big\{ \Phi_p(z) - \Phi_p(y) + \frac{a_{pp}}{2} \Big\}, \quad [14, p. 292], \tag{12}$$

$$\frac{\exp \Phi_i\big(T_p(z)\big)}{\exp \Phi_i(z)} \equiv \exp(n_1 a_{i1} + n_2 a_{i2} + \cdots + n_{l-1} a_{il-1}), \quad [13, p. 66]. \tag{13}$$

Here $T_p(z) = \frac{\alpha_p z + \beta_p}{\gamma_p z + \delta_p}$, $\alpha_p \delta_p - \beta_p \gamma_p = 1$ is the so-called *normal* form of $T_p(z)$, $a_{pq}$ are the values of the integrals

$$\int_{A'_q A_q} \theta(z, c_{p^{-1}})\, dz$$

over the paths $A'_q A_q$, which neither cut themselves nor each other, make the domain $\mathfrak{T}$ schlicht, and which are outside of circles $K_1,\dots,K_{l-1}, K'_1,\dots,K'_{l-1}$; $K'_j$ being symmetric to $K_j$ with respect to the real axis, $A_q = o_q + iR_q, A'_q = \bar{o}_q - iR_q, q = 1,\dots,l-1$. For the following we shall use also the notations $A_q = o_q - iR_q, A'_q = \bar{o}_q + iR_q, q = l,\dots,2l-2$. Furthermore, the function $\theta(z,a)$ is defined by the equation

$$\theta(z,a) = \sum_{i=0}^{\infty} \frac{(\gamma_i z + \delta_i)^{-2}}{\frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} - a}$$

(*i.e.*, as a Poincaré theta-series). The integers $n_1,\dots,n_{l-1}$ are defined as follows: if $z \in \mathfrak{T}$ then any path between $z$ and $T_i(z)$ which does not cut the barriers $A'_p A_p$, $p = 1,\dots,l-1$, includes portions which are reconcilable with homologues of some of the original barriers $A'_1 A_1,\dots,A'_{l-1}A_{l-1}$ taken either positively or negatively; then, finally, among these homologues that of $A'_1 A_1$ occurs $n_1$ times, that of $A'_2 A_2$ $n_2$ times and so on. In the following we shall use the fact that for $i = 1,\dots,l-1$ in (13) the numbers $n_k$ are equal to $-\delta_{k,p}$, $k, p = 1,\dots,l-1$, since for such $p$ the above path contains only $A_p A'_p$. The numbers $a_{pq}$ above have the following properties

1. $a_{pq} = a_{qp}$, $p, q = 1,\dots,l-1$ [13, p. 64–65],
2. $a_{pq} \in \mathbb{R}$, $p, q = 1,\dots,l-1$ (since the group $\Gamma$ is symmetric [13, Section 7]).

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

In fact the matrix

$$\Big(-\frac{1}{2}a_{pq}\Big)$$

is nothing else than the matrix of periods for the corresponding hyperelliptic Riemann surface [13, p. 71–72].

Finally we denote by $v_j \in G(K_1, \dots, K_{l-1})$, for $j = 1, \dots, \nu^*$, the points which correspond to the zeros $w_j$ of $\rho_\nu$, that is,

$$w_j = \phi(v_j) = (a_1 - a_2)\frac{\Omega^2(v_j, 0)}{\Omega(v_j, \xi)\Omega(v_j, -\xi)} + a_2. \tag{14}$$

Since the function $(x - w_j)(u)$ has poles at the points $\xi$ and $\bar{\xi}$ and zeros at the points $v_j$ and $-v_j$ we have by the Burnside Representation Theorem [14, p. 293],

$$x - w_j = \text{const}\, \frac{\Omega(u, v_j)\Omega(u, -v_j)}{\Omega(u, \xi)\Omega(u, -\xi)} \exp \sum_{k=1}^{l-1} m_{jk}\Phi_k(u), \quad j = 1, \dots, \nu^*.$$

Taking into account the variation of the argument along the circles $\partial K_i, i = 1, \dots, l-1$, one obtains with the help of (8), (9) and the observation that both $T_k\big(T_j^{-1}(\infty)\big)$ and $T_k(\infty)$ lie inside or outside $\partial K_j$ simultaneously, that $m_{jk} = 0$, $j = 1, \dots, \nu^*$; $k = 1, \dots, l-1$. It gives

$$\rho_\nu\big(\phi(u)\big) = \text{const} \prod_{j=1}^{\nu^*}\Big(\frac{\Omega(u, v_j)\Omega(u, -v_j)}{\Omega(u, \xi)\Omega(u, -\xi)}\Big)^{\nu_j}. \tag{15}$$

In an analogous way we get

$$R\big(\phi(u)\big) = \text{const} \prod_{j=1}^{r} \frac{\Omega^2(u, u_j)}{\Omega(u, \xi)\Omega(u, -\xi)} \exp\Big(-\sum_{j=1}^{r} r_j\Phi_j(u)\Big), \tag{16}$$

where $r_j$, $j = 0, \dots, l-1$, is the number of zeros of $R$ on $[a_{2j+1}, a_{2j+2}]$. Let us note that if $R(a_1) = 0$ then one of the $u_j$'s has to be infinite. In such a case the corresponding factor in the product at the right-hand side of (16) must be replaced by

$$\frac{\Omega^2(u, o_1)}{\Omega(u, \xi)\Omega(u, -\xi)} \exp\big(-2\Phi_1(u)\big).$$

Here one should take into account the automorphity of $x - a_1$ together with the variation of its argument along the circle $\partial K_1$.

For the following let us determine

$$\lim_{u\to\xi} \frac{1}{\phi(u)} \frac{\Omega(u, -\xi)}{\Omega(u, \xi)} = \tau^{-1}. \tag{17}$$

It follows easily by (11) that

$$\tau = (a_1 - a_2)\frac{\Omega^2(\xi, 0)}{\Omega^2(\xi, -\xi)}. \tag{18}$$

For the reader's convenience let us give also the expressions of some potential-theoretic functions in terms of the Schottky-Burnside functions.

***Proposition 1*** *The capacity of E, the Green function g of $\mathbb{C} \setminus E$, and the harmonic measures of $E_j$, $j = 1, \dots, l-1$, are defined in terms of the Schottky-Burnside functions by the relations*

$$\Re\Phi_p(\xi) = (\omega A)_p,\ p = 1, \dots, l-1, \tag{19}$$

$$\text{cap}\,(E) = \tau \exp(-\omega A \omega^T), \tag{20}$$

$$g_{\mathbb{C}\setminus E}(z, \infty) = \log\left| \frac{\Omega(u, -\xi)}{\Omega(u, \xi)} \exp \sum_{j=1}^{l-1} \omega_j(\infty)\Phi_j(u) \right|, \tag{21}$$

*where $\omega_j(\infty)$ is the harmonic measure of the interval $E_j$ at the point $\infty$ with respect to $\mathbb{C} \setminus E$, $-A$ is the matrix of periods for the Riemann surface corresponding to $\mathbb{C} \setminus E$, $\omega = \big(\omega_1(\infty), \dots, \omega_{l-1}(\infty)\big)$, $z = \phi(u)$ and cap$(E)$ denotes the (logarithmic) capacity of E.*

The proof of (19) will be given in Corollary 1, the proofs of (20) and (21) in the Appendix, since we do not use them in the following.

## 2 The Basic Results

The starting point of our investigations is the following characterization (due to the second author [34]) of the polynomials orthogonal with respect to $\Psi_{R,\rho_\nu,\varepsilon}$ by a quadratic equation, also called *Pell-equation*.

***Lemma 1*** *Let $R$, $\rho_\nu, \varepsilon_j \in \{-1, 1\}$, $j = 1, \dots, \nu^*$ be given. Then for $n \geq \max\{\nu + 1 - l, (\nu + 1 - r)/2\}$ the following assertions hold*

(a) *The monic polynomial $p_n$ is orthogonal to $\mathbb{P}_{n-1}$ with respect to the functional $\Psi_{R,\rho_\nu,\varepsilon}$ if and only if there exists a monic polynomial $q_{(n)}$ such that*

$$Rp_n^2 - Sq_{(n)}^2 = \rho_\nu g_{(n)}, \tag{22}$$

*and*

$$(\sqrt{R}p_n)^{(k_j)}(w_j) = \varepsilon_j(\sqrt{S}q_{(n)})^{(k_j)}(w_j) \neq 0 \tag{23}$$

*at the zeros $w_j$ of $\rho_\nu$, $j = 1 \dots, \nu^*$, where $k_j$ is the minimal integer such that (23) holds, and $g_{(n)} \in \mathbb{P}_{l-1}$ with a zero of multiplicity $2k_j$ at $w_j$ for $j = 1, \dots, \nu^*$.*

(b) *The integer $n$ is a basic integer for the functional $\Psi_{R,\rho_\nu,\varepsilon}$ if and only if the polynomials $Rp_n$ and $Sq_{(n)}$ from (a) have no common zeros. Note, in this case (23) is satisfied with $k_j = 0$ for $j = 1, \dots, \nu^*$.*

(c) *If $(i_n)$ is the sequence of basic integers for the functional $\Psi_{R,\rho_\nu,\varepsilon}$ then*

(i) *for $m = i_n$ the orthogonal polynomial is unique;*

(ii) *for $i_k + 1 \le m \le i_k + [(i_{k+1} - i_k + 1)/2] - 1$, any polynomial $p_n$ which is orthogonal with respect to the functional $\Psi_{R,\rho_\nu,\varepsilon}$ to $\mathbb{P}_{m-1}$ has the form $p_m(x) = p_{i_n}(x)\varpi_{m-i_n}(x)$, where $\varpi_{m-i_n} \in \mathbb{P}_{m-i_n}$ is arbitrary;*

(iii) *for $i_k + [(i_{k+1} - i_k + 1)/2] \le m \le i_{k+1} - 1$, there exist no orthogonal polynomials with respect to $\Psi_{R,\rho_\nu,\varepsilon}$ to $\mathbb{P}_{m-1}$.*

**Proof**

(c) is known ([15, Proposition 1.14])

(a) For $n$ being a basic integer the necessity part is exactly Theorem 3(a) from [34]. The sufficiency part under the additional assumption that $p_n$ and $q_{(n)}$ have no common zero follows from Theorem 1(a) and Corollary 1 from [34]. Moreover, it is proven there that $p_n$ is orthogonal to $\mathbb{P}_{n+l-2-\partial g_{(n)}}$. If $p_n$ and $q_{(n)}$ have common zeros then after dividing (22), (23) by $\varpi^2(x)$, where $\varpi(x)$ denotes the polynomial which vanishes exactly at the common zeros of $p_n$ and $q_{(n)}$ with corresponding multiplicities, one obtains the relations

$$R(\tilde{p}_{n-\partial\varpi})^2 - S(\tilde{q}_{(n-\partial\varpi)})^2 = \rho_\nu \tilde{g}_{(n-\partial\varpi)}$$

and

$$(\sqrt{R}\tilde{p}_{n-\partial\varpi})(w_j) = \varepsilon_j(\sqrt{S}\tilde{q}_{(n-\partial\varpi)})(w_j) \neq 0.$$

Hence by the proven part $\tilde{p}_{n-\partial\varpi}$ is orthogonal to $\mathbb{P}_{n+l-2-\partial\tilde{g}_{(n-\partial\varpi)}}$, but $n + l - 2 - \partial\tilde{g}_{(n-\partial\varpi)} = n + l - 2 - (\partial g_{(n)} - 2\partial\varpi) \ge n - 1 + \partial\varpi$, and the orthogonality of $p_n = \tilde{p}_{n-\partial\varpi}\varpi$ to $\mathbb{P}_{n-1}$ follows by the definition of $\Psi_{R,\rho_\nu,\varepsilon}$. Concerning the necessity part for nonbasic integers let us note that by (c)(ii) for any integer $m$, $i_k + 1 \le m \le i_k + [(i_{k+1} - i_k + 1)/2] - 1$, the polynomial $p_m(x)$ can be represented as $p_m(x) = p_{i_n}(x)\varpi_{m-i_n}(x)$, with $\varpi_{m-i_n} \in \mathbb{P}_{m-i_n}$. Since by [34, Theorem 3(a)]

$$Rp_{i_n}^2 - S(Yp_{i_n} + \rho_\nu p_{i_n}^{[1]})^2 = \rho_\nu g_{(i_n)} \tag{24}$$

with $\partial g_{(i_n)} = i_n + l - i_{n+1}$, and by [15, Proposition 1.19] $p_m^{[1]}(x) = p_{i_n}(x)\varpi_{m-i_n}(x)$, one obtains after multiplying (24) by $\varpi_{m-i_n}^2$ the relation

$$Rp_m^2 - S(Yp_m + \rho_\nu p_m^{[1]})^2 = \rho_\nu g_{(m)}, \tag{25}$$

where $g_{(m)} = g_{(i_n)}\varpi_{m-i_n}^2$ and $\partial g_{(m)} = \partial g_{(i_n)} + 2\partial\varpi_{m-i_n} \le l - 1$, which proves the necessity part for such $m$.

(b) follows easily from the proof of (a). ■

The basic theorems for what follows are the next two.

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

***Theorem 1*** *Let $R$, $\rho_\nu, \varepsilon_j \in \{-1, 1\}$, $j = 1, \dots, \nu^*$, be given, let $p_n(x) = x^n + \cdots \in \mathbb{P}_n$ and $q_m \in \mathbb{P}_m$, and assume that $Rp_n$ and $Sq_m$ have no common zero and satisfy the relations*

$$Rp_n^2 - Sq_m^2 = \rho_\nu g_{(n)}, \tag{26}$$

*and*

$$(\sqrt{R}p_n)(w_j) = \varepsilon_j(\sqrt{S}q_m)(w_j) \tag{27}$$

*at the zeros $w_j$, for $j = 1, \dots, \nu^*$, of $\rho_\nu$, where $g_{(n)} \in \mathbb{P}_{l-1}$. Write*

$$\Omega_n(u) = \frac{\Omega^n(u,-\xi)}{\Omega^n(u,\xi)} \prod_{j=1}^{\nu^*} \frac{\Omega^{\frac{\nu_j}{2}(1+\varepsilon_j)}(u,v_j)\Omega^{\frac{\nu_j}{2}(1-\varepsilon_j)}(u,-v_j)}{\Omega(u,-\xi)} \cdot \prod_{j=1}^{r} \frac{\Omega(u,-\xi)}{\Omega(u,u_j)} \prod_{j=1}^{\partial g_{(n)}} \frac{\Omega(u,b_j^{(n)})}{\Omega(u,-\xi)} \exp \sum_{j=1}^{l-1} \frac{m_j^{(n)}}{2} \Phi_j(u), \tag{28}$$

*where $b_j^{(n)} \in \mathfrak{T}$, $j = 1, \dots, \partial g_{(n)}$ and $m_j^{(n)} \in \mathbb{Z}$, $j = 1, \dots, l-1$ are given by the system of equations*

$$\exp\Big(2(2n - \nu - \partial g_{(n)} + r)\Phi_p(\xi) + 2\sum_{j=1}^{\nu^*} \varepsilon_j \nu_j \Phi_p(v_j) - \sum_{j=1}^{l-1} m_j^{(n)} a_{jp} - 2\sum_{j=1}^{\partial g_{(n)}} \Phi_p(b_j^{(n)})\Big) = 1, \quad p = 1, \dots, l-1. \tag{29}$$

*Then, for $2n + r \geq \nu + \partial g_{(n)}$,*

$$2p_n(x) = c_n\big(\Omega_n(u) + \Omega_n(-u)\big), \tag{30}$$

$$2\sqrt{\frac{S(x)}{R(x)}} q_m(x) = c_n\big(\Omega_n(u) - \Omega_n(-u)\big), \tag{31}$$

*where $c_n$ is given by*

$$2/c_n = \lim_{u \to \xi} \frac{\Omega_n(u)}{\phi^n(u)}. \tag{32}$$

*Furthermore, denoting by $G_n$ the leading coefficient of $g_{(n)}$, we have for $2n+r > \nu+\partial g_{(n)}$,*

$$G_n = 4\tau^{2n-\nu-\partial g_{(n)}+r} \prod_{j=1}^{\nu^*} \left(\frac{\Omega(\xi,v_j)}{\Omega(\xi,-v_j)}\right)^{\varepsilon_j \nu_j} \prod_{j=1}^{r} \frac{\Omega(\xi,u_j)}{\Omega(\xi,-u_j)} \cdot \prod_{j=1}^{\partial g_{(n)}} \frac{\Omega(\xi,-b_j^{(n)})}{\Omega(\xi,b_j^{(n)})} \exp\Big(-\sum_{j=1}^{l-1} m_j^{(n)} \Phi_j(\xi)\Big). \tag{33}$$

*If $R(a_1) = 0$ then one factor in (28) has to be replaced in the same way as in (16).*

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

**Proof** If there is no possibility of confusion, we omit the indices $n, m, \nu$. Put

$$\Psi(u) = \left( p(x) + \left( \sqrt{\frac{S(x)}{R(x)}} \right) q(x) \right)^2 \Big/ \rho(x) g(x), \tag{34}$$

where $x = \phi(u)$ is the mapping function from (11). Then the function $\Psi(u)$ is defined on the domain $G(K_1, \ldots, K_{l-1})$. The values of the function $\Psi(u)R(x)$ considered as a function of $x$ differ on different edges of cuts $[a_1, a_2], \ldots, [a_{2l-1}, a_{2l}]$ by complex conjugation only and have modulus one according to (26). Hence it is possible to extend by the Riemann-Schwartz Symmetry principle the function $\Psi(u)R\big(\phi(u)\big)$ up to an automorphic function with respect to the corresponding Schottky group $\Gamma$. Since $x(u)$ is automorphic, the extended function $\Psi(u)$ will be automorphic too. Moreover by the evenness of the function $x = \phi(u)$ and by applying twice the Riemann-Schwartz Symmetry principle (to the real and imaginary axes) we deduce

$$\Psi(-u) = \left( p(x) - \left( \sqrt{\frac{S(x)}{R(x)}} \right) q(x) \right)^2 \Big/ \rho(x) g(x) = \frac{1}{R^2(x)} \frac{1}{\Psi(u)}. \tag{35}$$

From (35) we conclude that if $u$ is a zero of $\Psi(u)$ and not a zero or pole of $R\big(\phi(u)\big)$ then $-u$ is a pole of $\Psi(u)$, and vice versa. Keeping this in mind we find from (34) that

(i) $u = \xi$ (which corresponds to $x = \infty$) is a pole of multiplicity $2n - (\nu + \partial g)$ of $\Psi(u)$.

In conjunction with (35) this shows that

(ii) $u = -\xi$ is a zero of multiplicity $2n - (\nu + \partial g) + 2r$ of $\Psi(u)$.

From the definition of $\rho$, from (27), (34), and (35) it follows that

(iii) $u = -\nu_j$ is a zero (pole) of multiplicity $\nu_j$ of $\Psi(u)$, if $\varepsilon_j = +1(-1)$;
(iv) $u = \nu_j$ is a zero (pole) of multiplicity $\nu_j$ of $\Psi(u)$, if $\varepsilon_j = -1(+1)$.

Furthermore, with the help of (16),(34) and (35) and recalling the usual convention for the automorphic functions theory (only one from the boundary points of $\mathfrak{T}$, which are homologues to each other, belongs to $\mathfrak{T}$), we obtain

(v) $u = u_j$ is a double pole of $\Psi(u)$.

Since by (26) $\Psi(u)$ or $\Psi(-u)$ vanishes at the zeros of $g\big(\phi(u)\big)$, if $\partial g \geq 1$, it follows with the help of (35) that

(vi) $\Psi(u)$ has also zeros $b_1, \ldots, b_{\partial g}$ and poles $-b_1, \ldots, -b_{\partial g}$.

Summing up (i)–(vi) we get by the Burnside Representation theorem for automorphic functions [14, p. 293] that $\Psi(u)$ has a representation of the form

$$\Psi(u) = d\left[\frac{\Omega(u,-\xi)}{\Omega(u,\xi)}\right]^{2n-\nu-\partial g} \prod_{j=1}^{\partial g} \frac{\Omega(u,b_j)}{\Omega(u,-b_j)} \prod_{j=1}^{\nu^*}\left[\frac{\Omega(u,-\nu_j)}{\Omega(u,\nu_j)}\right]^{\varepsilon_j\nu_j} \cdot \prod_{j=1}^{r} \frac{\Omega^2(u,-\xi)}{\Omega^2(u,u_j)} \cdot \exp\sum_{j=1}^{l-1} m_j' \Phi_j(u). \tag{36}$$

Next we obtain as in (15),(16) that $g$ has a representation of the form

$$g\big(\phi(u)\big) = c\prod_{j=1}^{\partial g} \frac{\Omega(u,b_j)\Omega(u,-b_j)}{\Omega(u,\xi)\Omega(u,-\xi)}, \tag{37}$$

where $c \in \mathbb{C}$, and that

$$\tilde{\Psi}(u) = g\big(\phi(u)\big)\,\Psi(u)R\big(\phi(u)\big)$$

is automorphic since $\phi(u)$ and $\Psi(u)$ are automorphic too.

Writing down the condition of automorphity for $\tilde{\Psi}(u)$ gives because of (12), (13) the following relation

$$\exp\Big((2n-\nu-\partial g+r)\big(\Phi_p(\xi)-\Phi_p(-\xi)\big) - 2\sum_{j=1}^{\partial g}\Phi_p(b_j) + \sum_{j=1}^{\nu^*}\varepsilon_j\nu_j\big(\Phi_p(\nu_j)-\Phi_p(-\nu_j)\big) - \sum_{j=1}^{l-1} m_j a_{pj}\Big) = 1, \quad p = 1,\dots,l-1, \tag{38}$$

where $m_j = m_j' - r_j$, $j = 1,\dots,l-1$. Relation (38) in fact is the same as (29), since $\Re\Phi_p$ is odd. Indeed, for any $j = 1,\dots,l-1$ and for any $z \in \mathbb{C}$ $T_j(-z) = -T_j(z)$, hence the collection of $T_k(-z), k = 1,\dots,$ coincides with $-T_k(z), k = 1,\dots.$ Now from (13) one obtains

$$\begin{aligned}\exp\Phi_j(-z) &= \frac{-z-o_j}{-z+o_j}\prod_{\substack{k=1\\k\neq j}}^{\infty}\frac{-z-T_k\big(T_j^{-1}(\infty)\big)}{-z-T_k(\infty)} \\ &= \frac{z-\overline{o_j}}{z-o_j}\prod_{\substack{k=1\\k\neq j^{-1}}}^{\infty}\frac{z-T_k\big(T_{j^{-1}}^{-1}(\infty)\big)}{z-T_k(\infty)} = \exp\Phi_{j^{-1}}(z).\end{aligned} \tag{39}$$

But

$$\Phi_{j^{-1}}(z) = \int_0^z \theta(z,o_j)\,dz = -\int_0^z \theta(z,o_{j^{-1}})\,dz = -\Phi_j(z),$$

proving the oddness of $\Re\Phi_p$.

Hence we obtain using the fact $\phi(-u) = \phi(u)$,

$$\Big(p_n \pm \sqrt{\frac{S}{R}} q_m\Big)^2 = \frac{\tilde{\Psi}(\pm u)\rho\big(\phi(u)\big)}{R\big(\phi(u)\big)} = \tilde{c}_n \Omega_n^2(\pm u),$$

where $\tilde{c}_n \in \mathbb{C}$ and $\Omega_n$ is defined in (28). Thus

$$\Big(p_n \pm \sqrt{\frac{S}{R}} q_m\Big)\big(\phi(u)\big) = c_n \Omega_n(\pm u). \tag{40}$$

Recalling that $p_n$, and thus by (26) also $q_m$, are monic, we obtain

$$2 = \lim_{x\to\infty} \Big(p_n + \sqrt{\frac{S}{R}} q_m\Big) \Big/ x^n = c_n \lim_{u\to\xi} \frac{\Omega_n(u)}{[\phi(u)]^n},$$

which gives relation (32).

In order to obtain (33) we observe firstly that by (26) and (37)

$$g\rho/R = c_n^2 \Omega_n(u)\Omega_n(-u),$$

and hence

$$G_n = \lim_{x\to\infty} \frac{g(x)\rho(x)}{R(x)x^{\nu+\partial g-r}} = c_n^2 \lim_{u\to\xi} \frac{\Omega_n(u)\Omega_n(-u)}{[\phi(u)]^{\nu+\partial g-r}}.$$

Straightforward calculation using (17) and

$$\Omega(-z, y) = \Omega(z, -y) \tag{41}$$

(which can be obtained analogously to the oddness of $\Re\Phi_p$) now gives relation (33).

To prove the uniqueness of the solution of (29) one should observe that all considerations in the proof of Theorem 1 may be inverted, that is, if (29) holds then $\Psi$ from (36) is automorphic and $p_n$ and $q_m$ given by (30) and (31) are polynomials which satisfy (26) and (27) hence the uniqueness follows from the assumption about the absence of common zeros and from Lemma 1. ■

For the case $l = 2$ analogue considerations were used in [38] to obtain corresponding statements in terms of elliptic theta functions. That result is in fact a particular case of Theorem 1, what can be seen as in [2] by degeneracy of automorphic functions for $l = 2$ into elliptic theta functions.

Using the fact that $-\exp\Phi_p(z)/2$, $p = 1, \dots, l-1$, represent independent Abelian integrals of the first kind [13, p. 66] it follows that (29) can also be considered as a Jacobi inversion problem (compare [33, 49, 51]) in the variables $b_j^{(n)}$. Let us observe that similar representations for $p_{i_n}$ for the case $\varepsilon = (1, 1, \dots, 1)$ were obtained in [33, Theorem 5.1] in terms of Riemann theta-functions.

Now we are ready to determine the number of zeros in the intervals $E_j = [a_{2j+1}, a_{2j+2}]$, $j = 0, \dots, l-1$ of the orthogonal polynomials.

***Notation*** Let $\mathcal{Z}(p_n, [a_{2j+1}, a_{2j+2}]) := \#\{x \in [a_{2j+1}, a_{2j+2}] : p_n(x) = 0\}$ denote the number of zeros of $p_n$ in the interval $[a_{2j+1}, a_{2j+2}]$.

***Theorem 2*** *Let $R$, $\rho_\nu, \varepsilon_j \in \{-1,+1\}$, $j = 1,\dots,\nu^*$ be given, let $(i_n)$ be the sequence of basic integers with respect to $\Psi_{R,\rho_\nu,\varepsilon}$ and let $(p_{i_n})$ be the sequence of monic polynomials of degree $i_n$ orthogonal with respect to $\Psi_{R,\rho_\nu,\varepsilon}$. Suppose that $i_{n+1} + i_n + r \geq \nu + l + 1$. Then $p_{i_n}$ satisfies an equation of the form*

$$Rp_{i_n}^2 - Sq_{(i_n)}^2 = \rho_\nu g_{(i_n)}, \tag{42}$$

*with $\partial g_{(i_n)} = \mathfrak{g}_{i_n} \in \{0,\dots,l-1\}$ and the following representations hold*

$$\frac{2Rp_{i_n}^2}{\rho_\nu g_{(i_n)}} - 1 = \frac{\psi_{i_n}(u) + \psi_{i_n}(-u)}{2}, \tag{43}$$

$$2\sqrt{H(x)}\frac{q_{(i_n)}(x)p_{i_n}(x)}{\rho_\nu(x)g_{(i_n)}(x)} = \frac{\psi_{i_n}(u) - \psi_{i_n}(-u)}{2}. \tag{44}$$

*where*

$$\psi_{i_n}(u) = \left(\frac{\Omega(u,-\xi)}{\Omega(u,\xi)}\right)^{2i_n-\nu-\mathfrak{g}_{i_n}+r} \prod_{j=1}^{\mathfrak{g}_{i_n}} \left(\frac{\Omega(u,-b_j^{(i_n)})}{\Omega(u,b_j^{(i_n)})}\right)^{\delta_j^{(i_n)}} \prod_{j=1}^{\nu^*}\left(\frac{\Omega(u,-\nu_j)}{\Omega(u,\nu_j)}\right)^{\varepsilon_j\nu_j} \exp\sum_{j=1}^{l-1}(m_j^{(i_n)} + r_j)\Phi_j(u), \tag{45}$$

*$x$ and $u$ are connected by (11). The points $b_j^{(i_n)} \in G(K_1,\dots,K_{l-1})$, with $\Re b_j^{(i_n)} = 0$, $j = 1,\dots,\mathfrak{g}_{i_n}$; the integers $m_j^{(i_n)} \in \mathbb{Z}$, $j = 1,\dots,l-1$, and the $\delta_j^{(i_n)} \in \{-1,+1\}$, $j = 1,\dots,\mathfrak{g}_{i_n}$ are given uniquely by the system of equations*

$$(2i_n - \nu - \mathfrak{g}_{i_n} + r)\Re\Phi_p(\xi) + \sum_{j=1}^{\nu^*}\varepsilon_j\nu_j\Phi_p(\nu_j) - \sum_{j=1}^{l-1} m_j^{(i_n)}\frac{a_{jp}}{2} + \sum_{j=1}^{\mathfrak{g}_{i_n}}\delta_j^{(i_n)}\Re\Phi_p(b_j^{(i_n)}) = 0, \quad p = 1,\dots,l-1. \tag{46}$$

*Finally, the number of zeros of $p_{(i_n)}$ and the $m_j^{(i_n)}$'s, $j = 1,\dots,l-1$, given by (46), are related in the following way:*

$$2\mathcal{Z}(p_{i_n},[a_{2j+1},a_{2j+2}]) = \begin{cases} m_j^{(i_n)} & \text{if } r_j = 2 \text{ and } p_{i_n}^2(a_{2j+1}) + p_{i_n}^2(a_{2j+2}) = 0,\\ m_j^{(i_n)} - r_j, & \text{otherwise.}\end{cases} \tag{47}$$

*Moreover,*

$$2\mathcal{Z}(p_{i_n},[a_1,a_2]) = -\sum_{j=1}^{l-1} m_j^{(i_n)} + \sum_{j=1}^{\nu^*}\varepsilon_j\nu_j + \sum_{j=1}^{\mathfrak{g}_{(i_n)}}\delta_j^{(i_n)} + (2i_n - \nu - \mathfrak{g}_{(i_n)} + r).$$

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

**Proof** In view of (42) we have at the zeros $x_j^{(i_n)} =: \phi(b_j^{(i_n)})$, where $b_j^{(i_n)} \in G(K_1, \dots, K_{l-1})$ and $\Re b_j^{(i_n)} = 0$,

$$p_{i_n}(x_{j,n}) = \delta_j^{(i_n)}(\sqrt{H}q_{i_n})(x_{j,n}), \quad j = 1, \dots, \mathfrak{g}_{i_n}, \tag{48}$$

where $\delta_j^{(i_n)} \in \{-1, 1\}$. Taking a look at the function $\Psi$ from (34) it follows that (36) becomes

(49)
$$\Psi_{i_n}(u) = d\left[\frac{\Omega(u,-\xi)}{\Omega(u,\xi)}\right]^{2i_n-\nu-\mathfrak{g}_{(i_n)}} \prod_{j=1}^{\mathfrak{g}_{(i_n)}} \left(\frac{\Omega(u, b_j^{(i_n)})}{\Omega(u,-b_j^{(i_n)})}\right)^{\delta_j^{(i_n)}} \prod_{j=1}^{\nu^*}\left[\frac{\Omega(u,-\nu_j)}{\Omega(u,\nu_j)}\right]^{\varepsilon_j\nu_j}$$
$$\cdot \prod_{j=1}^{r} \frac{\Omega^2(u,-\xi)}{\Omega^2(u,u_j)} \cdot \exp\sum_{j=1}^{l-1}(m_j^{(i_n)} + r_j)\Phi_j(u).$$

Putting

$$\psi_{i_n}(u) = \Psi_{i_n}(u) R\big(\phi(u)\big)$$

it follows by (34) and (35) that

$$\psi_{i_n}(u) \pm \psi_{i_n}(-u) = c\left(\frac{R\big(p_{i_n} + \sqrt{\frac{S}{R}}q_{(i_n)}\big)^2 \pm R\big(p_{i_n} - \sqrt{\frac{S}{R}}q_{(i_n)}\big)^2}{\rho_\nu g_{(i_n)}}\right)\big(\phi(u)\big) \tag{50}$$

which gives with the help of (24) relations (43), (44) up to a constant factor.

Now let us observe that

$$\lim_{x\to\infty}\left(\frac{2Rp_{i_n}^2}{\rho_\nu g_{(i_n)}} - 1\right)\cdot\frac{1}{x^{r+2i_n-\nu-\mathfrak{g}_{i_n}}} = \frac{2}{G_{i_n}},$$

where

$$G_{i_n} = 4\tau^{2i_n-\nu-\mathfrak{g}_{i_n}+r}\prod_{j=1}^{\nu^*}\left(\frac{\Omega(\xi,\nu_j)}{\Omega(\xi,-\nu_j)}\right)^{\varepsilon_j\nu_j}\prod_{j=1}^{\mathfrak{g}_{i_n}}\left(\frac{\Omega(\xi,b_j^{(i_n)})}{\Omega(\xi,-b_j^{i_n})}\right)^{\delta_j^{(i_n)}}$$
$$\cdot\ \exp-\sum_{j=1}^{l-1}(m_j^{(i_n)} + r_j)\Phi_j(\xi)$$

and

$$\lim_{u\to\xi}\frac{\psi_{i_n}(u)}{\big(\phi(u)\big)^{r+2i_n-\nu-\mathfrak{g}_{i_n}+r}} = \lim_{u\to\xi}\left(\frac{\Omega(u,-\xi)}{\Omega(u,\xi)\phi(u)}\right)^{2i_n-\nu-\mathfrak{g}_{i_n}+r}$$
$$\cdot\prod_{j=1}^{\mathfrak{g}_{i_n}}\left(\frac{\Omega(\xi,-b_j^{(i_n)})}{\Omega(\xi,b_j^{(i_n)})}\right)^{\delta_j^{(i_n)}}\prod_{j=1}^{\nu^*}\left(\frac{\Omega(\xi,-\nu_j)}{\Omega(\xi,\nu_j)}\right)^{\varepsilon_j\nu_j}\exp-\sum_{j=1}^{l-1}(m_j^{(i_n)} + r_j)\Phi_j(\xi),$$

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

hence (43) and (44) follow.

To find the numbers $m_j^{(i_n)}$ one has to count the variation of the argument of $\psi_{i_n}(u)$ along the circumferences $\partial K_1, \dots, \partial K_{l-1}$. Indeed, it follows from (9), (11) that

$$\Delta \arg_{u\in\partial K_j} \psi_{i_n}(u) = 2\pi m_j^{(i_n)}.$$

Now we calculate the variation of the argument in (45) in another way. Namely, by (45)

$$2\pi m_j^{(i_n)} = \Delta \arg_{x\in\mathcal{K}_j}\left( R\Big( p_{i_n} + \sqrt{\frac{S}{R}} q_{(i_n)}\Big)^2 \Big/ \rho_\nu g_{(i_n)}\right)(x),$$

where $\mathcal{K}_j$ is the interval $[a_{2j+1}, a_{2j+2}]$ which is run around twice (its lower edge in the positive direction and its upper edge in the negative direction). First let us observe that for any zero $x_k \in (a_{2j+1}, a_{2j+2})$ of $p_{i_n}$ one has locally by writing down the Taylor expansion for $R(p + \sqrt{S/R}q)^2/\rho g$ (omitting the indices) near a zero $x_k$ of $p$:

$$R(p + \sqrt{S/R}q)^2/\rho g = -1 + c(x - x_k)^\beta + \cdots . \tag{51}$$

Of course that expansion is valid for our fixed branch at only one side of the cut $[a_{2j+1}, a_{2j+2}]$. At the same side one has

$$R(p - \sqrt{S/R}q)^2/\rho g = -1 + c_1(x - x_k)^\beta + \cdots . \tag{52}$$

After multiplying both relations with the help of (42) it follows easily that $c_1 = c$. Substracting (52) from (51) gives

$$\begin{aligned} 2\sqrt{H}\frac{qp}{\rho g} &= R\Big( p + \sqrt{\frac{S}{R}}q\Big)^2 \Big/ \rho g - R\Big( p - \sqrt{\frac{S}{R}}q\Big)^2 \Big/ \rho g \\ &= 2c(x - x_k)^\beta + \cdots . \end{aligned} \tag{53}$$

But by assumption the polynomials $q$ and $p$ have no common zeros, hence $\beta = 1$. Now let us write relation (51) briefly as

$$R\Big( p_{i_n} + \sqrt{\frac{S}{R}} q_{(i_n)}\Big)^2 \Big/ \rho_\nu g_{(i_n)}(x) + 1 \sim x - x_k$$

and let us note that $x - x_k \sim u - \phi_{1,2}^{-1}(x_k)$. Hence, when $x$ moves along a small semicircle surrounding $x_k$ at one side of the real axis, $u = \phi^{-1}(x)$ will move around a simple curve near the circle $\partial K_j$ once. Thus the variation of the argument of $(x - x_k)$ will be equal to the variation of the argument of $u - \phi_{1,2}^{-1}(x_k)$, that is, equal to $\pi$. Analogously, if $a_k \in \{a_{2j+1}, a_{2j+2}\}$ with $R(a_k) = 0$ and $p_{i_n}(a_k) \neq 0$ we have, by taking a look at (53) that

$$R\Big( p_{i_n} + \sqrt{\frac{S}{R}} q_{(i_n)}\Big)^2 \Big/ \rho_\nu g_{(i_n)} + 1 \sim \sqrt{x - a_k} \sim u - \phi^{-1}(a_k);$$

and if $a_k \in \{a_{2j+1}, a_{2j+2}\}$ with $R(a_k) = 0$ and $p_{i_n}(a_k) = 0$

$$R\Big(p_{i_n} + \sqrt{\frac{S}{R}}q_{(i_n)}\Big)^2 \Big/ \rho_\nu g_{(i_n)} + 1 \sim (x - a_k)^{3/2} \sim \big(u - \phi^{-1}(a_k)\big)^3;$$

hence, putting $k_j^{(i_n)} = \mathcal{Z}(p_{i_n}, [a_{2j+1}, a_{2j+2}])$,

$$\Delta \arg_{x\in\mathcal{K}_j}\bigg(R\Big(p_{i_n} + \sqrt{\frac{S}{R}}q_{(i_n)}\Big)^2 \Big/ \rho_\nu g_{(i_n)}\bigg)(x) = \tag{54}$$

$$\begin{cases} 4\pi k_j^{(i_n)} & \text{for } R(a_{2j+1})R(a_{2j+2}) \neq 0, \\ 4\pi k_j^{(i_n)} + 2\pi & \text{for } R(a_{2j+1})R(a_{2j+2}) = 0,\ R^2(a_{2j+1}) + R^2(a_{2j+2}) \neq 0, \\ & \quad \text{and } p_{i_n}(a_{2j+1})p_{i_n}(a_{2j+2}) \neq 0, \\ 4\pi k_j^{(i_n)} & \text{for } R(a_{2j+1})R(a_{2j+2}) = 0,\ R^2(a_{2j+1}) + R^2(a_{2j+2}) \neq 0, \\ & \quad \text{and } p_{i_n}(a_{2j+1})p_{i_n}(a_{2j+2}) = 0, \\ 4\pi(k_j^{(i_n)} + 1) & \text{for } R^2(a_{2j+1}) + R^2(a_{2j+2}) = 0 \text{ and } p_{i_n}(a_{2j+1})p_{i_n}(a_{2j+2}) \neq 0, \\ 4\pi k_j^{(i_n)} + 2\pi & \text{for } R^2(a_{2j+1}) + R^2(a_{2j+2}) = 0,\ p_{i_n}(a_{2j+1})p_{i_n}(a_{2j+2}) = 0, \\ & \quad \text{and } p_{i_n}^2(a_{2j+1}) + p_{i_n}^2(a_{2j+2}) \neq 0, \\ 4\pi k_j^{(i_n)} & \text{for } R^2(a_{2j+1}) + R^2(a_{2j+2}) + p_{i_n}^2(a_{2j+1}) + p_{i_n}^2(a_{2j+2}) = 0. \end{cases}$$

Now let us note that the function $\Omega_{i_n}^2(u)$ has zeros and poles of even order, hence the function $\Omega_{i_n}(u) = \sqrt{\Omega_{i_n}^2(u)}$ is a single-valued automorphic function and in its representation

$$\Omega_{i_n}(u) = \frac{\Omega^{i_n}(u,-\xi)}{\Omega^{i_n}(u,\xi)} \prod_{j=1}^{\nu^*} \frac{\Omega^{\frac{\nu_j}{2}(1+\varepsilon_j)}(u,\nu_j)\Omega^{\frac{\nu_j}{2}(1-\varepsilon_j)}(u,-\nu_j)}{\Omega^{\nu_j}(u,-\xi)} \prod_{j=1}^{r} \frac{\Omega(u,-\xi)}{\Omega(u,u_j)} \tag{55}$$

$$\cdot \prod_{j=1}^{\mathfrak{g}_{i_n}} \frac{\Omega^{\frac{1+\delta_j^{(i_n)}}{2}}(u,-b_j^{(i_n)})\Omega^{\frac{1-\delta_j^{(i_n)}}{2}}(u,b_j^{(i_n)})}{\Omega(u,-\xi)} \exp \sum_{j=1}^{l-1} \frac{m_j^{(i_n)} + r_j}{2} \Phi_j(u),$$

the numbers $(m_j^{(i_n)} + r_j)/2$ have to be integers. Hence the third and the fifth cases in (54) are impossible, and the assertions of Theorem 2 about the $k_j^{(i_n)}$'s, $j = 1, \dots, l-1$ are proved.

As above we have

$$\Delta \arg_{x\in\mathcal{K}_0}\bigg(R\Big(p_{i_n} + \sqrt{\frac{S}{R}}q_{(i_n)}\Big)^2 \Big/ \rho_\nu g_{(i_n)}\bigg)(x) = -2\pi m_0^{(i_n)}.$$

On the other hand,

$$\Delta \arg_{x\in\mathcal{K}_0}\bigg(R\Big(p_{i_n} + \sqrt{\frac{S}{R}}q_{(i_n)}\Big)^2 \Big/ \rho_\nu g_{(i_n)}\bigg)(x) = \Delta \arg_{u\in\mathbb{R}} \psi_{i_n}(u)$$

$$= 2\pi \sum_{j=1}^{l-1} m_j^{(i_n)} - 2\pi \sum_{j=1}^{\nu^*} \varepsilon_j \nu_j - 2\pi \sum_{j=1}^{\mathfrak{g}_{i_n}} \delta_j^{(i_n)} - 2\pi(2i_n - \nu - \mathfrak{g}_{i_n} + r),$$

and the final part of Theorem 2 is proved. ∎

***Corollary 1*** *The quantities $\Phi_p(\xi)$ and $a_{pq}$ are connected by the relations*

$$2\Re\Phi_p(\xi) = \sum_{j=1}^{l-1} \omega_j(\infty) a_{jp}, \quad p = 1, \dots, l-1, \tag{56}$$

*where $\omega_j(\infty)$ is the harmonic measure of the interval $[a_{2j+1}, a_{2j+2}]$, $j = 1, \dots, l-1$, at the point $\infty$ with respect to $E$.*

**Proof** Let us take $\varepsilon = (1, 1, \dots, 1)$ and $R/h\rho > 0$. Then it is known [34] that $i_n = n$ and $\partial g_{(n)} = l-1$, $n \geq n_0$. Multiplying (46) by $1/n$ and taking the limit we obtain (56) by using the fact

$$k_j^{(n)}/n \to \int_{[a_{2j+1}, a_{2j+2}]} d\mu_E(x) = \omega_j(\infty), \quad j = 1, \dots, l-1,$$

where $\mu_E$ is the equilibrium measure (see, for instance, [54]). ∎

Let us note that it is possible to prove Corollary 1 with the help of the correspondence between the functions $\Phi_p$ and the Abelian integrals of the first kind, and [49, (26)].

Next let us characterize the case $g_{(n)} \in \mathbb{P}_0$ which will be of particular interest in obtaining explicit representations of minimal polynomials with respect to the max-norm.

***Theorem 3*** *Let $R$, $\rho_\nu$, $\varepsilon_j \in \{-1, +1\}$, $j = 1, \dots, \nu^*$ be given. Then there exist polynomials $p_n(x) = x^n + \cdots \in \mathbb{P}_n$ and $q_m \in \mathbb{P}_m$ with no common zero and satisfying the relations*

$$Rp_n^2 - Sq_m^2 = c\rho_\nu,$$

*where $c \in \mathbb{R}$, and*

$$(\sqrt{R}p_n)(w_j) = \varepsilon_j(\sqrt{S}q_m)(w_j)$$

*at the zeros $w_j (j = 1, \dots, \nu^*)$ of $\rho_\nu$, if and only if there are $m_1, \dots, m_{l-1} \in \mathbb{Z}$ such that*

$$\Re\Phi_p(\xi) = \Big(-\sum_{j=1}^{\nu^*} \varepsilon_j \nu_j \Re\Phi_p(\nu_j) + \frac{1}{2}\sum_{j=1}^{l-1} m_j a_{jp}\Big) \Big/ (2n - \nu + r), \tag{57}$$

$p = 1, \dots, l-1.$

**Proof** In proving Theorem 1 we have shown that relation (29) holds; together with the reality of $\exp \Phi_p$ this gives the necessity part of the theorem.

For the sufficiency part, consider the function

$$f(u) := R\big(\phi(u)\big)\,\big(\Omega_n(u) + \Omega_n(-u)\big), \tag{58}$$

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

where $\Omega_n(u)$ is defined in (28) after setting $\partial g_{(n)} = 0$ and (57) there. Since $f(u)$ is an even function, being automorphic with respect to the group $\Gamma$ by (11), (16), (28), and (57), it is a rational function of $x$. Furthermore, it follows from (28) on recalling that the $u_j$'s are the zeros of $R(\phi(u))$, that $f$ as a function of $x$ has its only singularity at $x = \infty$ which is a $(n+r)$-fold pole. Hence $f$ is a polynomial of degree $(n+r)$ which vanishes, by (58), at the zeros of $R$, and thus

$$\tilde{p}_n(x) := \Omega_n(u) + \Omega_n(-u)$$

is a polynomial of degree $n$ exactly. Now the proof proceeds in the same way as in [38, Theorem 2]. The assertion is proved. ■

From Theorem 3 we now obtain an explicit description of the polynomials which deviate least from zero on $E$ with respect to the max-norm and a weight function of the form $1/\sqrt{\rho}$, where $\rho$ is a polynomial with $\rho > 0$ on $E$, and which have a maximal number of extremal points on $E$.

***Corollary 2*** *Let $\rho_\nu \in \mathbb{P}$ with $\rho_\nu(x) > 0$ for $x \in E$ and let $n \in \mathbb{N}$ be such that $2n > \nu$. Then there exists a polynomial $p_n(x) = x^n + \cdots$ such that*

$$\max_{x\in E}\left|\frac{p_n(x)}{\sqrt{\rho_\nu(x)}}\right| = \min_{c_i\in\mathbb{R}}\max_{x\in E}\left|\frac{x^n + c_1x^{n-1} + \cdots + c_n}{\sqrt{\rho_\nu(x)}}\right| \tag{59}$$

*and all boundary points of $E$ are extremal points with*

$$\frac{p_n}{\sqrt{\rho}}(a_{2j}) = \frac{p_n}{\sqrt{\rho}}(a_{2j+1}), \quad \text{for } j = 1, \ldots, l-1,^{1} \tag{60}$$

*if and only if there exist $m_1, \ldots, m_{l-1} \in \mathbb{Z}$ such that*

$$\Re\Phi_p(\xi) = \Big(\frac{1}{2}\sum_{j=1}^{l-1} m_j a_{jp} - \sum_{j=1}^{\nu^*}\nu_j\Re\Phi_p(\nu_j)\Big)\Big/(2n-\nu), \tag{61}$$

*$p = 1, \ldots, l-1$. If the $\Phi_p(\xi)$'s are of the form (61) then the minimal polynomial $p_n$ in (59) is given by formula (30), with $\Omega_n$ given by (28) with $\partial g_{(n)} = r = 0$ and $\varepsilon_j = 1$ for $j = 1, \ldots, \nu^*$. In this case the minimum deviation is given by*

$$\max_{x\in E}\left|\frac{p_n(x)}{\sqrt{\rho_\nu(x)}}\right| = 2\tau^{(2n-\nu)/2}\prod_{j=1}^{\nu^*}\Big(\frac{\Omega(\xi,\nu_j)}{\Omega(\xi,-\nu_j)}\Big)^{\nu_j/2}\exp\Big(-\sum_{j=1}^{l-1}\frac{m_j}{2}\Phi_j(\xi)\Big).$$

**Proof** This follows from Corollary 2.9 in [34] and Theorem 3. ■

***Remark 1*** Analogous representations of the extremal polynomials with respect to the max-norm may be written not only for the case $R \equiv 1$, but also for other possible $R$.

[1] We take the opportunity to indicate that condition (60) was omitted in Corollary 1 from [38].

***Remark 2*** For $\rho_\nu \equiv 1$ the extremal polynomials will deviate least from zero on $E$ with respect to the max-norm and without weight. Hence the corollary gives in a certain sense the answer for the question from [16, p. 442]. Moreover, one can find the general case (without condition (60)) for the extremal polynomials in [31, Theorem 1] and [40, Theorem 2.3].

Let us mention that $p_n(x)/\sqrt{\rho_\nu(x)}$ was represented in another way in [30, Lemma 3], where conditions (61) were given in an essentially more complicated form.

## 3 Applications

Let us study once more the zeros of the polynomials orthogonal with respect to the weight function $R/h\rho_\nu$, $R/h\rho_\nu > 0$ on $\operatorname{int}(E)$. It is known (see, *e.g.* [34]) that the orthogonal polynomial $p_n$ has at most $l-1$ zeros in $\mathbb{R} \setminus E$. But almost nothing is known about the appearance of such zeros. Recently Suetin [49] has shown that there exists a subsequence $(n_k)$ such that all zeros of $(p_{n_k})$ accumulate on $E$. Here we show that for arbitrary given $m \in \{0, \dots, l-1\}$ there exists a subsequence $(n_j)$ such that $p_{n_j}$, $j = 1, 2, \dots$ has $m$ zeros in $\mathbb{C} \setminus E$ which do not accumulate to $E$ if $R \equiv 1$.

***Lemma 2*** **(Kronecker)** *Suppose that $\omega_1(\infty), \dots, \omega_{l-1}(\infty), 1$ are linearly independent over the rationals. Then for every $x_1, \dots, x_{l-1} \in \mathbb{R}$ and for every sequence $(\varepsilon_k)$, $\varepsilon_k \downarrow 0$, there exist a strictly monotonic subsequence $(q_k)$ of natural numbers and $l-1$ sequences of integers $(\tilde{m}_{k,j})$, $j = 1, \dots, l-1$, such that*

$$|2q_k\omega_j(\infty) - \tilde{m}_{k,j} - x_j| < \varepsilon_k, \quad j = 1, \dots, l-1, \ k = 1, 2, \dots. \tag{62}$$

**Proof** From the Kronecker approximation theorem (see, for example, [22, p. 23]) it follows that for any real numbers $y_1, \dots, y_{l-1}$ and for arbitrary $\varepsilon > 0$ and $C > 0$ it is possible to find an integer $q$ and integers $\tilde{m}_1, \dots, \tilde{m}_{l-1}$ such that

$$|2q\omega_j(\infty) - \tilde{m}_j - y_j| < \varepsilon, \quad j = 1, \dots, l-1,$$

where $|q| > C$ and the sign of $q$ can be chosen arbitrarily. Now let's take a sequence $\varepsilon_k \downarrow 0$. Then it follows by the Kronecker approximation theorem that for any real numbers $x_1, \dots, x_{l-1}$ there exist a strictly monotonic sequence of integers $(q_k)$ and $(l-1)$ sequences of integers $(\tilde{m}_{k,p})$, $p = 1, \dots, l-1$, such that for $k \in \mathbb{N}$ (62) holds. ∎

***Theorem 4*** *Assume that the numbers $\omega_1(\infty), \dots, \omega_l(\infty)$ are linearly independent over the rationals. Suppose also that $R/h\rho > 0$ on $\operatorname{int}(E)$, $\varepsilon = (1, 1, \dots, 1)$ and $r_j < 2$, $j = 0, \dots, l-1$. Then for any $m \in \mathbb{Z}$, $0 \le m \le l-1$, there exists a subsequence $(p_{n_k})$ of polynomials orthogonal with respect to the weight $R/h\rho$ such that $\mathbb{Z}(p_{n_k}, [a_1, a_{2l}] \setminus E) = m$, $k \in \mathbb{N}$. In particular, for $R \equiv 1$ the $m$ zeros outside of $E$ do not accumulate to $E$.*

**Proof** We claim firstly that for any point $\beta = (\beta_1, \dots, \beta_{l-1})$, $\beta_j \in \delta_j(o_{j-1} + iR_{j-1}, o_j - iR_j)$, $j = 2, \dots, l-1$, $\beta_1 \in \delta_1(0, o_1 - iR_1)$, $\delta_j \in \{-1, +1\}$, $j = 1, \dots, l-1$, and for any neighbourhood of $\beta$ it is possible to find a point $\beta^{(0)}$ in the neighbourhood of $\beta$ and a sequence of integers $(n_k)$ such that the polynomials $g_{(n_k)}$ which are by Theorem 2 associated to the polynomials $p_{n_k}$ have zeros at the points $\phi(\beta_j^{(k)})$, $j = 1, \dots, l-1$, $\lim_{k\to\infty} \beta_j^{(k)} = \beta_j^{(0)}$, $j = 1, \dots, l-1$, and relation (48) holds for $i_n = n_k$.

Indeed, let us put

$$z_p = -2(r - \nu - l + 1)\Re\Phi_p(\xi) + 2\sum_{j=1}^{l-1} \Re\Phi_p(\beta_j) - 2\sum_{j=1}^{\nu^*} \nu_j \Re\Phi_p(\nu_j), \quad p = 1, \dots, l-1. \tag{63}$$

The Jacobian of (63) considered as a system of equations with respect to $\beta$ has the form

$$\begin{vmatrix} \theta(\beta_1, c_1) & \cdots & \theta(\beta_{l-1}, c_1) \\ \vdots & & \vdots \\ \theta(\beta_1, c_{l-1}) & \cdots & \theta(\beta_{l-1}, c_{l-1}) \end{vmatrix}. \tag{64}$$

Since the functions $\theta(z, c_p)$ are analytic and linearly independent [13, p. 62–63], the Jacobian (64) may have only a finite number of zeros as function of $\beta_1$ with fixed $\beta_2, \dots, \beta_{l-1}$. Hence for any neighbourhood of $\beta$ it is possible to find a point $\beta^{(0)}$ in that neighbourhood such that for the system of equations

$$z_p^{(0)} = -2(r - \nu - l + 1)\Re\Phi_p(\xi) + 2\sum_{j=1}^{l-1} \Re\Phi_p(\beta_j^{(0)}) - 2\sum_{j=1}^{\nu^*} \nu_j \Re\Phi_p(\nu_j), \quad p = 1, \dots, l-1, \tag{65}$$

all conditions of the Implicit Function Theorem are satisfied at the point $\beta^{(0)}$. Now let $\Phi(\xi)$, $\omega$ denote the vectors

$$\bigl(\Re\Phi_p(\xi)\bigr)_{p=1}^{l-1}, \quad \bigl(\omega_j(\infty)\bigr)_{j=1}^{l-1}$$

and $A = (a_{pq}/2)$. Since $A$ differs from the period matrix of the corresponding hyperelliptic Riemann surface (see [13, p. 66] by the sign only, $A$ is real and nonsingular (see, for instance, [52, Theorem X.35]) and thus we get by (56)

$$A^{-1}\Phi(\xi) = \omega.$$

Then it follows by Kronecker's Lemma that for any real numbers $y_1, \dots, y_{l-1}$ and for every sequence $(\varepsilon_k)$, $\varepsilon_k \downarrow 0$, there exist a strictly monotonic subsequence $(q_k)$ of natural numbers and $l-1$ sequences of integers $(\tilde{m}_{k,j})$, $j = 1, \dots, l-1$, such that

$$|2q_k\omega_j(\infty) - \tilde{m}_{k,j} - y_j| < \varepsilon_k, \quad j = 1, \dots, l-1, \ k = 1, 2, \dots. \tag{66}$$

Multiplying (66) by $A$, one obtains that there exist a strictly monotonic sequence of integers $(q_k)$ and $(l-1)$ sequences of integers $(\tilde{m}_{k,p})$, $p = 1, \dots, l-1$, such that for $k \in \mathbb{N}$ the relations

$$(67) \qquad 4n_k \Re\Phi_p(\xi) = \sum_{j=1}^{l-1} m_j^{(k)} a_{jp} + z_p^{(0)} + \varepsilon_{k,p}, \quad p = 1, \dots, l-1,$$

with $|\varepsilon_{k,p}| < \varepsilon_k$, $p = 1, \dots, l-1$, hold.

By applying the Implicit Function Theorem to the system (65) we can find for $\varepsilon_k$-neighbourhood of $z_p^{(0)}$ a point $\beta^{(k)}$ in a neighbourhood of the point $\beta^{(0)}$ such that

$$(68) \qquad 2(r-\nu-l+1)\Re\Phi_p(\xi) - 2\sum_{j=1}^{l-1} \Re\Phi_p(\beta_j^{(k)}) + 2\sum_{j=1}^{\nu^*} \nu_j \Re\Phi_p(\nu_j) + z_p^{(0)} + \varepsilon_{k,p} = 0, \quad p = 1, \dots, l-1, \ k \geq k_0.$$

Hence (67) becomes

$$(69) \qquad 2(2n_k + r - \nu - l + 1)\Re\Phi_p(\xi) - 2\sum_{j=1}^{l-1} \Re\Phi_p(\beta_j^{(k)}) + 2\sum_{j=1}^{\nu^*} \nu_j \Re\Phi_p(\nu_j) - \sum_{j=1}^{l-1} m_j^{(k)} a_{jp} = 0, \quad p = 1, \dots, l-1, \ k \geq k_0,$$

and thus by Theorem 2 the first claim is proved.

Now let's put $\delta_j = -1$, $j = 1, \dots, \lambda$ and $\delta_j = 1$, $j = \lambda + 1, \dots, l-1$. Then by Theorem 2

$$\mathcal{Z}(p_{n_k}, E) = \sum_{j=0}^{l-1} k_j^{(n_k)} = \frac{1}{2}\sum_{j=0}^{l-1} m_j^{(n_k)} - \frac{r}{2} = n_k - \lambda.$$

Concerning the last statement one obtains putting $R \equiv 1$ in (42) that

$$p_n^2 - Hq_m^2 = \rho_\nu g_{(n)}.$$

Dividing the equation by $G_n$ we obtain

$$(70) \qquad |p_n|^2/G_n = |\rho_\nu \hat{g}_{(n)}| \text{ at the boundary points of } E,$$

where $\hat{g}_{(n)}$ is monic. By the claim proved at the beginning the zeros of $\hat{g}_{(n)}$ have no accumulation points in $E$. Hence the right hand side of (70) is bounded from below by a constant for all $k \in \mathbb{N}$, which gives the last statement. ■

***Corollary 3*** *Let the functional $\Psi_{R,\rho_\nu}$ be positive definite, and let $\omega_1(\infty), \dots, \omega_l(\infty)$ be linearly independent over the rationals. Then for any $m, 0 \leq m \leq l-1$, it is*

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

*possible to find a union of at least m gaps* $K_m = \bigcup_{j=1}^{m}[a_{2i_j}, a_{2i_j+1}]$ *such that there exists a subsequence of the sequence of the diagonal Padè approximants to the function*

$$f(z) = \left(\frac{R(z)}{\sqrt{H(z)}} - Y(z)\right)/\rho_\nu(z) \tag{71}$$

*which does not converge uniformly inside any domain* $\Omega \subset \mathbb{C} \setminus E$ *with* $\Omega \supset \operatorname{int}(K_m)$ *and converges to* $f$ *uniformly on compact subsets of* $\mathbb{C} \setminus (E \cup K_m)$.

**Proof** From Theorem 4 it follows that for any $m, 0 \le m \le l-1$, it is possible to find a subsequence $\{n_k\}$ such that all polynomials $p_{n_k}$, orthogonal with respect to $\Psi_{R,\rho_\nu}$, have exactly $m$ zeros in $\bigcup_{i=1}^{l-1}(a_{2i}, a_{2i+1})$. Moreover, since $p_{n_k}$ has at most one zero in each gap $(a_{2i}, a_{2i+1})$, $i = 1, \dots, l-1$, it is possible to find at least $m$ intervals $(a_{2i_j}, a_{2i_j+1})$, $j = 1, \dots, m$ and a subsequence $\Lambda$ of $\{n_k\}$ such that $p_n, n \in \Lambda$, has exactly one zero in each gap $(a_{2i_j}, a_{2i_j+1})$, $j = 1, \dots, m$ and no other zeros in $\mathbb{C} \setminus E$.

Now by [34, (4.9)] for any $n \ge n_0$ the rational function $\frac{p_n^{[1]}(z)}{p_n(z)}$ is just the $[n/n]$ diagonal Padé approximant of $f$. The function $f$ is, by definition of $Y$, holomorphic and single-valued in $\bar{\mathbb{C}} \setminus E$, and all assumptions of [48, Theorem 1.7] are satisfied, hence the sequence $\{\frac{p_n^{[1]}(z)}{p_n(z)}\}$ converges in capacity to $f$ in $\bar{\mathbb{C}} \setminus E$. Since for $n \in \Lambda$ all functions $\frac{p_n^{[1]}(z)}{p_n(z)}$ are holomorphic in $\bar{\mathbb{C}} \setminus (E \cup K_m)$, meromorphic in $\bar{\mathbb{C}}$ and have exactly one pole in $(a_{2i_j}, a_{2i_j+1})$, $j = 1, \dots, m$ (recall that $p_n^{[1]}$ and $p_n$ have no common zeros), the assertion follows from Gonchar's lemma [20, Lemma 1]. ■

***Remark 3*** For the case $m = 0$ the corollary was proved by different method in [49]; in fact it means that the Baker-Gammel-Wills conjecture holds for functions of the kind (71). But let us point out that it was very recently announced by D. S. Lubinsky in a talk at the conference "Computational methods and function theory 2001", University of Aveiro, Portugal, June 25–29, 2001, that the conjecture fails for a Rogers-Ramanujan continued fraction (compare also [28]).

Next we give explicit representations for the recurrence coefficients of polynomials orthogonal with respect to the functional $\Psi_{R,\rho,\varepsilon}$.

Combining Theorem 1 and results of the second author [34] we get the following theorem.

***Theorem 5*** *Let* $R$, $\rho_\nu, \varepsilon_j \in \{-1, +1\}$, $j = 1, \dots, \nu^*$ *be given, let* $(i_n)$ *be the sequence of basic integers with respect to* $\Psi_{R,\rho_\nu,\varepsilon}$, *and suppose that* $i_{n+1} + i_n + r \ge \nu + l + 1$. *Then the following propositions hold.*

(a) *We have* $\Psi_{R,\rho_\nu,\varepsilon}(x^{i_{n+1}-1} p_{i_n}) = \frac{1}{2} G_{i_n}$, *where* $G_{i_n}$ *is given by (33).*
(b) *The recurrence coefficients of the* $p_{i_n}$*'s are given explicitly by*

$$\alpha_{i_{n+1}} = \frac{1}{2}(a_1 + \cdots + a_{2l}) - \sum_{k=1}^{l-1} \phi(b_k^{(i_n)}) \quad \textit{if } i_{n+1} = i_n + 1, \tag{72}$$

*and*

$$(73)\quad \lambda_{i_{n+2}} = \tau^2 \prod_{j=1}^{\partial g_{(i_n)}} \frac{\Omega(\xi, b_j^{(i_n)})}{\Omega(\xi, -b_j^{(i_n)})} \prod_{j=1}^{\partial g_{(i_{n+1})}} \frac{\Omega(\xi; -b_j^{(i_{n+1})})}{\Omega(\xi, b_j^{(i_{n+1})})} \exp \sum_{j=1}^{l-1} (m_j^{(i_n)} - m_j^{(i_{n+1})}) \Phi_j(\xi),$$

*where the $b_j^{(i_n)}$'s and $m_j^{(i_n)}$'s are given by (29).*

(c) *For $i_{n+1} \geq \nu + 1$ the polynomials $q_{i_m}$, with $i_m = i_n + r - l$, given by (31) are orthogonal with respect to $\Psi_{S,\rho_\nu,\varepsilon}$ and the recurrence coefficients of $q_{i_m}$ denoted by $\tilde{\alpha}_{i_m}$ for $i_m = i_{m-1} + 1$ and $\tilde{\lambda}_{i_m}$ are given by*

$$(74)\qquad \tilde{\alpha}_{i_m} = \alpha_{i_n} \quad \textit{and} \quad \tilde{\lambda}_{i_m} = \lambda_{i_n}.$$

**Proof** In view of Theorem 3 in [34] it follows that there is a polynomial $q_{i_m}$ such that

$$(75)\qquad Rp_{i_n}^2 - Sq_{i_m}^2 = \rho_\nu g_{(i_n)}$$

where $g_{(i_n)} \in \mathbb{P}_{i_n+l-i_{n+1}}$ has leading coefficient $2\Psi_{R,\rho_\nu,\varepsilon}(x^{i_{n+1}-1}p_{i_n})$ and $(\sqrt{R}p_{i_n})(w_j) = \varepsilon_j(\sqrt{S}q_{i_m})(w_j)$ at the zeros $w_j$ of $\rho_\nu$, which, by Theorem 1, gives the statements (a) and (c).

Applying the functional $\Psi_{R,\rho_\nu,\varepsilon}$ to both sides of (75) we get in conjunction with (1), (2) and [34, Lemma 1(c)]

$$\int_E \frac{p_{i_n}^2 R}{\rho_\nu h}\, dx = -\int_E \frac{q_{i_m}^2 S}{\rho_\nu h}\, dx.$$

Thus in view of $\alpha_{i_{n+1}} = \Psi_{R,\rho_\nu,\varepsilon}(xp_{i_n}^2)/\Psi_{R,\rho_\nu,\varepsilon}(p_{i_n}^2)$ and from part (c)

$$(76)\qquad \int_E x\frac{p_{i_n}^2 R}{\rho_\nu h}\, dx = -\int_E x\frac{q_{i_m}^2 S}{\rho_\nu h}\, dx.$$

Multiplying relation (75) by $x$ and then applying the functional $\Psi_{R,\rho_\nu,\varepsilon}$ equation (72) follows with the help of (76) and [34, Lemma 1(a)]. Indeed we just demonstrated that

$$(77)\qquad g_{(i_n)}(x) = G_{i_n}\left(x^{l-1} + \left(\alpha_{i_{n+1}} - \frac{1}{2}(a_1 + a_2 + \cdots + a_{2l})\right)x^{l-2} + \cdots\right)$$

if $g_{(i_n)} \in \mathbb{P}_{l-1} \setminus \mathbb{P}_{l-2}$. From the fact (see (37)) that $\pm b_j^{(i_n)}$, $j = 1, \ldots, \partial g_{(i_n)}$ are the zeros of $g_{(i_n)}\big(\phi(u)\big)$ it follows that

$$-\alpha_{i_{n+1}} + \frac{1}{2}(a_1 + a_2 + \cdots + a_{2l}) = \sum_{j=1}^{\partial g_{(i_n)}} \phi(b_j^{(i_n)}),$$

which is assertion (72). Concerning $\lambda_{i_{n+2}}$ it is enough to note that by (5)

$$(78)\qquad \lambda_{i_{n+2}} = \frac{\Psi_{R,\rho_\nu,\varepsilon}(x^{i_{n+2}-1}p_{i_{n+1}}^2)}{\Psi_{R,\rho_\nu,\varepsilon}(x^{i_{n+1}-1}p_{i_n}^2)} = \frac{G_{i_{n+1}}}{G_{i_n}}.$$

Now formula (73) follows immediately from (78) and (33). ■

For $l = 2$ the analogue of Theorem 5 is given in [38, Theorem 3].
Concerning the sequence of basic integers we have the following corollary.

***Corollary 4*** *Let $n_0$ be such that $i_{n+1} + i_n + r \geq \nu + l + 1$ for $n \geq n_0$. Then we have, for $n \geq n_0$,*

(a) $\Psi_{R,\rho_\nu,\varepsilon}(x^{l-2} p_{i_n}^2) = 0$, *that is,* $i_{n+1} = i_n + l$, *if and only if there exist* $m_j^{(n)} \in \mathbb{Z}$, $j = 1, \ldots, l-1$, *such that*

$$-\sum_{j=1}^{\nu^*} \varepsilon_j \nu_j \Re\Phi_p(\nu_j) + \frac{1}{2}\sum_{j=1}^{l-1} m_j^{(n)} a_{jp} = (2i_n - \nu + r)\Re\Phi_p(\xi),$$

$p = 1, \ldots, l-1$.

(b) *If at least one from the quantities $\omega_j(\infty)$ is irrational then there exists at most one basic integer $i_{n^*+1}$, with $n^* \geq n_0$, such that $i_{n^*+1} = i_{n^*} + l$.*

**Proof** (a) In view of Theorem 3(a) in [34], $\Psi_{R,\rho_\nu,\varepsilon}(x^{l-2} p_{i_n}^2) = 0$ is equivalent to $g_{(i_n)}(x)$ being constant, which, by Theorem 3, establishes part (a).

(b) Suppose that there is another basic integer $i_{n^{**}}$ such that $i_{n^{**}+1} = i_{n^{**}} + l$. Then it follows from (a) that

$$2(i_{n^{**}} - i_{n^*})\Phi_p(\xi) = \sum_{j=1}^{l-1} (m_j^{(i_{n^{**}})} - m_j^{(i_{n^*})}) a_{jp}, \quad p = 1, \ldots, l-1,$$

or in vector form

$$(i_{n^{**}} - i_{n^*})\Phi(\xi) = -(\bar{m}^{(i_{n^{**}})} - \bar{m}^{(i_{n^*})})A. \tag{79}$$

Hence by multiplying (79) by $A^{-1}$ and with the help of Corollary 1 we obtain a contradiction. ■

As another consequence of Theorem 5 we obtain a result on the periodicity of the recurrence coefficients given in [35] in a different form.

***Corollary 5*** *Let $n_0$ be defined as in Corollary 4. Suppose that the $\omega_j(\infty)$'s are of the form*

$$\omega_j(\infty) = \frac{m_j}{N}, \quad m_j, N \in \mathbb{N}.$$

*Then the recurrence coefficients of the polynomials orthogonal with respect to $\Psi_{R,\rho_\nu,\varepsilon}$ have period $N$ for $n \geq n_0$, that is*

$$\lambda_{i_{n+1}} = \lambda_{jN+i_{n+1}}$$

*and*

$$\alpha_{i_n} = \alpha_{jN+i_n} \quad \textit{if } i_{n+1} = i_n + 1.$$

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

**Proof** First of all it follows from the condition of the corollary and from Corollary 1 that the relations

$$2N\Re\Phi_p(\xi) = \sum_{j=1}^{l-1} m_j a_{jp}, \quad p = 1, \ldots, l-1, \tag{80}$$

hold.

Now straightforward calculations using (29) show that $b_p^{(k)}$, $p = 1, \ldots, l-1$ satisfy (29) for $k = i_n + jN$, $j \in \mathbb{N}$ with $m_q^{(k)} = m_q^{(i_n)} + jm_q$, $q = 1, \ldots, l-1$. Hence by (72) and (73) we get the assertion. ∎

Naturally the question arises what can we say about the behaviour of the recurrence coefficients when the harmonic measures are not rational. We consider the case $R/h\rho_\nu > 0$ on $\operatorname{inte}(E)$ and $\varepsilon = (1, 1, \ldots, 1)$ only. Then we have for $n \in \mathbb{N}$ that $i_n = n$ and by (26) that $g_{(i_n)}$ has exactly one zero in each $[a_{2i}, a_{2i+1}]$, $i = 1, \ldots, l-1$, which implies that the $b_j^{(i_n)}$'s are on the imaginary axis.

***Theorem 6*** *Let $(p_n)$ be orthogonal on $E$ with respect to a weight function of the form $wR/h$ which is positive on* $\operatorname{inte}(E)$ *and such that $w \in C(E)$ has no zeros on $E$. Furthermore let $(\alpha_n), (\lambda_n)$ be the recurrence coefficients of $(p_n)$. If $\omega_1(\infty), \ldots, \omega_l(\infty)$ are linearly independent over the rationals, then the sequences $(\lambda_n), (\alpha_n)$ have nondegenerate intervals as the sets of limit points.*

**Proof** Let us consider firstly the case $w = 1/\rho_\nu$, where $\rho_\nu$ is a polynomial as before. Since $\exp \Phi_p(\xi)$, $\exp \Phi_p(b_j^{(n)})$ are real, and $\exp \Phi_p(\nu_j)$ are pairwise complex conjugate we can rewrite the condition (29) taking into account that $\Phi_p$ are uniquely defined only up to an additive constant of the type $2n\pi i$, $n \in \mathbb{Z}$, as follows:

$$(2n - \nu - l + r + 1)\Re\Phi_p(\xi) + \sum_{j=1}^{\nu^*} \nu_j \Re\Phi_p(\nu_j) - \sum_{j=1}^{l-1} \frac{m_j^{(n)}}{2} a_{jp} = \sum_{j=1}^{l-1} \Re\Phi_p(b_j^{(n)}), \tag{81}$$

$p = 1, \ldots, l-1$.

Now let us study the behaviour of the $b_j^{(n)}$'s and the $m_j^{(n)}$'s if the first term at the left hand side approaches given values $x_p \in \mathbb{R}$, $p = 1, \ldots, l-1$. More precisely, for given $x_1, \ldots, x_{l-1} \in \mathbb{R}$ let $(q_k)$, $(\tilde{m}_{k,j})_{k\in\mathbb{N}}$, $j = 1, \ldots, l-1$ and $\varepsilon_{k,p}$, $p = 1, \ldots, l-1$, be the values from Kronecker's Lemma. We will show that

$$b_j^{(q_k)} \to b_j^0(x), b_j^{(q_k+1)} \to b_j^1(x), m_j^{(q_k+1)} - m_j^{(q_k)} \to m_j(x), \tilde{m}_{k,j} - m_j^{(q_k)} \to \tilde{m}_j(x) \tag{82}$$

as $k \to \infty$. Indeed, putting $n = q_k$ and $n = q_k + 1$ in (29) we find

$$\begin{aligned}\exp\Bigl(&2(r - \nu - l + 1)\Phi_p(\xi) - 2\sum_{j=1}^{l-1} \Phi_p(b_j^{(q_k)}) + 2\sum_{j=1}^{\nu^*} \nu_j \Phi_p(\nu_j) + x_p \\ &+ \sum_{j=1}^{l-1} (\tilde{m}_{k,j} - m_j^{(q_k)}) a_{jp} + 2\varepsilon_{k,p}\Bigr) = 1, \quad p = 1, \ldots, l-1,\end{aligned} \tag{83}$$

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

and

$$(84)\qquad \exp\Big(2(r-\nu-l+2)\Phi_p(\xi)-2\sum_{j=1}^{l-1}\Phi_p(b_j^{(q_k+1)})+2\sum_{j=1}^{\nu^*}\nu_j\Phi_p(\nu_j)$$
$$+x_p+\sum_{j=1}^{l-1}(\tilde m_{k,j}-m_j^{(q_k+1)})a_{jp}+2\varepsilon_{k,p}\Big)=1,\quad p=1,\dots,l-1,$$

From those equations it follows that

$$(85)\qquad \exp\Big(2\Phi_p(\xi)-2\sum_{j=1}^{l-1}\big(\Phi_p(b_j^{(q_k+1)})-\Phi_p(b_j^{(q_k)})\big)$$
$$-\sum_{j=1}^{l-1}(m_j^{(q_k+1)}-m_j^{(q_k)})a_{jp}\Big)=1,\quad p=1,\dots,l-1.$$

Since $b_j^{(n)}\in\pm[o_j+iR_j,o_{j+1}-iR_{j+1}]$, $j=1,\dots,l-2$, $b_1^{(n)}\in[-o_1+iR_1,o_1-iR_1]$, we can find subsequences from $(q_k)$ (we keep the notation) such that (82) holds.

Next let us take the limits in (83) and (85) which gives

$$(86)\qquad \exp\Big(2(r-\nu-l+1)\Phi_p(\xi)-2\sum_{j=1}^{l-1}\Phi_p\big(b_j^0(x)\big)+2\sum_{j=1}^{\nu^*}\nu_j\Phi_p(\nu_j)+x_p$$
$$+\sum_{j=1}^{l-1}\tilde m_j(x)a_{jp}\Big)=1,\quad p=1,\dots,l-1,$$

and

$$(87)\qquad \exp\Big(2\Phi_p(\xi)-2\sum_{j=1}^{l-1}\Big(\Phi_p\big(b_j^1(x)\big)-\Phi_p(b_j^0(x))\Big)-\sum_{j=1}^{l-1}m_j(x)a_{jp}\Big)=1,$$

$p=1,\dots,l-1$. Moreover taking limits in (73) we find with help of (82)

$$(88)\qquad \lim_{k\to\infty}\lambda_{q_k+2}=\tau^2\prod_{j=1}^{l-1}\frac{\Omega\big(\xi,b_j^0(x)\big)\,\Omega\big(\xi,-b_j^1(x)\big)}{\Omega\big(\xi,-b_j^0(x)\big)\,\Omega\big(\xi,b_j^1(x)\big)}\exp-\sum_{j=1}^{l-1}m_j(x)\Phi_j(\xi).$$

As it was mentioned above, system (86) is nothing else than the Jacobi inversion problem with respect to $\phi\big(b_j^0(x)\big)$. Since $\phi\big(b_j^0(x)\big)\in[a_{2j},a_{2j+1}]$, the relation

$$\phi\big(b_j^0(x)\big)\neq\phi\big(b_k^0(x)\big),\quad j\neq k,x\in\mathbb{R},$$

holds. Hence by [33, Lemma], $\phi\big(b_j^0(x)\big)$ are analytic functions, and the right-hand side of (88) is a meromorphic function with respect to the variables $x_1,\dots,x_{l-1}$, which is non-constant. Of contraries let it be constant. Consider the function

$$f(z,x)=\frac{\Omega^2(z,0)}{\Omega^2(z,-\xi)}\prod_{j=1}^{l-1}\frac{\Omega\big(z,b_j^0(x)\big)\,\Omega\big(z,-b_j^1(x)\big)}{\Omega\big(z,-b_j^0(x)\big)\,\Omega\big(z,b_j^1(x)\big)}\exp-\sum_{j=1}^{l-1}m_j(x)\Phi_j(z).$$

The function $f$ is automorphic with respect to the group $\Gamma$ as a function of $z$ for any $x \in \mathbb{R}^{l-1}$. Indeed, the condition of automorphity for $f$ is

$$\exp\left(2\Phi_p(-\xi) - 2\Phi_p(0) + 2\sum_{j=1}^{l-1}\left(\Phi_p\left(b_j^1(x)\right) - \Phi_p\left(b_j^0(x)\right)\right) + \sum_{j=1}^{l-1} m_j(x)a_{jp}\right) = 1,$$

$p = 1, \dots, l-1$, which is the same as (87). By assumption $f(\xi, x)$ is a constant function with respect to $x$. Hence

$$\frac{\partial f(z,x)}{\partial x_k} = 0 \tag{89}$$

for $z = \xi, k = 1, \dots, l-1$. All functions $\frac{\partial f(z,x)}{\partial x_k}$, $k = 1, \dots, l-1$ are still automorphic with respect to the group $\Gamma$ as functions of $z$. We can take a point $x^0$ such that for a neighbourhood $U$ of $x^0$ $m_j(x)$ would be constant for $x \in U$, $j = 1, \dots, l-1$. Let us observe also that for any $x^1, x^2 \in U$

$$\frac{f(z,x^1)}{f(z,x^2)} = \prod_{j=1}^{l-1} \frac{\Omega\left(z, b_j^0(x^1)\right)\Omega\left(z, -b_j^1(x^1)\right)\Omega\left(z, -b_j^0(x^2)\right)\Omega\left(z, b_j^1(x^2)\right)}{\Omega\left(z, -b_j^0(x^1)\right)\Omega\left(z, b_j^1(x^1)\right)\Omega\left(z, b_j^0(x^2)\right)\Omega\left(z, -b_j^1(x^2)\right)}$$

is automorphic with respect to $\Gamma$ so

$$\frac{f(-z,x^1)}{f(-z,x^2)} = \frac{f(z,x^2)}{f(z,x^1)} \tag{90}$$

(since $\Omega(-z, y) = \Omega(z, -y)$). Hence

$$\begin{aligned}\frac{\partial f(-z,x)}{\partial x_k}\Big/ f(-z,x) &= \lim_{\Delta x_k \to 0} \frac{f(-z, x+\Delta x_k) - f(-z,x)}{\Delta x_k \cdot f(-z,x)} \\ &= \frac{f(z,x) - f(z, x+\Delta x_k)}{\Delta x_k \cdot f(z, x+\Delta x_k)} = -\frac{\partial f(z,x)}{\partial x_k}\Big/ f(z,x).\end{aligned}$$

Thus any linear combination

$$G(z,x) = \left(B_1 \frac{\partial f(z,x)}{\partial x_1} + \dots + B_{l-1}\frac{\partial f(z,x)}{\partial x_{l-1}}\right)\Big/ f(z,x)$$

is an odd automorphic function of $z$. Moreover, because of (90) and automorphity of $f(z,x^2)/f(z,x^1)$ we have

$$\frac{f(A_p,x^2)}{f(A_p,x^1)} = \frac{f(A_p',x^1)}{f(A_p',x^2)} = \frac{f(A_p',x^2)}{f(A_p',x^1)}$$

hence (neither $A_p$ nor $A_p'$ are zeros or poles of $f$) $f(A_p,x^2) = \pm f(A_p,x^1)$ and because of the continuity, $f(A_p,x^1) = f(A_p,x^2) = f(A_p',x^1) = f(A_p',x^2)$. By the same reason $f(\infty,x^1) = f(\infty,x^2)$.

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

Thus $G(z,x)$ is an automorphic function of $z$ with zeros $A_1,\dots,A_{2l-2},0,\infty$ (let us recall the usual convention about zeros and poles on the boundary of the fundamental domain of a discontinuous group) and with poles $\pm b_j^0(x), \pm b_j^1(x)$, $j=1,\dots,l-1$.

Now let $x$ be fixed. Taking any $l-2$ points $z_1,\dots,z_{l-2}$ we can find constants $B_1,\dots,B_{l-1}, B_1^2+\cdots+B_{l-1}^2\neq 0$ such that $G(z_j,x)=0$, $j=1,\dots,l-2$, and because of oddness $G(-z_j,x)=0$, $j=1,\dots,l-2$. Moreover, by (89), $G(\pm\xi,x)=0$. Finally the automorphic function $G(z,x)$ has at most $4(l-1)$ poles and at least $4l-2$ zeros in the fundamental domain $\mathfrak{T}$, which is a contradiction. So the first part of the statement (about the sequence $(\lambda_n)$) is proved.

By the same considerations as in the proof of Theorem 4 one can prove that for any point $\beta$ it is possible to find in any neighbourhood of $\beta$ a point $\beta^y$ such that $\phi(\beta_j^y)$, $j=1,\dots,l-1$ are the zeros of a polynomial $g_{(n)}$ which is associated by Theorem 1 to the orthogonal polynomial $p_n$. Hence by Theorem 5(b)

$$\alpha_n=\frac{1}{2}(a_1+\cdots+a_{2l})-\sum_{j=1}^{l-1}\phi(\beta_j^y).$$

Since the point $\beta$ was arbitrary we proved that the possible values of $\alpha_n$ have as a limit set the interval

$$\begin{aligned}\mathfrak{A}&=\Big\{\frac{1}{2}(a_1+\cdots+a_{2l})-\sum_{j=1}^{l-1}x_j : x_k\in[a_{2k},a_{2k+1}], k=1,\dots,l-1\Big\}\\&=\Big[(a_1+a_{2l})/2-\sum_{j=1}^{l-1}(a_{2j+1}-a_{2j})/2,\ (a_1+a_{2l})/2+\sum_{j=1}^{l-1}(a_{2j+1}-a_{2j})/2\Big].\end{aligned}$$

To prove the assertion for the case of general $w$ one has to repeat the considerations used in the proof of Theorem 4 from [38]. ■

***Remark 4*** The quasiperiodicity of the sequences $(\alpha_n)$, $(\lambda_n)$ follows easily from [18, Theorem 4.14], [34, Theorem 5(c)] and from the definition of the associated polynomials of any order. Using this property it could be possible to prove Theorem 6 in a shorter way. But let us point out that one needs to prove also that the "quasiperiods" are the harmonic measures $\omega_1(\infty),\dots,\omega_{l-1}(\infty)$ and in particular that the function which corresponds in the theory of Riemann's theta functions to the function $f(z,x)$ in the above proof is non-constant.

Let us note that the assumption of Theorem 6 is stronger than the condition of Corollary 5(b). That situation differs from the case $l=2$, when analogues of Theorem 6 and Corollary 5(b) were established in [38].

***Remark 5*** Let us note that a preliminary announcement of the results of the paper given in abstracts of the conference held in Kazan, September 1999, contains the formulation of Theorem 6 with different (wrong) conditions on $E$ and should be changed according to this paper.

## Appendix

Proof of (20). From [45, Theorem I] it follows that for a positive definite functional $\Psi_{\rho,\nu}$ one has the equality

$$\operatorname{cap}(E) = \lim_{n\to\infty} \sqrt[2n]{\Psi_{\rho,\nu}(p_n^2)}.$$

Hence by Theorem 5(a) and by (33),

$$\operatorname{cap}(E) = \lim_{n\to\infty} \tau \cdot \exp\Big(-\sum_{j=1}^{l-1} \frac{m_j^{(n)}}{2n} \Phi_j(\xi)\Big).$$

Now taking into account Theorem 2 and Corollary 1 the assertion is proved. ■

Proof of (21). Since by the definition of $T_j$ we have $T_j(u) = \bar{u}$ for $u \in \partial K_j$, $j = 1, \dots, l-1$, hence

$$\text{(91)}\quad \frac{\Omega\big(T_j(u), -\xi\big)}{\Omega\big(T_j(u), \xi\big)} \exp \sum_{k=1}^{l-1} \omega_k(\infty)\Phi_k\big(T_j(u)\big) = \frac{\Omega(\bar{u}, -\xi)}{\Omega(\bar{u}, \xi)} \exp \sum_{j=1}^{l-1} \omega_j(\infty)\Phi_j(\bar{u}).$$

Using the properties $\exp \Phi_j(-\bar{u}) = \overline{\exp \Phi_j(u)}$ and $\Omega(-\bar{u}, y) = -\overline{\Omega(u, -\bar{y})}$, which can be obtained in the same way as, for instance, (39), gives with the help of (41)

$$\text{(92)}\qquad \frac{\Omega(\bar{u}, -\xi)}{\Omega(\bar{u}, \xi)} \exp \sum_{k=1}^{l-1} \omega_k(\infty)\Phi_k(\bar{u}) = \overline{\left(\frac{\Omega(u, -\xi)}{\Omega(u, \xi)} \exp \sum_{j=1}^{l-1} \omega_j(\infty)\Phi_j(u)\right)^{-1}}.$$

On the other hand the left-hand side of (91) can be rewritten with the help of (13), (12) and (19) as

$$\text{(93)}\qquad \frac{\Omega(u, -\xi)}{\Omega(u, \xi)} \exp \sum_{j=1}^{l-1} \omega_j(\infty)\Phi_j(u) \cdot \exp\Big(2\Phi_j(\xi) - \sum_{k=1}^{l-1} \omega_k(\infty) a_{kj}\Big)$$
$$= \frac{\Omega(u, -\xi)}{\Omega(u, \xi)} \exp \sum_{j=1}^{l-1} \omega_j(\infty)\Phi_j(u).$$

Combining (91)–(93) gives

$$\text{(94)}\qquad \left|\frac{\Omega(u, -\xi)}{\Omega(u, \xi)} \exp \sum_{j=1}^{l-1} \omega_j(\infty)\Phi_j(u)\right| = 1, \quad u \in \partial K_j,\ j = 1, \dots, l-1.$$

Analogously one can check (94) for $u \in \mathbb{R}$. Note, that the function

$$\frac{\Omega(u, -\xi)}{\Omega(u, \xi)} \exp \sum_{j=1}^{l-1} \omega_j(\infty)\Phi_j(u)$$

https://doi.org/10.4153/CJM-2003-024-6 Published online by Cambridge University Press

is not single-valued, but its modulus is single-valued, and it has in $G(K_1, \dots, K_{l-1})$ the only simple pole at the point $\xi$, which corresponds to $\infty$. Hence (21) follows. Of course, (20) can be deduced from (21), (19) and the well-known connection between Green's function and the logarithmic capacity. ∎

***Remark 6*** During the preparation of the paper we have learned about the recent work [16], where the capacity for three intervals is given in an absolutely different form. We have learned also about the paper [44], where the Green function for the complement of a union of disjoint closed intervals was studied with the help of the Schwarz-Christoffel map.

## References


[1] N. H. Abel, *Über die Integration der differential-Formeln $\frac{\rho dx}{\sqrt{R}}$, wenn R und ρ ganze Funktionen sind.* J. Reine Angew. Math. **1**(1826), 186–221.
[2] N. I. Akhieser, *Über einige Funktionen, die in gegebenen Intervallen am wenigsten von Null abweichen*. Bull. Soc. Phys.-Math. Kazan (3) (2)**3**(1928), 1–69.
[3] ———, *Vorlesungen über Approximationstheorie*. Akademie Verlag, Berlin, 1953.
[4] ———, *Orthogonal polynomials on several intervals*. Soviet Math. Dokl. **1**(1960), 989–992.
[5] N. I. Akhieser and Yu. Ya. Tomčuk, *On the theory of orthogonal polynomials over several intervals.* Soviet Math. Dokl. **2**(1961), 687–690.
[6] A. I. Aptekarev, *Asymptotic properties of polynomials orthogonal on a system of contours, and periodic motions of Toda lattices*. Mat. Sb. **53**(1986), 233–260.
[7] H. Baker, *Abel's theorem and the allied theory*. 2nd ed. Cambridge Univ. Press, Cambridge, UK, 1995.
[8] H. Bateman and A. Erdélyi, *Higher transcendental functions.* Vol. 2. McGraw-hill, New York, NY, 1955.
[9] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its and V. B. Matveev, *Algebro-geometric approach to nonlinear integrable equations.* Springer, Berlin, 1994.
[10] S. N. Bernstein, *Sur les polynomes orthogonaux relatifs à un segment fini, I.* J. Math. Pure Appl. **9**(1930), 127–177.
[11] A. B. Bogatyrev, *On the efficient computation of Chebyshev polynomials for several intervals.* Sb. Math. **190**(1999), 1571–1605.
[12] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*. Springer, Berlin, 1995.
[13] W. Burnside, *On a class of automorphic functions.* Proc. London Math. Soc. **23**(1892), 49–88.
[14] ———, *Further note on automorphic functions*. Proc. London Math. Soc. **23**(1892), 281–295.
[15] A. Draux, *Polynômes orthogonaux formels—applications*. Springer, Berlin, 1983.
[16] T. Falliero and A. Sebbar, *Capacité d'une union de trois intervalles et fonctions thêta de genre 2*. J. Math. Pure Appl. **80**(2001), 409–443.
[17] L. R. Ford, *Automorphic functions*. McGraw-hill, New York, 1929.
[18] J. S. Geronimo and R. Johnson, *An inverse problem associated with polynomials orthogonal on the unit circle*. Comm. Math. Phys. **193**(1998), 125–150.
[19] Ya. L. Geronimus, *On some finite difference equations and corresponding systems of orthogonal polynomials*. Zap. Mat. Otdel. Fiz.-Mat. Fak. i Khar'kov Mat. Obshch., Ser. 4. **25**(1957), 87–100[Russian].
[20] A. A. Gončar, *On the convergence of generalized Padé approximants of meromorphic functions*. Mat. Sb. **97**(1975), 607–629.
[21] G. H. Halphen, *Traité des fonctions elliptiques et leurs applications*. tom. II. Gauthier-Villars, Paris, 1888.
[22] E. Hlawka, J. Schoissengeier and R. Tascher, *Geometric and analytic number theory*. Springer, Berlin, 1991.
[23] C. G. J. Jacobi, *Note sur une nouvelle application de l'analyse des fonctions elliptiques à l'algèbre*. J. Reine Angew. Math. **7**(1830), 41–43.
[24] M. G. Krein, B. Ya. Levin, and A. A. Nudel'man, *On special representation of polynomials that are positive on a system of closed intervals, and some applications.* In: Functional Analysis, Optimization, and Mathematical Economics: A Collection of Papers Dedicated to the Memory of Leonid Vital'evich Kantorovich, (ed., L. J. Leifman), Oxford Univ. Press, New York, 1990, 56–114.

[25] V. I. Lebedev, *Extremal polynomials with restrictions and optimal algorithms*. In: Advanced Mathematics: Computations and Applications, (eds., A. S. Alekseev and N. S. Bakhvalov), NCC Publ., Novosibirsk, 1995, 491–502.
[26] ———, *On the solution of inverse problems and trigonometric forms for the Geronimus polynomials. Application to the theory of iterative methods*. Russian J. Numer. Anal. Math. Modelling **15**(2000), 73–93.
[27] D. S. Lubinsky, *Zeros of orthogonal and biorthogonal polynomials: some old, some new*. In: Nonlinear numerical methods and rational approximation, II, (ed., A. Cuyt), Kluwer Acad. Publ., Dordrecht, 1994, 3–15.
[28] ———, *Will Ramanujan kill Baker-Gammel-Wills? (A selective survey of Padé approximation)*. In: New developments in approximation theory, (eds., M. W. Müller et al.), Birkhäuser, Basel, 1999, 159–174.
[29] ———, *Asymptotics of orthogonal polynomials: some old, some new, some identities*. Acta. Appl. Math. **61**(2000), 207–256.
[30] A. L. Lukashov, *The algebraic fractions of Chebyshev and Markov on several segments*. Anal. Math. **24**(1998), 111–130 [Russian].
[31] ———, *On Chebyshev-Markov rational fractions over several intervals*. J. Approx. Theory **95**(1998), 333–352.
[32] A. Magnus, *Recurrence coefficients for orthogonal polynomials on connected and nonconnected sets*. In: Padé Approximation and its Application, Lecture Notes in Math. **765**(1979), 150–171.
[33] J. Nuttall and S. R. Singh, *Orthogonal polynomials and Padé approximations associated with a system of arcs*. J. Approx. Theory **21**(1977), 1–42.
[34] F. Peherstorfer, *On Bernstein-Szegö orthogonal polynomials on several intervals*. SIAM J. Math. Anal. **21**(1990), 461–482.
[35] ———, *On Bernstein-Szegö orthogonal polynomials on several intervals,II: Orthogonal polynomials with periodic recurrence coefficients*. J. Approx. Theory **64**(1991), 123–161.
[36] ———, *Orthogonal and extremal polynomials on several intervals*. J. Comput. Appl. Math. **48**(1993), 187–205.
[37] ———, *Positive and orthogonal polynomials on several intervals*. Rend. Circ. Mat. Palermo (2) (33)(1993), 399–414.
[38] ———, *Elliptic orthogonal and extremal polynomials*. Proc. London Math. Soc. (3) **70**(1995), 605–624.
[39] ———, *Minimal polynomials on several intervals with respect to the maximum-norm—a survey*. In: Complex methods in approximation theory, (eds., A. M. Finkelshtein et al.), Univ. Alméria, Alméria, 1997, 137–159.
[40] F. Peherstorfer and K. Schiefermayr, *Description of extremal polynomials on several intervals and their computation, I, II*. Acta Math. Hungar. **83**(1999), 27–58, 59–83.
[41] F. Peherstorfer and R. Steinbauer, *On polynomials orthogonal on several intervals*. Ann. of Numer. Math. **2**(1995), 353–370.
[42] I. L. Ptashitskiĭ, *On integration in finite form of irrational differentials*. Saint-Petersburg, Thesis, 1881. [Russian]
[43] V. N. Rusak, *Rational functions as an apparatus of approximation*. Byelorussian Univ. Press, Minsk, 1979. [Russian]
[44] J. Shen, G. Strang and A. J. Wathen, *The potential theory of several intervals and its applications*. Appl. Math. Optim. **44**(2001), 67–85.
[45] J. Shohat, *On the continued fractions associated with, and corresponding to, the integral* $\int_a^b \frac{p(y)}{x-y}\,dy$. Amer. J. Math. **55**(1933), 218–230.
[46] M. L. Sodin and P. M. Yuditskiĭ, *Functions deviating least from zero on closed subsets of the real axis*. St. Petersburg Math. J. **4**(1993), 201–249.
[47] ———, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*. J. Geom. Anal. **7**(1997), 387–435.
[48] H. Stahl, *The convergence of Padé approximants to functions with branch points*. J. Approx. Theory **91**(1997), 139–204.
[49] S. P. Suetin, *Uniform convergence of Padé diagonal approximants for hyperelliptic functions*. Sb. Math. **191**(2000), 1339–1373.
[50] G. Szegö, *Orthogonal polynomials*. 4th ed. Amer. Math. Soc., Providence, RI, 1975.
[51] Yu. Ya. Tomčuk, *Orthogonal polynomials on a system of intervals of the real axis*. Zap. Mekh.-Mat. Fak. i Khar'kov Mat. Obshch., Ser. 4 **29**(1964), 93–128 [Russian].
[52] M. Tsuji, *Potential theory in modern function theory*. 2nd ed. Chelsea Publ., New York, NY, 1975.

[53] R. K. Vasiliev, *Chebyshev polynomials and approximation theory on compact subsets of the real axis*. Saratov Univ. Publ., Saratov, 1998.

[54] H. Widom, *Extremal polynomials associated with a system of curves in the complex plane*. Adv. in Math. **3**(1969), 127–232.

*Permanent Address:*
*Department of Mechanics and Mathematics*
*Saratov State University*
*Astrakhanskaya, 83*
*410026, Saratov*
*Russia*
*e-mail: LukashovAL@info.sgu.ru*

*Institut für Mathematik*
*Johannes Kepler Universität Linz*
*A-4040 Linz*
*Austria*
*e-mail: alexei.loukachov@jk.uni-linz.ac.at*

*Institut für Mathematik*
*Johannes Kepler Universität Linz*
*A-4040 Linz*
*Austria*
*e-mail: franz.peherstorfer@jk.uni-linz.ac.at*