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THE PERIODIC RADICAL OF GROUP RINGS AND INCIDENCE ALGEBRAS

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ABSTRACT. Let R be a ring with 1 and $P(R)$ the periodic radical of R. We obtain necessary and sufficient conditions for $P(\mathrm{RG}) = 0$ when RG is the group ring of an FC group G and R is commutative. We also obtain a complete description of $P\big(I(X, R)\big)$ when $I(X, R)$ is the incidence algebra of a locally finite partially ordered set X and R is commutative.

An element x in a ring R is called *periodic* if there exist distinct positive integers m and n such that $x^m = x^n$, while a ring consisting entirely of periodic elements is itself called *periodic*. In [2], Bell and Klein showed that periodicity is a radical property in the sense of Kurosh and Amitsur—specifically, they showed that any ring R has a maximum periodic ideal $P(R)$, defined as the sum of all periodic ideals, and that $R/P(R)$ has no nontrivial periodic ideals. Guo [4] continued the study of this periodic radical by showing that $P(R)$ is an intersection of suitable prime ideals and, consequently, that the periodic radical is a special radical (in the sense of Andrunakievic, see Divinsky [3] for details).

In Sections 2 and 3 of this paper, we investigate periodic radicals of group rings and incidence algebras. In the case of incidence algebras, a complete description is obtained whenever the coefficient ring is commutative with 1. In the case of a group ring RG, necessary and sufficient conditions for $P(\mathrm{RG}) = 0$ are obtained whenever R is commutative with 1 and G is an FC group. The group ring results are similar in spirit to Theorem 2 of Lawrence [5], where the algebraic radical of RG is studied when R is a field of characteristic p and G is an FC p'-group.

Section 1 contains general results about $P(R)$ which are needed later on. However, we feel that some of these results are of interest on their own—for example, Proposition 1.3 is an extension of Theorem 1 of Guo [4] in the case where R has 1.

For all necessary definitions, notation and background information on group rings or incidence algebras, the reader is referred to [8] and [9]. We assume for the rest of this paper that R has 1, although some of the observations do apply more generally.

1. **Introduction.** The following observations are well known (see, for example, Bell [1]).

LEMMA 1.1. *Let x belong to a ring R.*
(i) x is periodic if and only if x^n is an idempotent for some $n \geq 1$.

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(ii) If x is periodic and R has no nonzero nilpotent elements, then $x^n = x$ for some $n \geq 2$.

PROOF. (i) Say $x^m = x^n$ with $d = m - n > 0$. Then $x^n = x^{n+sd}$ for all $s \geq 1$. Hence $x^n = x^{2n+r}$ for some $r \geq 0$, in which case x^{n+r} is an idempotent.

(ii) Say $x^m = x^n$ with $m > n$. Then $x^n(x^{m-n} - 1) = 0$. Hence $x(x^{m-n} - 1)$ is nilpotent, and the result follows. ∎

The next result describes an important relationship among $P(R)$, the Jacobson radical $J(R)$, and the upper nil radical $U(R)$.

PROPOSITION 1.2. *$P(R) \cap J(R) = U(R)$.*

PROOF. $U(R) \subseteq P(R) \cap J(R)$ is immediate. For the converse, let $x \in P(R) \cap J(R)$. Since $x \in P(R)$, x^n is an idempotent for some $n \geq 1$ by Lemma 1.1(i). But then $x^n \in J(R)$ forces $x^n = 0$. Hence $P(R) \cap J(R)$ is a nil ideal and must be contained in $U(R)$. ∎

In Section 2, we will need the following sharper (when R has 1) version of Guo's Theorem 1 [4].

PROPOSITION 1.3. *$P(R) = \bigcap_\alpha P_\alpha$, where the intersection is taken over the set of prime ideals P_α such that R/P_α contains no nontrivial periodic ideals and such that if an integer z is a non zero divisor in R, then it is still a non zero divisor in R/P_α.*

PROOF. Go through exactly the same steps as in Guo's proof, but in the second paragraph change the definition of H to the following: $H = \big\{z\big(b^n - b^{n+1}p(b)\big) \mid n \in \mathbb{Z}^+, p(x) \in \mathbb{Z}[x], z \text{ a non zero divisor in } R\big\}$. ∎

Let I be an index set and put $R_i = R$ for all $i \in I$. In Section 3, we will need to know the structure of $P(A)$ when $A = \Pi_{i\in I} R_i$, the direct product of copies of R, in the case where R is commutative. Although such a ring A can be viewed as a very special type of incidence algebra, we will give a description of $P(A)$ here.

Let $x \in P(R)$. For $y \in R$, define $e_x(y)$ to be the smallest positive integer such that $(xy)^{e_x(y)}$ is an idempotent (see Lemma 1.1). Then define e_x as follows:

$$e_x = \begin{cases} \max\{e_x(y) \mid y \in R\} & \text{if it exists} \\ \infty & \text{otherwise.} \end{cases}$$

PROPOSITION 1.4. *Assume R is commutative and $A = \Pi_{i\in I} R_i$, with $R_i \cong R$ for each i. Let $a = (a_i)_{i\in I}$ belong to A. Then $a \in P(A)$ if and only if the following conditions all hold.*

(i) $a_i \in P(R)$ for all $i \in I$.
(ii) $|\{i \mid e_{a_i} = \infty\}| < \infty$.
(iii) There exists an integer N such that whenever $e_{a_i} < \infty$, then $e_{a_i} < N$.

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PROOF. Suppose $a \in P(A)$. It is clear that $a_i \in P(R)$ for all $i \in I$ (if $r \in R$, multiply by $(r)_{i\in I}$). If either (ii) or (iii) fails to hold, then we can find a subset $\{i_1, i_2, \ldots\}$ of I and elements $b_{i_1}, b_{i_2}, \ldots$ of R such that

$$e_{a_{i_1}}(b_{i_1}) < e_{a_{i_2}}(b_{i_2}) < \cdots.$$

For $k \in I$, let $c_k = b_{i_j}$ if $k = i_j$ for some j and $c_k = 0$ otherwise. Set $c = (c_i)_{i\in I}$. Since $ac \in P(A)$, Lemma 1.1 says that there must exist a positive integer n such that $(ac)^n = (ac)^{2n}$. But this forces $(a_ic_i)^n = (a_ic_i)^{2n}$ for all i, contradicting the fact that $e_{a_{i_j}}(b_{i_j}) > n$ for some j.

The converse follows easily from the simple observation that if x^m is an idempotent, then so is x^{mn} for all $n \geq 1$. ∎

2**. Group rings.** The following was proved by Schneider and Weissglass [7].

PROPOSITION 2.1. *If $U(R) = 0$ and $|H|$ is a non zero divisor in R for any finite subgroup H of a group G, then $U(\mathrm{RG}) = 0$.*

Our first result is a periodic radical analogue of Proposition 2.1.

PROPOSITION 2.2. *Let R be commutative. If $P(R) = 0$ and $|H|$ is a non zero divisor for any finite subgroup H of G, then $P(\mathrm{RG}) = 0$.*

PROOF. Assume to the contrary that the conditions hold but $0 \neq y \in P(\mathrm{RG})$. Choose a set $\{P_\alpha\}$ of prime ideals of R, following the description in Proposition 1.3, such that $0 = \bigcap_\alpha P_\alpha$. Then $\bigcap_\alpha(P_\alpha G) = 0$, and hence $0 \neq \bar{y} \in P\big((R/P_\alpha)G\big)$ for some P_α. Since $P(R/P_\alpha) = 0$ and $|H|$ is still a non zero divisor in R/P_α for all finite subgroups H of G, we may assume from now on that R is an integral domain. We also know that $U(\mathrm{RG}) = 0$ by Proposition 2.1, so we can assume that y is not nilpotent. Lemma 1.1 then tells us that we can assume y is an idempotent.

Choose any $0 \neq r \in R$. Since $ry \in P(\mathrm{RG})$, we must have $(ry)^n = (ry)^{2n}$ for some $n \geq 1$. Since R is an integral domain and y is an idempotent, this says that $r^ny = y$. But again using the fact that R is an integral domain, we conclude that every nonzero element of R is a unit of finite order, contradicting $P(R) = 0$. ∎

In general, the converse to Proposition 2.2 is not true (if $R = \mathbb{Z}_2$ and $G = \langle a, b \mid a^2 = 1, ab = b^{-1}a\rangle$, then $P(\mathrm{RG}) = 0$ but both conditions fail). However, the following is true and is easy to prove.

PROPOSITION 2.3. *Let R be commutative. If $P(\mathrm{RG}) = 0$, then $U(R) = 0$ and $|H|$ is a non zero divisor for any finite normal subgroup H of G.*

When G is an FC group, we are able to obtain necessary and sufficient conditions for $P(\mathrm{RG}) = 0$. Let $T(G)$ denote the torsion subgroup of an FC group G (see [6] for relevant background information).

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THEOREM 2.4. *Assume that R is commutative and that G is an* FC *group. Then $P(\mathrm{RG}) = 0$ if and only if $|H|$ is a non zero divisor for any finite subgroup H of G and one of the following holds.*

(i) $G = T(G)$ and $P(R) = 0$.

(ii) $G \neq T(G)$ and $U(R) = 0$.

PROOF. First assume $P(\mathrm{RG}) = 0$. Since G is FC, any finite subgroup of G is contained in a finite normal subgroup of G, so the condition on $|H|$ follows from Proposition 2.3. Proposition 2.3 also tells us that $U(R) = 0$, so we may assume from now on that $G = T(G)$. In this case, assume that $P(R) \neq 0$ and let $\sum \alpha_g g$ be a nonzero element in $P(R)G$. Since $U(R) = 0$, Lemma 1.1 tells us that for each g there exists $n_g \geq 2$ such that $\alpha_g^{n_g} = \alpha_g$. Consequently, since $z\alpha_g$ is in $P(R)$ for each g and for every integer $z \geq 1$, we get that α_g is of finite additive order for each g. It follows that the subring of R generated by $\{\alpha_g\}$ is a finite ring. Since $G = T(G)$ is an FC group, the subring of RG generated by $\sum \alpha_g g$ must therefore be finite, and hence periodic. We conclude that $P(R)G \subseteq P(\mathrm{RG})$, contradicting $P(\mathrm{RG}) = 0$.

For the converse, assume that the condition on $|H|$ holds and also that one of (i) and (ii) holds. Because of Proposition 2.2, we may also assume we are in case (ii), *i.e.* that $G \neq T(G)$ and $U(R) = 0$. The rest of the proof is an adaptation of the argument given for Theorem 2 in Lawrence [5].

Assume to the contrary that $P(\mathrm{RG}) \neq 0$ and let $0 \neq \alpha = x_1 s_1 + \cdots + x_n s_n$ be in $P(\mathrm{RG})$, where the $x_i \in R\big(T(G)\big)$ and the s_i come from a complete set of coset representatives S of $T(G)$ in G. Since $G/T(G)$ is ordered, we may assume that $\overline{s_1} > \overline{s_2} > \cdots > \overline{s_n}$, and we may also choose α so that $\overline{s_1} > 1$. We will reach a contradiction by showing that x_1 is nilpotent (in fact, this will show that the ideal generated by x_1 is nil, contradicting Proposition 2.1).

Since the subgroup generated by the group elements in the support of x_1 lies in a finite normal subgroup of G, some power of s_1, say s_1^k, commutes with x_1. Let $\beta = \alpha s_1^{k-1}$. Since $\beta \in P(\mathrm{RG})$, we must have $\beta^l = \beta^m$ for some $l > m \geq 1$. But $\beta^l = x_1^l s_1^{kl} + \sum y_\sigma s_\sigma$ where $y_\sigma \in R\big(T(G)\big)$ and $\overline{s_1^{kl}} > \overline{s_\sigma}$ for all σ, and similarly $\beta^m = x_1^m s_1^{km} + \sum y_\epsilon s_\epsilon$. Since $\overline{s_1^{kl}} > \overline{s_1^{km}}$, we can only conclude that $x_1^l = 0$. ∎

In the first half of the above proof, we needed to show at one point that if $U(R) = 0$ and G is a torsion FC group, then $P(R)G \subseteq P(\mathrm{RG})$. A direct argument was given for this, using the fact that R is commutative. Alternatively, we could have applied the following more general result.

PROPOSITION 2.5. *Assume R has the property that for any element $x \in R$ there exists a positive integer n_x such that $x^{n_x} = x$, and let S be a free normalizing extension of R. Then the ideal generated by $P(R)$ in S is contained in $P(S)$.*

PROOF. If k is the number of elements in a normalizing basis of S over R, then S can

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be viewed in a natural way as a subring of the matrix ring $M_k(R)$, with $P(R)S$ contained in $M_k\big(P(R)\big)$, so it is enough to prove that $M_k\big(P(R)\big) \subseteq P\big(M_k(R)\big)$.

Let A be in $M_k\big(P(R)\big)$, and let T be the subring of R generated by the entries of A. Then T is finite ([10], Theorem 3.4) and hence A, being in the finite ring $M_k(T)$, is periodic. ∎

It is not hard to see that the opposite inclusion to that mentioned in the above proof, namely $P\big(M_k(R)\big) \subseteq M_k\big(P(R)\big)$, always holds, so we have shown that $M_k\big(P(R)\big) = P\big(M_k(R)\big)$ whenever R satisfies the given conditions.

A natural question to ask is whether this equality always holds. It turns out, however, that this is yet another in a long series of ring theoretic statements which, if correct, would imply the truth of the Koethe conjecture. To see this just observe that since $M_k\big(J(R)\big) = J\big(M_k(R)\big)$, Proposition 1.2 tells us that the statement would imply $M_k\big(U(R)\big) = U\big(M_k(R)\big)$. But it is well known that this last equality implies the Koethe conjecture.

3**. Incidence algebras.** Following the notation in [9], we will let $I(X,R)$ denote the incidence algebra of a locally finite partially ordered set X over a ring R. Recall that the elements of $I(X,R)$ are functions f from $X \times X$ to R such that $f(x,y) = 0$ if $x \not\leq y$. Addition in $I(X,R)$ is pointwise and multiplication is given by $fg(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$. The reader is asked to refer to [9] for more details, but it is perhaps worth noting that it is very possible to have $f(x,y) \neq 0$ for infinitely many ordered pairs (x,y).

We will need to recall the following results concerning the Jacobson and upper nil radicals of $I(X,R)$.

THEOREM 3.1 ([9], P. 170). *If R is commutative, then the Jacobson radical $J\big(I(X,R)\big)$ is the set of all functions $f \in I(X,R)$ such that $f(x,x) \in J(R)$ for each $x \in X$.*

If R is commutative, a function $f \in I(X,R)$ is said to be *fully nilpotent* ([9], p. 174) if there is a positive integer n with the property that given any chain of the form $x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_n \leq y_n$ in X, then $\Pi_{i=1}^{n} f(x_i,y_i) = 0$.

THEOREM 3.2 ([9], PP. 183 AND 185). *If R is commutative, then the upper nil radical $U\big(I(X,R)\big)$ is precisely the set of fully nilpotent functions in $I(X,R)$. Moreover the prime radical of $I(X,R)$ is equal to $U\big(I(X,R)\big)$.*

We now proceed to describe $P\big(I(X,R)\big)$. First, some new notation is required.

If $f \in I(X,R)$, then we can decompose f uniquely as $f = f_D + f_E$ where $f_D(x,y) = 0$ if $x \neq y$ and $f_E(x,y) = 0$ if $x = y$. Note that f_D can be conveniently thought of as an element of $\Pi_{x \in X} R$.

DEFINITION 3.3. Let R be commutative. An element $f \in I(X,R)$ is called *fully periodic* if

(i) $f_D \in P(\Pi_{x \in X} R)$ and

(ii) there exists a positive integer n such that if $x_1 \leq y_1 < x_2 \leq y_2 < \cdots < x_n \leq y_n$ in X, then $\Pi_{i=1}^{n} f(x_i,y_i) = 0$.

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Recall that necessary and sufficient conditions for (i) to hold were obtained in Proposition 1.4. In particular, fully nilpotent elements are fully periodic. It is also clear from the definition that if f is fully periodic, then f_E is fully nilpotent.

Our main result is as follows.

THEOREM 3.4. *If R is commutative, then $P\big(I(X,R)\big)$ is precisely the set of fully periodic elements of $I(X,R)$.*

PROOF. First note that if we let K denote the set of fully periodic elements of $I(X,R)$, then K is an ideal of $I(X,R)$. Let $f \in K$. Condition (i) of Definition 3.3 tells us that there exist distinct positive integers m, n such that $f^m - f^n = (f^m - f^n)_E$. Since $f^m - f^n$ is fully periodic, it follows that $f^m - f^n$ is fully nilpotent and therefore in $U\big(I(X,R)\big)$ (Theorem 3.2). Hence $\bar{K} \subseteq P\Big(I(X,R)/U\big(I(X,R)\big)\Big) = \frac{P(I(X,R))}{U(I(X,R))}$, and so $K \subseteq P\big(I(X,R)\big)$.

Conversely, assume $f \in P\big(I(X,R)\big)$ is not fully periodic. Since condition (i) of Definition 3.3 is clearly satisfied, we conclude that for each positive integer n there exists a chain

$$x_{n,1} \le y_{n,1} < x_{n,2} \le y_{n,2} < \cdots < x_{n,n} \le y_{n,n}$$

such that

$$\Pi_{i=1}^{n} f(x_{n,i}, y_{n,i}) \neq 0.$$

Local finiteness allows us also to assume that the intervals $[x_{n,1}, y_{n,n}]$ and $[x_{m,1}, y_{m,m}]$ are disjoint whenever $m \neq n$.

Let $h \in I(X,R)$ be defined by $h(y_{n,i}, x_{n,i+1}) = 1$ for $i = 1, 2, \ldots, n-1$ and for all $n \geq 2$. Also set $h(x,y) = 0$ in all other cases. Then $(fh)_D = 0$, so Theorem 3.1 and Proposition 1.2 say that $fh \in J\big(I(X,R)\big) \cap P\big(I(X,R)\big) = U\big(I(X,R)\big)$, and therefore fh is fully nilpotent by Theorem 3.2. However, applying fh to the chains

$$x_{n,1} < x_{n,2} \le x_{n,2} < x_{n,3} \le x_{n,3} \cdots < x_{n,n-1} \le x_{n,n-1} < x_{n,n}$$

gives a contradiction. ∎

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