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ON STABILIZERS OF SOME TEICHMÜLLER DISKS IN POINTED MAPPING CLASS GROUPS

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Abstract. We prove that for each Riemann surface $\tilde{S}$ of finite analytic type (p, n) with $p \geq 2$, there exist uncountably many Teichmüller disks Δ in the Teichmüller space $T(S)$, where $S = \tilde{S} - \{\text{a point } a\}$, with these properties: (1) the natural projection $j : T(S) \to T(\tilde{S})$ defined by forgetting a induces an isometric embedding of each Δ into $T(\tilde{S})$; and (2) the stabilizer of each Teichmüller disk Δ in the a-pointed mapping class group of S is trivial.

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1. Introduction. Let $\tilde{S}$ be a Riemann surface of type (p, n), where p is the genus and n is the number of punctures. Assume that $3p - 3 + n > 0$. Let $\text{Homeo}(\tilde{S})$ be the group of orientation preserving self-homeomorphisms of $\tilde{S}$ and let $\text{Homeo}_0(\tilde{S})$ be the subgroup consisting of those maps isotopic to the identity. The Teichmüller space $T(\tilde{S})$ is the space of all conformal structures on $\tilde{S}$ quotient by $\text{Homeo}_0(\tilde{S})$. The mapping class group $\text{Mod}_{\tilde{S}}$ defined by the quotient $\text{Homeo}(\tilde{S})/\text{Homeo}_0(\tilde{S})$ acts on $T(\tilde{S})$ as a group of holomorphic automorphisms; and also as a group of isometries if $T(\tilde{S})$ is endowed with the standard Teichmüller distance $d(\cdot, \cdot)$. See Abikoff [**1**] for more details.

There is a smooth fibre bundle $V(\tilde{S}) \to T(\tilde{S})$ whose fibre V_x over a point $x \in T(\tilde{S})$ is a Riemann surface representing x. The Bers fibre space $\pi : F(\tilde{S}) \to T(\tilde{S})$ is defined as the composition

$$F(\tilde{S}) \to V(\tilde{S}) \to T(\tilde{S}), \tag{1.1}$$

where $F(\tilde{S}) \to V(\tilde{S})$ is the universal covering map. The central fibre of $F(\tilde{S})$ is the hyperbolic plane $\mathbf{H} = \{z \in \mathbf{C} : \text{Im } z > 0\}$. Let G be the covering group of the universal covering map $\varrho : \mathbf{H} \to \tilde{S}$. Then G is a Fuchsian group such that $\mathbf{H}/G = \tilde{S}$. Note that every point $x \in T(\tilde{S})$ is represented by a conformal structure σ on $\tilde{S}$ that is lifted to a measurable function $\sigma(z)$ on $\mathbf{H}$ with these properties: (i) ess. sup $\{|\sigma(z)|;\ z \in \mathbf{H}\} < 1$ and (ii) $\sigma(g(z))\overline{g'(z)}/g'(z) = \sigma(z)$ for all $g \in G$ and $z \in \mathbf{H}$. The fibre $\mathscr{F}_x \subset F(\tilde{S})$ over a point $x \in T(\tilde{S})$ is the quasi-disk $w^\sigma(\mathbf{H})$, where $w^\sigma : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is, according to Ahlfors and Bers [**2**], a quasi-conformal map fixing $0, 1, \infty$ and satisfying

$$\frac{\partial_{\bar{z}} w^\sigma(z)}{\partial_z w^\sigma(z)} = \begin{cases} \sigma(z) & \text{if } z \in \mathbf{H}, \\ 0 & \text{if } z \in \mathbf{C} - \overline{\mathbf{H}}. \end{cases} \tag{1.2}$$

Let $\tilde{\theta} \in \mathrm{Mod}_{\tilde{S}}$ be a hyperbolic element; that is, there is a point $x \in T(\tilde{S})$ such that $d(x,\ \tilde{\theta}(x)) > 0$ and

$$\inf d(x',\ \tilde{\theta}(x')) = d(x,\ \tilde{\theta}(x)),$$

where the infimum is taken over all $x' \in T(\tilde{S})$. This implies that $\tilde{\theta}$ is represented by an absolutely extremal Teichmüller self-map $\tilde{\omega}$ on a surface (also called $\tilde{S}$) that determines a holomorphic quadratic differential $\phi_{\tilde{\omega}}$ on $\tilde{S}$, which may have simple poles at punctures of $\tilde{S}$ and satisfies

$$\iint_{\tilde{S}} |\phi_{\tilde{\omega}}(z)|\, dx\, dy = 1. \tag{1.3}$$

Note that all zeros of $\phi_{\tilde{\omega}}$ may be punctures of $\tilde{S}$. Denote by $\mu = \bar{\phi}_{\tilde{\omega}}/|\phi_{\tilde{\omega}}|$. Then μ is a $(1, 1)$-form on $\tilde{S}$. Let $[\nu]$ denote the equivalence class of a conformal structure ν on $\tilde{S}$. Let $\mathbf{D} = \{t : |t| < 1\}$ be the unit disk. We define a Teichmüller disk

$$\tilde{\Delta}_{\tilde{\omega}} = \{[t\mu] : t \in \mathbf{D}\} \subset T(\tilde{S}).$$

By Theorem 5 of Bers [**6**], $\tilde{\Delta}_{\tilde{\omega}}$ is the unique invariant disk under the action of $\tilde{\theta}$. Associated to each point $\hat{z} \in \mathbf{H}$ there is a disk

$$\mathscr{D}_{\tilde{\omega}}(\hat{z}) = \{([t\mu], w^{t\mu}(\hat{z})) :\ t \in \mathbf{D}\} \subset F(\tilde{S}). \tag{1.4}$$

Let $a \in \tilde{S}$ be a point and let $S = \tilde{S} - \{a\}$. The Bers isomorphism theorem [**5**] states that there is an isomorphism $\varphi : F(\tilde{S}) \to T(S)$ that is determined up to a modular transformation of $T(S)$ so that

$$j = \pi \circ \varphi^{-1} : T(S) \to T(\tilde{S}) \tag{1.5}$$

is the natural forgetful map. Due to the Bers isomorphism theorem, one proves (Kra [**11**]) that the embedding

$$\Delta_{\tilde{\omega}}(\hat{z}) = \varphi(\mathscr{D}_{\tilde{\omega}}(\hat{z})) \subset T(S) \tag{1.6}$$

is a Teichmüller disk so that $j(\Delta_{\tilde{\omega}}(\hat{z})) = \tilde{\Delta}_{\tilde{\omega}}$. Let Mod_S^a denote the a-pointed mapping class group Mod_S^a. That is, Mod_S^a consists of mapping classes fixing the puncture a. Let $V_{\tilde{\omega}}(\hat{z})$ denote the stabilizer of $\Delta_{\tilde{\omega}}(\hat{z})$ in Mod_S^a. Then $V_{\tilde{\omega}}(\hat{z})$ is a subgroup of the Veech group of $\Delta_{\tilde{\omega}}(\hat{z})$ of finite index. In particular, if $\tilde{S}$ is compact, then $V_{\tilde{\omega}}(\hat{z})$ is the Veech group of $\Delta_{\tilde{\omega}}(\hat{z})$. See Veech [**17**] and Hubert–Lanneau [**8**] for discussions on Veech groups of general Teichmüller disks in a Teichmüller space. It is well known (Earle and Gardiner [**7**]) that there exist Teichmüller disks in $T(S)$ whose Veech groups are trivial.

The main purpose of this article is to prove the following result.

THEOREM 1.1. *Let $\tilde{S}$ be a Riemann surface of finite analytic type (p, n) with the genus $p \geq 2$. Then,*

(1) *For each hyperbolic mapping class $\tilde{\theta}$ of $\mathrm{Mod}_{\tilde{S}}$, there exists an uncountable subset $\Omega_0 \subset \mathbf{H}$ with a full measure such that for each point $\hat{z}_0 \in \Omega_0$, the Teichmüller disk $\Delta_{\tilde{\omega}}(\hat{z}) \subset T(S)$ defined as* (1.6) *satisfies $j(\Delta_{\tilde{\omega}}(\hat{z})) = \tilde{\Delta}_{\tilde{\omega}}$ and the stabilizer $V_{\tilde{\omega}}(\hat{z})$ of $\Delta_{\tilde{\omega}}(\hat{z})$ in Mod_S^a is trivial.*

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(2) *There exist hyperbolic mapping classes* $\tilde{\theta}$ *in* $Mod_{\tilde{S}}$ *such that for every point* $\hat{z} \in \mathbf{H}$, *the group* $V_{\tilde{\omega}}(\hat{z})$ *either is trivial or contains a purely hyperbolic subgroup of finite index.*

This article is organized as follows. In Section 2, we review the Teichmüller existence and uniqueness theorem. In Section 3, we investigate the mapping class group $\mathrm{Mod}_{\tilde{S}}$ as well as its extension to act on the corresponding Bers fibre space $F(\tilde{S})$. Properties of invariant disks by some mapping classes are also discussed in the section. Section 4 is devoted to the proof of Theorem 1.1. Section 5 includes some remarks.

2. Fixed point sets of special quasi-conformal self-maps of Riemann surfaces. Let $\tilde{S}$ be defined as before, and let $\tilde{S}_0$ be a Riemann surface of the same type (p, n). Then $\tilde{S}$ and $\tilde{S}_0$ are diffeomorphic. Let $f : \tilde{S} \to \tilde{S}_0$ be a quasi-conformal map. The Teichmüller existence and uniqueness theorem states that in the homotopy class of f there is either a unique conformal map or a unique quasi-conformal map f_0, called an extremal quasi-conformal map, such that the Beltrami coefficient

$$\frac{\partial_{\bar{z}} f_0}{\partial_z f_0} = k\,\frac{\bar{\phi}}{|\phi|} \quad \text{for some real number } k \text{ with } 0 < k < 1,$$

where ϕ is a quadratic differential on $\tilde{S}$ that satisfies (1.3) and may have simple poles at some punctures of $\tilde{S}$. Associated to f_0 there is another quadratic differential ψ on $\tilde{S}_0$. The quadratic differentials ϕ and ψ have the same number of zeros and some zeros of ϕ and ψ may be punctures of $\tilde{S}$ and $\tilde{S}_0$, respectively. If z is a zero of ϕ, then $f_0(z)$ is a zero of ψ of the same order. The quadratic differential ϕ (resp. ψ) determines a pair (Φ_h, Φ_v) (resp. (Ψ_h, Ψ_v)) of transverse trajectories on $\tilde{S}$ (resp. $\tilde{S}_0$). The map f_0 sends Φ_h to Ψ_h and Φ_v to Ψ_v via a stretching map and a compressing map. More precisely, if $P \in \tilde{S}$ is not a zero of ϕ and z is a ϕ-coordinate about P, then there is a ψ-coordinate ζ at $f_0(z) \in \tilde{S}_0$ such that

$$\zeta \circ f_0 = \frac{z + k\bar{z}}{\sqrt{1 - k^2}}. \tag{2.1}$$

From (2.1) the map f_0 can be realized as

$$\mathrm{Re}\,\zeta = \left(\frac{1+k}{1-k}\right)^{1/2} \mathrm{Re}\,z \ \text{ and } \ \mathrm{Im}\,\zeta = \left(\frac{1+k}{1-k}\right)^{-1/2} \mathrm{Im}\,z. \tag{2.2}$$

In literature (see e.g. Strebel [**13**]), ϕ and ψ are called the initial and terminal quadratic differentials of f_0, respectively.

Consider the case $\tilde{S}_0 = \tilde{S}$. Then f_0 defines a mapping class of $\mathrm{Mod}_{\tilde{S}}$ and ϕ and ψ are differentials on $\tilde{S}$. As an example, for any simple closed geodesic $\tilde{c}$ on $\tilde{S}$, the Dehn twist $t_{\tilde{c}}$ determines a parabolic mapping class of $\mathrm{Mod}_{\tilde{S}}$ (for the definition of parabolic mapping classes, see Bers [**6**]). According to the Teichmüller theorem, in the homotopy class of $t_{\tilde{c}}$, there is a unique extremal Teichmüller self-map whose corresponding initial and terminal quadratic differentials ϕ and ψ are simple Jenkins–Strebel differentials. See Strebel [**15**] for an exposition.

As another example, for every irreducible self-map f of $\tilde{S}$ (in the sense that it does not keep any curve simplex invariant), we can modify the conformal structure of $\tilde{S}$ so that f is isotopic to an f_0 that is an absolutely extremal Teichmüller map on a new surface $\tilde{S}_0$. The choice of $\tilde{S}_0$ is not unique and all the surfaces $\tilde{S}_0$ so obtained determine a set of points in $T(\tilde{S})$ that constitutes a real Teichmüller geodesic. In this case, we have $\phi = \psi$ and thus $\Phi_h = \Psi_h$ and $\Phi_v = \Psi_v$. See Bers [**6**] and Thurston [**16**] for more details.

For any quasi-conformal self-map f of $\tilde{S}$, we denote by

$$\mathscr{S}(f) = \{z \in \tilde{S} : \quad f(z) = z\}. \tag{2.3}$$

Let $\tilde{\chi}$ be a non-trivial non-elliptic hyperbolic mapping class of $\tilde{S}$, and let f_0 be the Teichmüller self-map representing $\tilde{\chi}$. We claim that the set $\mathscr{S}(f_0)$ is finite. In fact more careful examination reveals that $\mathscr{S}(f_0)$ consists of zeros of the corresponding quadratic differential ϕ. See Lemma 3.1.

More generally, if $f_0 : \tilde{S} \to \tilde{S}$ is a quasi-conformal map whose Beltrami coefficient is $t\mu$ for $t \in \mathbf{D} - \{0\}$ and $\mu = \bar{\phi}/|\phi|$, where ϕ is a holomorphic quadratic differential on $\tilde{S}$. We claim that $\mathscr{S}(f_0)$ has measure zero. Suppose that $\mathscr{S}(f_0)$ has measure > 0. We denote by $z = z(P)$ a local coordinate chart at a point $P \in \tilde{S}$. For each $P \in \mathscr{S}(f_0)$, the function $F(z) = f_0(z) - z = 0$. This implies that $\partial_z F(z) = 0$ and $\partial_{\bar{z}} F(z) = 0$ for almost all points $z = z(P) \in \mathscr{S}(f_0)$, where the derivatives are taken in the sense of distribution. Since $\partial_z F(z) = \partial_z f_0(z) - 1$, we conclude that $\partial_z f_0(z) \neq 0$ for almost all points $z = z(P) \in \mathscr{S}(f_0)$. On the other hand, $\partial_{\bar{z}} F(z) = \partial_{\bar{z}} f_0(z)$. It follows that $\partial_{\bar{z}} f_0(z) = 0$ for almost all points $z = z(P)$ in $\mathscr{S}(f_0)$. Therefore, the Beltrami coefficient of f_0 vanishes at almost all points in $\mathscr{S}(f_0)$. But we know that $\mathscr{S}(f_0)$ has measure > 0 and the Beltrami coefficient of f_0 is $t\bar{\phi}/|\phi|$ which can not be zero on a set with positive measure (since ϕ has isolated zeros). This is a contradiction. We summarize the result in the following lemma.

LEMMA 2.1. *Let $\tilde{\chi}$ be a non-trivial non-elliptic mapping class of $\tilde{S}$, and let f_0 be a quasi-conformal self-map of $\tilde{S}$ representing $\tilde{\chi}$. Assume that f_0 has a Beltrami coefficient with form $t\bar{\phi}/|\phi|$ for a complex number $t \in \mathbf{D} - \{0\}$ and a holomorphic quadratic differential ϕ on $\tilde{S}$. Then the set $\mathscr{S}(f_0)$ defined as in* (2.3) *has measure zero.*

3. Mapping class groups acting on Bers fibre spaces. The mapping class group $\mathrm{Mod}_{\tilde{S}}$ extends to the group $\mathrm{mod}(\tilde{S})$ that acts on $F(\tilde{S})$ and preserves the fibre structure of $F(\tilde{S})$. Every element $\theta \in \mathrm{mod}(\tilde{S})$ can be represented by a self-map w of $\mathbf{H}$. Two such self-maps $\hat{w}_1$, $\hat{w}_2$ of $\mathbf{H}$ represent the same element of $\mathrm{mod}(\tilde{S})$ if $\hat{w}_1 = \hat{w}_2$ on $\partial\mathbf{H}$ and they both project to maps $w_1, w_2 : \tilde{S} \to \tilde{S}$ isotopic to each other; or equivalently, $\hat{w}_1 h(\hat{w}_1)^{-1} = \hat{w}_2 h(\hat{w}_2)^{-1}$ for all elements $h \in G$, write $\theta = [\hat{w}_1] = [\hat{w}_2]$. Then $\theta \in \mathrm{mod}(\tilde{S})$.

With the aid of the Bers isomorphism $\varphi : F(\tilde{S}) \to T(S)$, the group $\mathrm{mod}(\tilde{S})$ is isomorphic to Mod_S^a by a conjugation φ^*:

$$\mathrm{mod}(\tilde{S}) \ni [\hat{w}] \xrightarrow{\varphi^*} \varphi \circ [\hat{w}] \circ \varphi^{-1} \in \mathrm{Mod}_S^a.$$

It follows that the forgetful map $j : T(S) \to T(\tilde{S})$ defined as (1.5) also induces a natural projection

$$i : \mathrm{Mod}_S^a \to \mathrm{Mod}_{\tilde{S}} \tag{3.1}$$

obtained by forgetting the puncture a. For simplicity, we denote throughout the article $[\hat{w}]^* = \varphi^*([\hat{w}])$. In particular, the group G can be regarded as a normal subgroup of mod($\tilde{S}$) and thus it keeps each fibre of $F(\tilde{S})$ invariant. The group G is isomorphic to a subgroup of Mod_S^a that consists of all elements $\theta = h^*$ with $h \in G$ so that $i(h^*) =$ identity, i.e., $G \cong i^{-1}(\mathrm{id})$. In [**11**], Kra proved that $h \in G$ is simple hyperbolic if and only if h^* is represented by a spin map (that is of the form $t_\alpha \circ t_\beta^{-1}$, where t_α and t_β are the positive Dehn twists along curves α and β that are components of an a-punctured cylinder on S); and is parabolic if and only if h^* is represented by a Dehn twist along a curve that bounds a twice punctured disk enclosing a.

Now let $\tilde{\theta}$, $\tilde{\omega}$ and $\phi_{\tilde{\omega}}$ be as introduced in the previous section.

LEMMA 3.1. *If $\tilde{\omega}(z_0) = z_0$ for some $z_0 \in \tilde{S}$, then z_0 is a zero of $\phi_{\tilde{\omega}}$. Further, $\tilde{\omega}$ leaves invariant the set of non-puncture zeros of $\phi_{\tilde{\omega}}$.*

Proof. The lemma follows directly from the definition of absolutely extremal Teichmüller self-map. See also (2.1) or (2.2). □

Assume that there exists a hyperbolic $\theta \in \mathrm{Mod}_S^a$ such that $i(\theta) = \tilde{\theta}$. By Bers [**6**], there exists a unique Teichmüller disk in $T(S)$, denoted by Δ_θ, that is invariant under θ.

LEMMA 3.2. *With the above conditions, $j(\Delta_\theta) \subset T(\tilde{S})$ is an invariant disk (which may not be a Teichmüller disk) under the action of $\tilde{\theta}$.*

Proof. For any point $x \in T(S)$, we have

$$j(\theta(x)) = \tilde{\theta}(j(x)). \tag{3.2}$$

Now let $y \in j(\Delta_\theta)$, and let $x \in \Delta_\theta$ be such that $j(x) = y$. Since θ is hyperbolic and $x \in \Delta_\theta$, $\theta(x) \in \Delta_\theta$. Thus $j(\theta(x)) \in j(\Delta_\theta)$. It follows from (3.2) that $\tilde{\theta}(y) = \tilde{\theta}(j(x)) = j(\theta(x)) \in j(\Delta_\theta)$. Since $y \in j(\Delta_\theta)$ is arbitrary, $\tilde{\theta}$ keeps $j(\Delta_\theta)$ invariant. □

The following lemma was proved in Kra [**11**].

LEMMA 3.3. *Assume that $\phi_{\tilde{\omega}}$ has a non-puncture zero z_0 so that $\tilde{\omega}(z_0) = z_0$. Then there are infinitely many (countable) hyperbolic elements $\theta \in \mathrm{Mod}_S^a$ such that $i(\theta) = \tilde{\theta}$ and the invariant disk Δ_θ of θ is of form $\Delta_{\tilde{\omega}}(\hat{z})$ for some $\hat{z} \in \mathbf{H}$.*

From Lemma 3.2 and Lemma 3.3, we obtain the following lemma.

LEMMA 3.4. *Assume that $\phi_{\tilde{\omega}}$ has a non-puncture zero z_0 with $\tilde{\omega}(z_0) = z_0$. There exist infinitely many (countable) hyperbolic elements $\theta \in \mathrm{Mod}_S^a$ such that $i(\theta) = \tilde{\theta}$ and the projection* (1.5) *realizes an isometric embedding of Δ_θ into $T(\tilde{S})$ with the property that $j(\Delta_\theta)$ is the invariant Teichmüller disk of $\tilde{\theta}$.*

REMARK 3.1. It is not clear if there is a hyperbolic element $\theta \in \mathrm{Mod}_S^a$ with $i(\theta) = \tilde{\theta}$ being hyperbolic such that Δ_θ is not of the form $\Delta_{\tilde{\omega}}(\hat{z})$, but $j : T(S) \to T(\tilde{S})$ still induces an isometric embedding of Δ_θ into $T(\tilde{S})$.

LEMMA 3.5. *Let $\hat{z}_1, \hat{z}_2 \in \mathbf{H}$ be two points with $\hat{z}_1 \neq \hat{z}_2$. Let $\mathscr{D}_{\tilde{\omega}}(\hat{z}_1)$, $\mathscr{D}_{\tilde{\omega}}(\hat{z}_2)$ be defined as in* (1.4). *Then $\mathscr{D}_{\tilde{\omega}}(\hat{z}_1)$ is disjoint from $\mathscr{D}_{\tilde{\omega}}(\hat{z}_2)$. Further, every such $\mathscr{D}_{\tilde{\omega}}(\hat{z})$ intersects $\mathbf{H}$ exactly one point.*

Proof. To see that $\mathscr{D}_{\tilde{\omega}}(\hat{z}_1)$ is disjoint from $\mathscr{D}_{\tilde{\omega}}(\hat{z}_2)$, we notice that

$$\mathscr{D}_{\tilde{\omega}}(\hat{z}_i) = \left\{([t\mu], w^{t\mu}(\hat{z}_i)) : \ t \in \mathbf{D}\right\}.$$

Suppose that

$$([t_1\mu], w^{t_1\mu}(\hat{z}_1)) = ([t_2\mu], w^{t_2\mu}(\hat{z}_2)), \quad \text{for some } t_1, t_2 \in \mathbf{D}. \tag{3.3}$$

From Lemma 3.4, we see that

$$\pi(\mathscr{D}_{\tilde{\omega}}(\hat{z}_1)) = \pi(\mathscr{D}_{\tilde{\omega}}(\hat{z}_2))$$

is the Teichmüller disk $\Delta_{\tilde{\theta}}$, which means that there is an isometric embedding $\iota : \mathbf{D} \to \Delta_{\tilde{\theta}}$ that sends $t \in \mathbf{D}$ to $[t\mu]$, from which it follows that $t_1 = t_2 = t$. Now from (3.3) we obtain that $w^{t\mu}(\hat{z}_1) = w^{t\mu}(\hat{z}_2)$. Since $\hat{z}_1 \neq \hat{z}_2$ and $w^{t\mu} : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is a quasi-conformal map, $w^{t\mu}(\hat{z}_1) \neq w^{t\mu}(\hat{z}_2)$. This contradicts (3.3). Thus there are no points of $F(\tilde{S})$ lying in both $\mathscr{D}_{\tilde{\omega}}(\hat{z}_1)$ and $\mathscr{D}_{\tilde{\omega}}(\hat{z}_2)$, which says that $\mathscr{D}_{\tilde{\omega}}(\hat{z}_1)$ is disjoint from $\mathscr{D}_{\tilde{\omega}}(\hat{z}_2)$.

Suppose that $\mathscr{D}_{\tilde{\omega}}(\hat{z})$ intersects $\mathbf{H}$ at least twice for some $\hat{z} \in \mathbf{H}$, which implies that the projection $\pi : F(\tilde{S}) \to T(\tilde{S})$ does not define an embedding of $\mathscr{D}_{\tilde{\omega}}(\hat{z})$ into $T(\tilde{S})$. Thus $\pi(\mathscr{D}_{\tilde{\omega}}(\hat{z}))$ is not the Teichmüller disk invariant under $\tilde{\theta}$. This contradicts Lemma 3.4. □

Let $\chi \in \mathrm{Mod}_S^a$ be a non-trivial mapping class such that

$$\chi(\Delta_{\tilde{\omega}}(\hat{z})) = \Delta_{\tilde{\omega}}(\hat{z}) \tag{3.4}$$

for a point $\hat{z} \in \mathbf{H}$. Assume that $\tilde{\chi} = i(\chi) \in \mathrm{Mod}_{\tilde{S}}$ is non-trivial and non-elliptic. Let $[\hat{f}] \in \mathrm{mod}(\tilde{S})$ be the element corresponding to χ. Then we have

$$[\hat{f}](\mathscr{D}_{\tilde{\omega}}(\hat{z})) = \mathscr{D}_{\tilde{\omega}}(\hat{z}). \tag{3.5}$$

By definition (1.4) and Lemma 3.5. $\mathscr{D}_{\tilde{\omega}}(\hat{z}) \cap \mathbf{H} = \hat{z}$.

Caution. We may choose a representative (also denoted by $\hat{f}$) of $[\hat{f}]$ so that $\hat{f}(\hat{z}) \neq \hat{z}$. That is to say, there are quasi-conformal maps $\hat{f}$ such that $\hat{f}(\hat{z}) \neq \hat{z}$ and (3.5) still holds. On the other hand, for quasi-conformal maps $\hat{f}$ with $\hat{f}(\hat{z}) = \hat{z}$, there is no guarantee that (3.5) is satisfied.

Despite the complexity mentioned above, when $\hat{f}$ is a quasi-conformal self-map of $\mathbf{H}$ whose Beltrami coefficient is $t\mu$ for $t \in \mathbf{D} - \{0\}$ and $\mu = \overline{\phi_{\tilde{\omega}}}/|\phi_{\tilde{\omega}}|$, (3.5) is equivalent to the condition $\hat{f}(\hat{z}) = \hat{z}$. More precisely, we establish the following result.

LEMMA 3.6. *With the above notation, assume that* (3.4) *or* (3.5) *holds. The mapping classes* χ *and* $\tilde{\chi}$ *are represented by quasi-conformal maps* $\hat{f}_0$ *and* f_0*, respectively, so that* $\hat{f}_0(\hat{z}) = \hat{z}$, $f_0(z) = z$ *for* $z = \varrho(\mathscr{D}_{\tilde{\omega}}(\hat{z}) \cap \mathbf{H})$ *and both have the Beltrami coefficient* $t\mu$ *for a* $t \in \mathbf{D} - \{0\}$ *and* $\mu = \overline{\phi_{\tilde{\omega}}}/|\phi_{\tilde{\omega}}|$. *Conversely, if* χ *is represented by* $\hat{f}_0$ *that has the Beltrami coefficient* $t\mu$ *and satisfies* $\hat{f}_0(\hat{z}) = \hat{z}$, *then* (3.4) *or* (3.5) *holds. In particular, if* $\tilde{\chi} = \tilde{\theta}$, *then* $z = z_0$ *is a non-puncture zero of* $\phi_{\tilde{\omega}}$ *so that* $\tilde{\omega}(z_0) = z_0$.

Proof. Since χ leaves $\Delta_{\tilde{\omega}}(\hat{z})$ invariant, Lemma 3.2 says that $\tilde{\chi} = i(\chi)$ leaves $j(\Delta_{\tilde{\omega}}(\hat{z}))$ invariant. From Lemma 3.4, $\tilde{\Delta}_{\tilde{\omega}} = j(\Delta_{\tilde{\omega}}(\hat{z}))$ is the Teichmüller disk determined by the quadratic differential $\phi_{\tilde{\omega}}$. So $\tilde{\chi}$ leaves the Teichmüller disk $\tilde{\Delta}_{\tilde{\omega}}$ invariant.

Since $\tilde{\chi}([0]) \in \tilde{\Delta}_{\tilde{\omega}}$, we can write $\tilde{\chi}([0]) = [t_1\mu]$ for a complex number $t_1 \in \mathbf{D} - \{0\}$, where $\mu = \overline{\phi_{\tilde{\omega}}}/|\phi_{\tilde{\omega}}|$. There is a quasi-conformal self-map f_0 of $\tilde{S}$ such that f_0 represents $\tilde{\chi}$ and has Beltrami coefficient $t_1\mu$. An easy computation shows that f_0^{-1} has Beltrami coefficient $t_2\mu$, where t_2 is a complex number with $|t_2| < 1$ and $|t_1/t_2| = 1$.

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Let $\hat{f}_0 : \mathbf{H} \to \mathbf{H}$ denote the lift of f_0 satisfying $[\hat{f}_0]^* = \chi$. Then we know that

$$\tilde{\chi}([0]) = \left[\text{Beltrami coefficient of } \hat{f}_0^{\,-1}\right] \in \tilde{\Delta}_{\tilde{\omega}}. \tag{3.6}$$

Now (3.4) tells us that $\chi \circ \varphi(\mathscr{D}_{\tilde{\omega}}(\hat{z})) = \varphi(\mathscr{D}_{\tilde{\omega}}(\hat{z}))$. Thus

$$[\hat{f}_0](\mathscr{D}_{\tilde{\omega}}(\hat{z})) = \mathscr{D}_{\tilde{\omega}}(\hat{z}). \tag{3.7}$$

From direct computations similar to the proof of Theorem 1 of [**11**], we know that the Beltrami coefficient of $w^{t\mu} \circ \hat{f}_0^{-1}$ for any complex number $t \in \mathbf{D}$ is $\xi(t)\mu$, where $\xi(t)$ is a complex number defined as

$$\xi(t) = -\frac{t_2}{t_1}\,\frac{t - t_1}{1 - t\bar{t}_1}.$$

Hence $\xi : \mathbf{D} \to \mathbf{D}$ is a conformal automorphism.

Notice that $\hat{z} = \mathscr{D}_{\tilde{\omega}}(\hat{z}) \cap \mathbf{H}$. It then follows from the argument of Proposition 3 of [**11**] that the action of $[\hat{f}_0]$ can be written as

$$[\hat{f}_0]([t\mu],\ w^{t\mu}(\hat{z})) = ([\xi(t)\mu],\ w^{\xi(t)\mu} \circ \hat{f}_0(\hat{z})). \tag{3.8}$$

As t runs over $\mathbf{D}$, $\xi(t)$ runs over $\mathbf{D}$ as well. This says that

$$[\hat{f}_0](\mathscr{D}_{\tilde{\omega}}(\hat{z}))$$

is a disk of the form (1.4) and passes through the point $\hat{f}_0(\hat{z}) \in \mathbf{H}$.

Denote by $\mathscr{D}_0 = [\hat{f}_0](\mathscr{D}_{\tilde{\omega}}(\hat{z}))$. Assume that $\hat{f}_0(\hat{z}) \neq \hat{z}$. By Lemma 3.5, $\mathscr{D}_0 \neq \mathscr{D}_{\tilde{\omega}}(\hat{z})$. Hence $\varphi(\mathscr{D}_0) \neq \varphi(\mathscr{D}_{\tilde{\omega}}(\hat{z})) = \Delta_{\tilde{\omega}}(\hat{z})$. But

$$\varphi(\mathscr{D}_0) = \varphi \circ [\hat{f}_0](\mathscr{D}_{\tilde{\omega}}(\hat{z})) = [\hat{f}_0]^* \circ \varphi(\mathscr{D}_{\tilde{\omega}}(\hat{z})) = [\hat{f}_0]^*(\Delta_{\tilde{\omega}}(\hat{z})).$$

From (3.7), we have $[\hat{f}_0]^*(\Delta_{\tilde{\omega}}(\hat{z})) = \Delta_{\tilde{\omega}}(\hat{z})$. This leads to a contradiction, proving that $\hat{f}_0(\hat{z}) = \hat{z}$. Set $z = \varrho(\hat{z})$. From $\varrho \circ \hat{f}_0 = f_0 \circ \varrho$ we see that $f_0(z) = z$.

The converse part of the result is included in the argument of Theorem 1 of [**11**]. Finally, if $f_0 = \tilde{\omega}$, then $\tilde{\omega}(z) = z$. In particular, z cannot be a puncture of $\tilde{S}$. Since $\tilde{\omega}$ is an absolutely extremal Teichmüller map on $\tilde{S}$, by Lemma 3.1, $z = z_0$ is a non-puncture zero of the quadratic differential $\phi_{\tilde{\omega}}$. □

From Lemma 3.3, we know that if there is a point $z_0 \in \tilde{S}$ that is a non-puncture zero of $\phi_{\tilde{\omega}}$ with $\tilde{\omega}(z_0) = z_0$, then for any point $\hat{z}_0 \in \mathbf{H}$ with $\varrho(\hat{z}_0) = z_0$, the stabilizer $V_{\tilde{\omega}}(\hat{z}_0)$ of $\Delta_{\tilde{\omega}}(\hat{z}_0)$ in the pointed mapping class group Mod_S^a contains a hyperbolic mapping class θ with $i(\theta) = \tilde{\theta}$. This particularly implies that $V_{\tilde{\omega}}(\hat{z}_0)$ is not empty and contains at least an infinite cyclic subgroup generated by θ.

4. Proof of Theorem 1.1. By Hurwitz's theorem (see e.g. Theorem V.1.3 of Farkas and Kra [**9**]), the group of conformal automorphisms of $\tilde{S}$ is finite, and the order of the group is $< 84(p-1)$. For each conformal automorphism ζ of $\tilde{S}$, the quotient $\tilde{S}/\langle\zeta\rangle$ is an orbifold of finite type. So by a theorem of Kravetz [**12**], the number of fixed points of ζ on $\tilde{S}$ is finite. Let A be the set of points z of $\tilde{S}$ such that there is a non-trivial conformal automorphism ζ with $\zeta(z) = z$. We conclude that A is finite.

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It is well known (see Abikoff [1]) that the mapping class group $\mathrm{Mod}_{\tilde{S}}$ is discrete when viewed as a group of holomorphic automorphisms of $T(\tilde{S})$. By Lemma 2.1, for each non-conformal quasi-conformal self-map f of $\tilde{S}$, whose Beltrami coefficient is $t\overline{\phi_{\tilde{\omega}}}/|\phi_{\tilde{\omega}}|$, the set

$$B_f = \{P \in \tilde{S} : f(P) = P\}$$

has measure zero. We conclude that the union

$$B = \bigcup B_f$$

for all such maps f has measure zero. By combining A and B, we see that $A \cup B$ has measure zero. Hence the set

$$\Sigma = \{\hat{z} \in \mathbf{H} : \varrho(\hat{z}) \in A \cup B\} \tag{4.1}$$

has measure zero. Denote by

$$\Omega = \mathbf{H} - \Sigma,$$

where Σ is defined as (4.1). Then Ω is uncountable with a full measure. We prove the following lemma.

LEMMA 4.1. *For any $\hat{z} \in \Omega$, the group $V_{\tilde{\omega}}(\hat{z})$ is either trivial or finitely cyclic.*

Proof. Suppose that $V_{\tilde{\omega}}(\hat{z})$ contains an element $\chi \in \mathrm{Mod}_S^a$ with infinite order. That is,

$$\chi(\Delta_{\tilde{\omega}}(\hat{z})) = \Delta_{\tilde{\omega}}(\hat{z}) \text{ and } \chi^n \neq \mathrm{id} \text{ for any integer } n.$$

Since χ is not elliptic, from the Bers classification for mapping classes [6], χ is either parabolic, hyperbolic, or pseudo-hyperbolic.

From Theorem 1 of Kra [11], any pseudo-hyperbolic mapping class χ cannot keep $\Delta_{\tilde{\omega}}(\hat{z})$ invariant. To see this fact, we note that for a Teichmüller disk $\Delta_{\tilde{\omega}}(\hat{z})$, there is an isometry

$$\iota : \mathbf{D} \to \Delta_{\tilde{\omega}}(\hat{z}) \tag{4.2}$$

with respect to the hyperbolic metric on $\mathbf{D}$ and the Teichmüller metric on $\Delta_{\tilde{\omega}}(\hat{z})$. If there exists a pseudo-hyperbolic mapping class χ keeping $\Delta_{\tilde{\omega}}(\hat{z})$ invariant, then $\iota^{-1} \circ \chi \circ \iota \in \mathrm{Aut}(\mathbf{D})$ is a non-trivial Möbius transformation that is neither elliptic, nor parabolic, nor hyperbolic. This is absurd.

Via the isometry (4.2), χ determines a Möbius transformation A_χ on $\mathbf{D}$. So A_χ is either trivial or non-trivial. If A_χ is trivial, then $\Delta_{\tilde{\omega}}(\hat{z})$ is contained in a fixed-point locus of χ. In this case, if we write $[\hat{f}]^* = \chi$, then $[\hat{f}] \in \mathrm{mod}(\tilde{S})$ determines a conformal automorphism ζ of $\tilde{S}$. Since $\Delta_{\tilde{\omega}}(\hat{z}) \cap \mathbf{H} = \hat{z}$, ζ fixes the point $z = \varrho(\hat{z}) \in \tilde{S}$. It follows by definition that $z \in A$. Hence $\hat{z} \in \Sigma$. But this contradicts that $\hat{z} \in \Omega$. If A_χ is non-trivial, there are three cases to consider: A_χ is either hyperbolic, parabolic, or elliptic.

Case 1. A_χ is hyperbolic. Then A_χ has an invariant geodesic $l \subset \mathbf{D}$. Hence χ has an invariant Teichmüller geodesic $\iota(l)$. From Theorem 6 of Bers [6], χ is hyperbolic.

If $\tilde{\chi} = i(\chi) \in \mathrm{Mod}_{\tilde{S}}$ is non-trivial and non-elliptic, then we notice that $\mathscr{D}_{\tilde{\omega}}(\hat{z}) \cap \mathbf{H} = \hat{z}$. From Lemma 3.6, we assert that $\tilde{\chi}$ is represented by a quasi-conformal map

f_0 that has a Beltrami coefficient $t\mu$ for some $t \in \mathbf{D} - \{0\}$ and satisfies $f_0(z) = z$. This means $z \in B$. Thus $\hat{z} \in \Sigma$ and $\hat{z} \notin \Omega$. This is a contradiction.

If $\tilde{\chi}$ is trivial, then there is an element $g \in G$ such that $g^* = \chi$ (in fact, by Theorem 2 of [**11**], g is an essential hyperbolic element). Now g acts on $\mathbf{H}$ without any fixed points, so from Lemma 3.5, $g(\mathscr{D}_{\tilde{\omega}}(\hat{z}))$ is disjoint from $\mathscr{D}_{\tilde{\omega}}(\hat{z})$, which in turn implies that $g^*(\Delta_{\tilde{\omega}}(\hat{z}))$ is disjoint from $\Delta_{\tilde{\omega}}(\hat{z})$. But $g^* = \chi$. It turns out that $\chi(\Delta_{\tilde{\omega}}(\hat{z}))$ is disjoint from $\Delta_{\tilde{\omega}}(\hat{z})$, contradicting that $\chi(\Delta_{\tilde{\omega}}(\hat{z})) = \Delta_{\tilde{\omega}}(\hat{z})$.

If $\tilde{\chi}$ is elliptic, then there is an integer $k \geq 2$ such that $\tilde{\chi}^k(\tilde{x}) = \tilde{x}$ for a point $\tilde{x} \in T(\tilde{S})$. Let $x \in \mathscr{D}_{\tilde{\omega}}(\hat{z})$ be such that $\pi(x) = \tilde{x}$. Since χ leaves invariant $\Delta_{\tilde{\omega}}(\hat{z})$, and since $i(\chi) = \tilde{\chi}$, we obtain

$$\pi \circ \chi^k(x) = \tilde{\chi}^k(\tilde{x}) = \tilde{x} = \pi(x). \tag{4.3}$$

But we know that the restriction of π to $\mathscr{D}_{\tilde{\omega}}(\hat{z})$ is an embedding. It follows from (4.3) that $\chi^k(x) = x$. This implies that A_χ is a non-trivial elliptic Möbius transformation. So Case 1 can not occur, and we conclude that χ is not hyperbolic.

Case 2. A_χ is parabolic. By taking a suitable power if necessary, we may assume that χ is represented by multi twists along a system

$$\{c_1, \ldots, c_s\}, \quad s \geq 1 \tag{4.4}$$

of disjoint simple closed geodesics on S. If $s \geq 2$, or $s = 1$ but c_1 projects a non-trivial geodesic on $\tilde{S}$, then $\tilde{\chi} = i(\chi)$ is non-trivial. We can use Lemma 3.6 to conclude that $\tilde{\chi}$ can be represented by a quasi-conformal map f_0 that has a Beltrami coefficient $t\mu$ for some $t \in \mathbf{D} - \{0\}$ and satisfies $f_0(z) = z$ for $z = \varrho(\hat{z})$. This means $z \in B$ and so $\hat{z} \in \Sigma$; that is, $\hat{z} \notin \Omega$. Once again, this is a contradiction.

If $s = 1$ and c_1 projects to the trivial mapping class of $\tilde{S}$, then in this case, c_1 bounds a twice punctured disk on S that encloses a and another puncture of $\tilde{S}$. This implies that there is a parabolic element $g \in G$ with $g^* = \chi$. By the same argument as in Case 1, we see that $\chi(\Delta_{\tilde{\omega}}(\hat{z}))$ is disjoint from $\Delta_{\tilde{\omega}}(\hat{z})$, contradicting the fact that $\chi(\Delta_{\tilde{\omega}}(\hat{z})) = \Delta_{\tilde{\omega}}(\hat{z})$. Hence χ cannot be parabolic.

Case 3. A_χ is elliptic. We can conclude the proof of Lemma 4.1 by proving the following result.

LEMMA 4.2. *The group $V_{\tilde{\omega}}(\hat{z})$, $\hat{z} \in \Omega$, only contains elliptic elements with a common fixed point in $\Delta_{\tilde{\omega}}(\hat{z})$.*

Proof. If $V_{\tilde{\omega}}(\hat{z})$ contains another elliptic element χ' with distinct fixed points, then via (4.2), we obtain two elliptic Möbius transformations M and M' with distinct fixed points. According to Theorem 7.39.2 of Beardon [**4**], the commutator $[M, M']$ is hyperbolic. Hence $\iota \circ [M, M'] \circ \iota^{-1} \in V_{\tilde{\omega}}(\hat{z})$ is hyperbolic and by the above argument, this is impossible. Therefore, $V_{\tilde{\omega}}(\hat{z})$ is at most finitely cyclic. □

This also completes the proof of Lemma 4.1. □

Denote by

$$\mathscr{F} = \pi^{-1}(\tilde{\Delta}_{\tilde{\omega}}) = \{x \in F(\tilde{S}) : \pi(x) \in \tilde{\Delta}_{\tilde{\omega}}\}.$$

To complete the proof of Theorem 1.1 (1), we need to identify those points $\hat{z}_n \in \Omega$ for which $\Delta_{\tilde{\omega}}(\hat{z}_n)$ may be invariant under some elliptic mapping classes in Mod_S^a. We then remove those $\hat{z}_n$ from Ω. To accomplish this goal, we first prove the following lemma.

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LEMMA 4.3. *Let* $\chi \in Mod_S^a$ *be elliptic and keeps* $\Delta_{\tilde{\omega}}(\hat{z})$ *invariant. Let* A_χ *be defined under the isometry* (4.2). *Assume that* A_χ *is a non-trivial elliptic element. Let* $y_0 \in \mathbf{D}$ *be the fixed point of* A_χ. *Then* $v_0 = \iota(y_0)$ *is the only fixed point of* χ *in* $\varphi(\mathscr{F})$.

Proof. Obviously, $\chi(v_0) = v_0$ and χ fixes no other points of $\Delta_{\tilde{\omega}}(\hat{z})$. Let $x_0 \in \mathscr{D}_{\tilde{\omega}}(\hat{z})$ be such that $\varphi(x_0) = v_0$. Denote by $[\nu] = \pi(x_0) \in T(\tilde{S})$.

Suppose that χ also fixes a point $v \in \varphi(\mathscr{F})$ with $v \neq v_0$. Write $v = \varphi(x)$ for some $x \in \mathscr{F} \subset F(\tilde{S})$. Since the element $[\hat{f}] \in \text{mod}(\tilde{S})$, where $[\hat{f}]^* = \chi$, is fibre preserving, and since $[\hat{f}]$ keeps $\mathscr{D}_{\tilde{\omega}}(\hat{z})$ invariant, x and x_0 lie in the same fibre $\pi^{-1}([\nu]) \subset \mathscr{F}$ over $[\nu]$. From the assumption, the restriction of $[\hat{f}]$ to the fibre $\pi^{-1}([\nu])$ is conformal and fixes the two points x and x_0. So the restriction of $[\hat{f}]$ to $\pi^{-1}([\nu])$ must be the identity. On the other hand, the action of $[\hat{f}]$ on the fibre is written as

$$[\hat{f}]([\nu], \hat{z}) = ([\nu], w^\nu \circ \hat{f} \circ (w^\nu)^{-1}(\hat{z})).$$

We see that $\hat{z} = w^\nu \circ \hat{f} \circ (w^\nu)^{-1}(\hat{z})$, or $\hat{f} \circ (w^\nu)^{-1}(\hat{z}) = (w^\nu)^{-1}(\hat{z})$. Since $(w^\nu)^{-1}(\hat{z}) \in \mathbf{H}$ is an arbitrary point, we conclude that $[\hat{f}]$ is trivial. This is again a contradiction. □

Now from Lemma 4.3, we know that there exist at most countably many elliptic mapping classes χ_n each of which, when viewed as an automorphism of $\varphi(\mathscr{F})$, has a single distinct fixed point v_n in $\varphi(\mathscr{F})$. Assume that χ_n keeps $\Delta_{\tilde{\omega}}(\hat{z}_n)$ invariant. We claim that the only fixed point v_n of χ_n must lie in $\Delta_{\tilde{\omega}}(\hat{z}_n)$.

Indeed, by $\iota_n : \mathbf{D} \to \Delta_{\tilde{\omega}}(\hat{z}_n)$ we denote the corresponding isometry. Then χ_n defines a trivial or elliptic Möbius transformation A_{χ_n}. If A_{χ_n} is trivial, we use the previous argument to show that this is impossible. If A_{χ_n} is non-trivial, A_{χ_n} has a unique fixed point y_n in $\mathbf{D}$. Thus $\iota_n(y_n) = v_n$ is fixed by χ_n and $v_n \in \Delta_{\tilde{\omega}}(\hat{z}_n)$.

Moreover, from Lemma 4.2 we conclude that $\hat{z}_n \neq \hat{z}_m$ whenever $m \neq n$. Hence by Lemma 3.5, $\Delta_{\tilde{\omega}}(\hat{z}_n)$ is disjoint from $\Delta_{\tilde{\omega}}(\hat{z}_m)$ whenever $m \neq n$. It follows that there exist at most countably many points $\hat{z}_n \in \Omega$ such that $\Delta_{\tilde{\omega}}(\hat{z}_n)$ contains v_n and does not contain any other v_m. Therefore, if we set

$$\Omega_0 = \Omega - \bigcup_{n \geq 0} \{z_n\},$$

then Ω_0 is an uncountable subset of $\mathbf{H}$ with a full measure and for any $\hat{z} \in \Omega_0$, $\Delta_{\tilde{\omega}}(\hat{z})$ avoids all single fixed points of elliptic mapping classes of Mod_S^a. Hence $\Delta_{\tilde{\omega}}(\hat{z})$, $\hat{z} \in \Omega_0$, can not be an invariant disk by any elliptic mapping class of Mod_S^a. This proves (1) of Theorem 1.1.

(2) For a polynomial

$$P(x) = x^n - x^{n-1} - \cdots - x - 1 \tag{4.5}$$

with degree $n \geq 3$, there is a real root $c > 1$ and all other roots are within the unit circle. In literature c is called the Pisot root. The main result of Arnoux and Yoccoz [**3**] states that there exists a non-conformal absolutely extremal Teichmüller self-mapping $\tilde{\omega}$ on a surface whose maximal dilatation $K(\tilde{\omega})$ satisfies the condition that $K(\tilde{\omega})^{1/2} = c$.

Let $\tilde{\theta}$ be the hyperbolic mapping class represented by $\tilde{\omega}$. Let $\tilde{\Delta}_{\tilde{\omega}}$ be the Teichmüller disk in $T(\tilde{S})$ invariant under the action of $\tilde{\theta}$. We see that for any $\hat{z} \in \mathbf{H}$, $\Delta_{\tilde{\omega}}(\hat{z})$ defined as (1.6) is a Teichmüller disk in $T(S)$. For any $\hat{z} \in \Omega_0$, from (1) of Theorem 1.1, $V_{\tilde{\omega}}(\hat{z})$ is trivial. So it remains to consider those points $\hat{z}$ in $\mathbf{H} - \Omega_0$. We claim that $V_{\tilde{\omega}}(\hat{z})$ for $\hat{z} \in \mathbf{H} - \Omega_0$ does not contain any parabolic mapping classes.

Suppose that there is a parabolic mapping class $\chi \in \mathrm{Mod}_S^a$ such that $\chi(\Delta_{\tilde{\omega}}(\hat{z})) = \Delta_{\tilde{\omega}}(\hat{z})$ for $\hat{z} \in \mathbf{H} - \Omega_0$. By taking a suitable power if necessary, we assume that χ is reduced by the curve system (4.4). If $s = 1$ and c_1 bounds a twice punctured disk D, then χ is represented by the Dehn twist $t_{\partial D}$. So $i(\chi)$ is trivial. By Theorem 2 of [**11**], there is a parabolic element $g \in G$ with $g^* = \chi$. Then we can use the same argument of Lemma 4.1 to conclude that this cannot happen. If $s > 1$ or $s = 1$ but c_1 does not bound any twice punctured disk, then $\tilde{\chi} = i(\chi)$ is a non-trivial parabolic mapping class. From Lemma 3.4, we have $j(\Delta_{\tilde{\omega}}(\hat{z})) = \tilde{\Delta}_{\tilde{\omega}}$. By Lemma 3.2, $\tilde{\chi}$ leaves invariant the disk $\tilde{\Delta}_{\tilde{\omega}}$. It follows that the Veech group of $\tilde{\Delta}_{\tilde{\omega}}$ contains a parabolic element $\tilde{\chi}$. This contradicts Corollary 1.3 of Hubert and Lanneau [**8**].

We conclude that for any point $\hat{z} \in \mathbf{H}$, the group $V_{\tilde{\omega}}(\hat{z})$ is either trivial or only contains hyperbolic and elliptic elements. Now the argument of Corollary 1.5 of [**8**] can be carried over to our case: any elliptic mapping class is induced by a conformal automorphism ζ of some surface. So by Hurwitz's theorem (see Theorem V.1.3 of [**9**]), the order of ζ is bounded above by $84(p-1)$. According to Theorem 7 of Purzitsky [**14**], We conclude that for any point $\hat{z} \in \mathbf{H}$, the group $V_{\tilde{\omega}}(\hat{z})$, if non-trivial, must contain a finite index subgroup consists of only hyperbolic elements. This proves (2) of Theorem 1.1.

5. Some remarks. (1) We outline the proof of Corollary 1.3 of [**8**] as follows. Details can be found in [**8**]. The polynomial (4.5) of degree $n \geq 3$ has two real roots if n is even and only one real root if n is odd. Since c is the Pisot root, one proves that $\mathbb{Q}[c]$ is not totally real. A computation shows that $\mathbb{Q}[c + c^{-1}]$ is not totally real either. On the other hand, if $\Delta_{\tilde{\omega}}$ is stabilized by the hyperbolic mapping class $\tilde{\theta}$ that is represented by a product of two parabolic mapping classes, then Theorem 1.1 of [**8**] asserts that (by using the results of Arnoux and Yoccoz [**3**] and Kenyon and Smillie [**13**]) $\mathbb{Q}[c + c^{-1}]$ can be identified with the trace field which must be totally real, which leads to a contradiction.

(2) The argument of Theorem 1.1 (1) yields that there is an uncountable set Ω_0 of $\mathbf{H}$ with a full measure such that the Veech group of each disk $\Delta_{\tilde{\omega}}(\hat{z})$ for $\hat{z} \in \Omega_0$ is at most finitely cyclic with order $n+1$. Indeed, by Theorem 10 of Bers [**5**], the group Mod_S^a is a subgroup of Mod_S with index $n+1$. Suppose that there is a non-trivial $\chi \in \mathrm{Mod}_S$ such that $\chi(\Delta_{\tilde{\omega}}(\hat{z})) = \Delta_{\tilde{\omega}}(\hat{z})$. If χ is hyperbolic or parabolic, then $\chi^{n+1} \in \mathrm{Mod}_S^a$ is also hyperbolic or parabolic, and the argument remains the same. If χ is elliptic with χ^{n+1} being non-trivial, our argument is still valid. It follows that for $\hat{z} \in \Omega_0$, the Veech group V of $\Delta_{\tilde{\omega}}(\hat{z})$ only consists of elliptic elements χ_i with $\chi_i^{n+1} = \mathrm{id}$. Hence, by the Nielsen realization theorem (see Kerckhoff [**10**]), V itself is finitely cyclic of order $n+1$.

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