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Dr THIRD, President, in the Chair.

Some Methods applicable to Identities and Inequalities of Symmetric Algebraic Functions of n Letters.

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The methods explained here are applicable to a large number of problems relating to the symmetric algebraic functions of n letters, and the special results here deduced from them are merely specimens to indicate some of the ways of applying these methods.

In the first section the main principle adopted is that of taking the standard form of a symmetric function to be a sum extending over all the cases of the typical term got by permuting the letters involved in all possible ways, whether they are different or not; and the main result reached is an Inequality Theorem arrived at by expressing the excess of the greater over the less in an explicitly positive form.

The method of the second section, while closely related to that of the first, is in some cases more easily applicable.

Both methods, though leading to rather complicated considerations in certain of their applications, are essentially elementary in their character, *i.e.*, they are based immediately on elementary ideas, and do not involve any of the more abstract conceptions of Higher Algebra.

§ 1.

Consider a term of the form $a^\alpha b^\beta c^\gamma \ldots l^\lambda$, involving n letters $a, b, c \ldots l$, each of the indices $\alpha, \beta, \ldots \lambda$ being either a positive integer or zero.

(1) Let us denote by $\Sigma!(a^\alpha b^\beta \ldots l^\lambda)$, the sum of all the terms that can be got by all possible permutations of $a, b, c \ldots$ in the typical

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term (which is taken to involve all the letters, even if the indices of some of them are zero), while the indices $\alpha, \beta, \ldots \lambda$ remain unchanged; so that the sum contains $n!$ terms, and is homogeneous and symmetric as to $a, b, c \ldots l$.

For example,

$$\Sigma!(a^2b^2c^0) \equiv a^2b^2c^0 + a^2c^2b^0 + b^2a^2c^0 + b^2c^2a^0 + c^2a^2b^0 + c^2b^2a^0$$

and is $= 2\Sigma a^2b^2c^0$ or $2\Sigma a^2b^2$, when Σ has the usual signification of summation over all *different* terms of the type a^2b^2.

In general, if the indices are not all unequal, there being p of them having one common value, q of them having another common value, etc., we have

$$(2) \qquad \Sigma!(a^\alpha b^\beta c^\gamma \ldots l^\lambda) = (p!\,q!\ldots) \times \Sigma a^\alpha b^\beta \ldots l^\lambda.$$

This notation may be extended to symmetric sums of *any* functions of $a, b, c \ldots l$ (each of which is taken to involve *all* the letters, as explained above); so that we shall have

$$(3) \qquad \Sigma!\,\mathrm{F}(a, b, c \ldots l) = \mathrm{N}\Sigma\mathrm{F}(a, b, c \ldots l)$$

when F is a function of $a, b, c \ldots l$, and N is the number of times each particular *different value* of $\mathrm{F}(a, b, c \ldots l)$ occurs in the $\Sigma!$

We shall use the abbreviation

$$(4) \qquad [\alpha, \beta, \gamma \ldots \lambda] \equiv \Sigma!(a^\alpha b^\beta \ldots l^\lambda)$$

and for brevity the commas may sometimes be omitted.

Fundamental Inequality Theorem for expressions of the type $[\alpha, \beta, \gamma \ldots \lambda]$.

The expression $\Sigma!\{a^\beta b^\beta c^\gamma \ldots l^\lambda (a^\rho - b^\rho)(a^\sigma - b^\sigma)\}$ may be expanded into

$$\Sigma!\{a^{\beta+\rho+\sigma} b^\beta c^\gamma \ldots l^\lambda + a^\beta b^{\beta+\rho+\sigma} c^\gamma \ldots l^\lambda$$
$$- a^{\beta+\sigma} b^{\beta+\rho} c^\gamma d^\delta \ldots l^\lambda - a^{\beta+\rho} b^{\beta+\sigma} c^\gamma \ldots l^\lambda\}$$
$$= 2\Sigma!(a^{\beta+\rho+\sigma} b^\beta c^\gamma \ldots l^\lambda) - 2\Sigma!(a^{\beta+\rho} b^{\beta+\sigma} c^\gamma \ldots l^\lambda).$$

Hence, writing α for $\beta+\rho+\sigma$, we have, subject to the restriction $\alpha>\beta+\sigma$,

(5) $$\Sigma\,!\,(a^{\alpha}b^{\beta}c^{\gamma}\ldots l^{\lambda})-\Sigma(a^{\alpha-\sigma}b^{\beta+\sigma}c^{\gamma}\ldots l^{\lambda})$$

$$=\tfrac{1}{2}\Sigma\,!\,\{a^{\beta}b^{\beta}c^{\gamma}\ldots l^{\lambda}(a^{\alpha-\beta-\sigma}-b^{\alpha-\beta-\sigma})(a^{\sigma}-b^{\sigma})\}$$

whence, if $a, b, c\ldots$ are all positive, we have

(6) $$\Sigma\,!\,a^{\alpha}b^{\beta}c^{\gamma}\ldots l^{\lambda}>\Sigma\,!\,a^{\alpha-\sigma}b^{\beta+\sigma}c^{\gamma}\ldots l^{\lambda}$$

unless $a, b, c\ldots$ are all equal.

This fundamental inequality may be applied to prove the more general theorem, that if the conditions

(7) $$\alpha+\beta+\gamma\ldots+\lambda=\alpha'+\beta'+\gamma'\ldots+\lambda'$$

(8) $$\alpha\nless\alpha',\ \alpha+\beta\nless\alpha'+\beta',\ \alpha+\beta+\gamma\nless\alpha'+\beta'+\gamma',\ \text{etc.}$$

(one at least of the signs $\nless$ being $>$) are fulfilled then we have

(9) $$[\alpha,\ \beta,\ \gamma\ldots\lambda]>[\alpha',\ \beta',\ \gamma\ldots\lambda']$$

unless $a, b, c\ldots$ are all equal; and to express the difference

$$[\alpha\ \beta\ \gamma\ldots\lambda]-[\alpha'\ \beta'\ \gamma'\ldots\lambda']$$

in a form that is essentially positive.

Take, for example, $(6, 3, 2, 0, 0)-(4, 4, 1, 1, 1)$.

It may be written $(6, 3, 2, 0, 0)-(5, 4, 2, 0, 0)$

$+(5, 4, 2, 0, 0)-(4, 4, 2, 1, 0)$

$+(4, 4, 2, 1, 0)-(4, 4, 1, 1, 1)$

when each of the three lines is expressible by (5) as an essentially positive quantity, a, b, c, d, e being positive and not all equal.

In fact, for five letters a, b, c, d, e we have

$$\Sigma\,!\,a^{6}b^{3}c^{2}d^{0}e^{0}-\Sigma\,!\,a^{4}b^{4}cde$$

$$=\tfrac{1}{2}\Sigma!(a-b)(a^{2}-b^{2})a^{3}b^{3}c^{2}d^{0}e^{0}+\tfrac{1}{2}\Sigma!(a-b)(a^{4}-b^{4})c^{4}d^{2}e^{0}+\tfrac{1}{2}\Sigma\,!(a-b)^{2}c^{4}d^{4}e.$$

Similarly, to prove the general theorem (7), we must show that it is possible to insert between $[\alpha\beta\gamma\ldots\lambda]$ and $[\alpha'\beta'\gamma'\ldots\lambda']$ a series of quantities of the same type, but of continually descending order as distinguished by conditions (8), such that any two successive members of the complete series formed by the two given expressions and the intermediates have a difference of the form (5). Now suppose, for example, that β is the first index in $[\alpha\beta\gamma\ldots\lambda]$ which is greater than its correspondent in $[\alpha'\beta'\gamma'\ldots\lambda']$, and that ϵ' is the first of the set α', $\beta'\ldots\lambda'$ which is greater than its correspondent ϵ. Then the quantity $[\alpha,\ \beta-1,\ \gamma,\ \delta,\ \epsilon+1,\ldots\lambda]$ is of the same degree as $[\alpha,\ \beta,\ldots\lambda]$ or $[\alpha'\ \beta',\ldots\lambda]$ but its order as tested by (8) is lower than that of $[\alpha,\ \beta,\ldots\lambda]$ but not lower than that of $(\alpha',\ \beta',\ldots\lambda']$; while the difference $[\alpha,\ \beta,\ \gamma\ldots\lambda]-[\alpha,\ \beta-1,\ \gamma,\ \delta,\ \epsilon+1,\ldots\lambda]$ is of the form (5). The expression $[\alpha,\ \beta-1,\ \gamma,\ \delta,\ \epsilon+1,\ldots\lambda]$, then, would be the first of the intermediates required to bridge the interval between $[\alpha\beta\ldots\lambda]$ and $[\alpha'\beta'\ldots\lambda']$; and it is obvious that a finite number of these will suffice.

Thus Theorem (9) is proved. We can also show that for the difference $[\alpha\beta\gamma\ldots\lambda]-[\alpha'\beta'\gamma'\ldots\lambda']$ to be a quantity necessarily positive for all *positive* values of a, b, $c\ldots$ which are not all equal, the conditions (7) and (8) are *necessary* as well as sufficient. This is done by showing that by suitably choosing the values of a, b, $c\ldots$ we can, if (7) and (8) are not satisfied, make the difference positive or negative at will.

If (7) does not hold, *i.e.*, if the quantities are of different *degree*, it is clear that by making a, b, c all equal to a sufficiently large number, the quantity of higher degree would be the greater, which if we make a, b, $c,\ldots$ all equal to a sufficiently small fraction, the quantity of lower degree would be the greater.

If (8) does not hold: if, for example, $\alpha+\beta+\gamma<\alpha'+\beta'+\gamma'$, then by making a, b, c each equal to m a sufficiently great multiple of the greatest of the remaining quantities d, $e\ldots l$, it is clear that the only important terms in $[\alpha\beta\gamma\ldots\lambda]-[\alpha'\beta'\gamma'\ldots\lambda']$ will be those having m in its highest powers, say $\mathrm{P}m^{\alpha+\beta+\gamma}-\mathrm{Q}m^{\alpha'+\beta'+\gamma'}$, which for a sufficiently large value of m is negative, $\alpha+\beta+\gamma$ being less than $\alpha'+\beta'+\gamma'$.

Thus the conditions (7) and (8) are *necessary* as well as *sufficient* to ensure that $[\alpha\beta\ldots\lambda]-[\alpha'\beta'\ldots\lambda']$ should have a value necessarily positive for all positive values of $a, b, c\ldots l$ which are not all equal.

Expression for the Excess of the Arithmetic Mean of n quantities over their Geometric Mean.

As a special case of the preceding let us take

(11) $\quad [n-p+1,\ 1,\ 1,\ldots\ 0,\ 0]-[n-p,\ 1,\ 1,\ldots 0,\ 0]$

when in the former there are $p-1$ indices equal to 1 and $n-p$ equal to 0; and in the latter, p equal to 1 and $n-p-1$ equal to 0.

(12) The difference is $=\frac{1}{2}\Sigma\,!\,\{(a^{n-p}-b^{n-p})(a-b)c^1d^1\ldots k^0l^0\}$

where there are $p-1$ letters $c, d\ldots$ with index 1 and $n-p-1$ letters with index 0.

It may also be written

(13) $\quad \frac{1}{2}\Sigma\,!\,\{(a-b)^2\mathrm{H}_{n-p-1}c^1d^1\ldots k^0l^0\}$

$$=(n-p-1)\,!\,(p-1)\,!\,\Sigma(a-b)^2\mathrm{H}_{n-p-1}c^1d^1\ldots\ldots$$

when $\mathrm{H}_{n-p-1}\equiv$ sum of all homogeneous products of a and b of $n-p-1$ dimensions. Note that the factor $\frac{1}{2}$ is cancelled, since $\Sigma(a-b)^2=\frac{1}{2}\Sigma\,!\,(a-b)^2$.

Writing the identity (12) for the values $1, 2, \ldots \overline{n-1}$ of p and combining the results, we get

(14) $\quad [n,\ 0,\ 0\ldots]-[1,\ 1,\ 1\ldots]$

$$\begin{aligned}=\ &(n-2)\,!\,\Sigma(a-b)(a^{n-1}-b^{n-1})\\ &+(n-3)\,!\,1\,!\,\Sigma(a-b)(a^{n-2}-b^{n-2})c\\ &+(n-4)\,!\,2\,!\,\Sigma(a-b)(a^{n-3}-b^{n-3})cd\\ &+\ldots\ldots\quad +1\,!\,(n-3)\,!\,\Sigma(a-b)(a^2-b^2)cd\ldots k\\ &\qquad\qquad +(n-2)\,!\,\Sigma(a-b)(a-b)cd\ldots kl.\end{aligned}$$

But $[n,\ 0,\ 0\ldots]\equiv(n-1)\,!\,\Sigma a^n$, and $[1,\ 1,\ldots]\equiv n\,!\,abc\ldots kl.$

Hence, dividing by $n!$ we get

(15) $\frac{1}{n}\Sigma a^n - abc\ldots kl$

$$= \frac{1}{n(n-1)}\Sigma(a-b)(a^{n-1}-b^{n-1}) + \frac{1}{n(n-1)(n-2)}\Sigma(a-b)(a^{n-2}-b^{n-2})c$$
$$+ \frac{1.2}{n(n-1)(n-2)(n-3)}\Sigma(a-b)(a^{n-3}-b^{n-3})cd + \ldots\ldots\ldots\ldots$$
$$+ \frac{1}{n(n-1)(n-2)}\Sigma(a-b)(a^2-b^2)cd\ldots k + \frac{1}{n(n-1)}\Sigma(a-b)^2cd\ldots kl,$$

a formula which expresses the excess of the Arithmetic Mean of n positive quantities a^n, $b^n\ldots l^n$ over their Geometric Mean, in a form which shows it to be necessarily positive.

(16) It may be noted that the factor $(a-b)(a^p-b^p)$ may be written $(a-b)^2\mathrm{H}_{p-1}$, where $\mathrm{H}_r \equiv$ the sum of all possible homogeneous products of a and b of the rth degree.

Consider the difference

(17) $$\frac{[6,\ 5,\ 1]}{[3,\ 3,\ 2]} - \frac{[5,\ 5,\ 3]}{[4,\ 3,\ 2]}$$

where each of the fractions is of degree 4 in $abc\ldots$.

Bringing it to a common denominator, the numerator is

(18) $[6,\ 5,\ 1]\times[4,\ 3,\ 2]-[5,\ 5,\ 3]\times[3,\ 3,\ 2]$

$$=[10, 8, 3]+[10, 7, 4]+[9, 9, 3]+[9, 7, 5]+[8, 9, 4]+[8, 8, 5]$$
$$-[8, 8, 5]-[8, 7, 6]-[8, 8, 5]-[8, 7, 6]-[7, 8, 6]-[7, 8, 6].$$

This shows, in virtue of (9), that (16) is positive.

Under what circumstances can we prove that

$$\frac{[\alpha,\ \beta,\ \gamma\ldots]}{[p,\ q,\ r\ldots]} - \frac{[\alpha',\ \beta',\ \gamma'\ldots]}{[p',\ q',\ r'\ldots]}$$

is essentially positive for positive values of a, b, $c\ldots$ that are not all equal?

If we expand $[\alpha,\ \beta,\ \gamma\ldots]\times[p',\ q',\ r'\ldots]$ we get

$$\Sigma[\alpha+p',\ \beta+q',\ \gamma+r',\ldots],$$

where the summation extends over the $n!$ values got by keeping α, β, $\gamma\ldots$ in fixed order, and permuting p', q', $r'\ldots$ in all possible ways. A similar expansion of $[\alpha', \beta', \gamma' \ldots] \times [p, q, r \ldots]$ can be got; and the former expansion will be necessarily greater than the latter only if we can couple each $[\alpha + p', \beta + q', \gamma + r' \ldots]$ with a $[\alpha' + p, \beta' + q, \gamma' + r \ldots]$ in such a way that conditions similar to (7) and (8) are satisfied for each couple. (This remark is applicable to more general cases.)

In certain cases the possibility of this will be obvious without working out the expansions; for example, when the order of $[\alpha + p', \beta + q' \ldots]$ with $p', q', \ldots$ in *ascending* order, is higher (as tested by (8)) than that of $[\alpha' + p, \beta' + q \ldots]$ with $p, q, \ldots$ in descending order of magnitude. In that case every member of the expansion $\Sigma[\alpha + p', \beta + q', \ldots]$ will be of higher order than any member of the expansion $\Sigma[\alpha' + p, \beta' + q, \ldots]$ using 'order' for the moment to denote the relation defined by (8).

§ 2.

Method for extending to n letters certain symmetrical Identities which may be known to hold for a restricted number of letters.

Let $a, b, c, \ldots$ be n letters, and let $F(a, b, c \ldots)$ or simply $F(p)$ be a function involving only p of these letters.

(19) Let $\binom{n}{p} \equiv \dfrac{n!}{p!(n-p)!}$, and $\binom{n}{0} \equiv 1$.

(20) Let $S(n, p) \equiv \Sigma F(p)$ where the summation Σ extends to all the different values of $F(p)$ got by taking different selections of the n letters to form it.

For example, if $F(2) = ab$, then $S(n, 2) \equiv ab + ac + bc + \ldots$

and if $F(2) = a^2b$, then $S(n, 2) \equiv a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 + \ldots$.

Thus S is a certain symmetric function of $a, b, c \ldots$.

Then if $n > r \nless p$, we have the following identity:

(21) $$\Sigma S(r, p) = \binom{n-p}{r-p} S(n, p)$$

where Σ here denotes the summation of all cases of $S(r, p)$ got by the $\binom{n}{r}$ combinations of r letters taken from $a, b, \ldots$.

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The identity may be seen to be true by observing that each particular $F(p)$ occurs but once in $S(n, p)$, while in $\Sigma S(r, p)$ it occurs as often as there are cases of $S(r, p)$ containing it, *i.e.*, as often as its p letters can be associated with $r-p$ others out of the remaining $n-p$ letters.

Next let there be an identical relation of the form

$$AS(r, p) + BS(r, q) + \ldots\ldots = 0$$

r being not less than any of the numbers $p, q\ldots$, which is known to be true for r letters, where $n > r$ and A, B, C... are independent of $a, b, c\ldots.$ We can by (21) generalize it so as to apply to n letters, the result being:

$$(22) \qquad \binom{n-p}{r-p} AS(n, p) + \binom{n-q}{r-q} BS(n, q) + \ldots\ldots = 0.$$

In most cases it will be simplest to prove the identity first for $r =$ the greatest of the numbers $p, q\ldots.$ For example, we have for four letters, the identity

$$\Sigma a^2b^2(c-d)^2 = 3\Sigma a^2b^2c^2 - 2\Sigma a^2b^2cd.$$

Hence for n letters we get

$$\binom{n-4}{0}\Sigma a^2b^2(c-d)^2 = \binom{n-3}{1} 3 \,.\, \Sigma a^2b^2c^2 - \binom{n-4}{0} 2 \,.\, \Sigma a^2b^2cd$$

or
$$\Sigma a^2b^2(c-d)^2 = 3(n-3)\Sigma a^2b^2c^2 - 2\Sigma a^2b^2cd.$$

As another example, we have for six letters

$$\Sigma a^2b^2(cd-ef)^2 = 6\Sigma a^2b^2c^2d^2 - 6\Sigma a^2b^2cdef \qquad \text{(i)}$$

$$\Sigma a^2b^2cd(e-f)^2 = 3\Sigma a^2b^2c^2de - 12\Sigma a^2b^2cdef \qquad \text{(ii)}$$

and for five letters

$$\Sigma a^2b^2c^2(d-e)^2 = 4\Sigma a^2b^2c^2d^2 - 2\Sigma a^2b^2c^2de$$

$\therefore$ for six letters

$$\Sigma a^2b^2c^2(d-e)^2 = 2 \,.\, 4\Sigma a^2b^2c^2d^2 - 2\Sigma a^2b^2c^2de \qquad \text{(iii)}$$

From (i), (ii), (iii) we deduce, for six letters,

$$4\Sigma a^2b^2(cd-ef)^2 = 2\Sigma a^2b^2cd(e-f)^2 + 3\Sigma a^2b^2c^2(d-e)^2.$$

Hence by (22) we deduce for n letters $abc\ldots$

$$4\Sigma a^2b^2(cd-ef)^2 = 2\Sigma a^2b^2cd(e-f)^2 + 3(n-5)\Sigma a^2b^2c^2(d-e)^2.$$

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In expanding a function such as

$$(abc + abd + acd + bcd\ldots)^2$$

in terms of $\Sigma a^2b^2c^2$, Σa^2b^2cd, etc., where the Σ applies to all the different terms of the type indicated that can be formed from n letters a, b, $c\ldots$; *First* we note that the numerical coefficients of $\Sigma a^2b^2c^2$, etc., in the required expansion do not depend on the number n, so that the coefficient

of $\Sigma a^2b^2c^2$ in $(\Sigma abc)^2$ is the same as in $(abc)^2$, and that
,, Σa^2b^2cd ,, ,, ,, ,, ,, ,, ,, $(abc + abd + acd + bcd)^2$
,, Σa^2bcde ,, ,, ,, ,, ,, ,, ,, $(\sum_5 abc)^2$

where the symbol $\sum_5$ means summation with reference to a, b, c, d, e only; and so on. *Secondly*, the numerical coefficient can be determined by reckoning the number of ways in which the typical product under the Σ can arise in forming the expansion.

Thus, in the above case, we have

$$(\Sigma abc)^2 = \mathrm{A}\Sigma a^2b^2c^2 + \mathrm{B}\Sigma a^2b^2cd + \mathrm{C}\Sigma a^2bcde + \mathrm{D}\Sigma abcdef.$$

Here $\mathrm{A} = 1$ obviously; and $\mathrm{B} = 2$ since a^2b^2cd can only arise from abc and abd, whose product, in expanding the square is multiplied by 2. Again $\mathrm{C} = 6$, since a^2bcde can be separated into two factors in $\frac{1}{2} \times \binom{4}{2}$ ways, and the products of these are doubled. By similar reasoning, $\mathrm{D} = 2 \times \frac{1}{2} \times \binom{6}{3} = 20.$

Thus $\qquad (\Sigma abc)^2 = \Sigma a^2b^2c^2 + 2\Sigma a^2b^2cd + 6\Sigma a^2bcde + 20\Sigma abcdef.$

Let us study the relations between the different cases of the general function thus defined :—

(23) $\quad (r, p, t) \equiv \Sigma\{(a^2b^2\ldots$ to $r - p - t$ factors$)$ $(d.e\ldots$ to $2p$ factors$)$

$$(gh\ldots \text{ to } t \text{ factors} - lm\ldots \text{ to } t \text{ factors})^2\}$$

where the summation extends to all cases that can be formed from the n letters a, b, $c,\ldots$, each case containing $r + p + t$ different letters, and being of dimensions $2r$ in the letters a, b, $c\ldots$.

(24) Note that $(r, p, 0) \equiv \Sigma(ab\ldots$ to $2p$ factors$)$ $(k^2l^2\ldots$ to $r - p$ factors$)$.

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It is easy to see that for $r+p+t$ letters $abc\ldots$ we have

$$(25)\qquad (r,\ p,\ t)=\binom{r-p}{t}(r,\ p,\ 0)-\binom{2p+2t}{2t}\binom{2t}{t}(r,\ p+t,\ 0)$$

$$\therefore\quad (r,\ p+t-1,\ 1)=(r-p-t+1)(r,\ p+t-1,\ 0)$$
$$-\binom{2p+2t}{2}\binom{2}{1}(r,\ p+t,\ 0)\qquad (26)$$

and for $r+p+t-1$ letters

$$(r,\ p,\ t-1)=\binom{r-p}{t-1}(r,\ p,\ 0)-\binom{2p+2t-2}{2p}\binom{2t-2}{t-1}(r,\ p+t-1,\ 0).$$

Hence by (2), for $r+p+t$ letters

$$(27)\ \ (r,p,t-1)=t\binom{r-p}{t-1}(r,p,0)-\binom{2p+2t-2}{2p}\binom{2t-2}{t-1}(r,p+t-1,0).$$

Eliminating the expressions $(r,\ p,\ 0)$, $(r,\ p+t,\ 0)$ from (25), (26), (27), we find that $(r,\ p+t-1,\ 0)$ also disappears, and we get, for $r+p+t$ letters,

$$(28)\quad (r,\ p,\ t)t\,!\,t\,!\,2p\,!=(r,\ p,\ t-1)(r-p-t+1)(t-1)\,!\,(t-1)\,!\,2p\,!$$
$$+(r,\ p+t-1,\ 1)(2p+2t-2)\,!$$

Hence, by (22), for n letters we have

$$(29)\quad (r,\ p,\ t)t\,!\,t\,!\,2p\,!$$
$$=(r,\ p,\ t-1)(n-r-p-t+1)(r-p-t+1)(t-1)\,!\,(t-1)\,!\,2p\,!$$
$$+(r,\ p+t-1,\ 1)(2p+2t-2)\,!$$

Thus any $(r,\ p,\ t)$ can be expressed in terms of $(r,\ p,\ t-1)$ and $(r,\ p+t-1,\ 1)$. And by successive reductions of the letter t, we can finally express $(r,\ p,\ t)$ in terms of

$$(r,\ p+t-1,\ 1),\ (r,\ p+t-2,\ 1),\ (r,\ p+t-3,\ 1)\ldots(r,\ p+1,\ 1).$$

The expression will be developed presently.

The identity (25) can by means of (22) be generalized as follows to apply to n letters.

$$(30)\ \ (r,\ p,\ t)=\binom{n-r-p}{t}\binom{r-p}{t}(r,\ p,\ 0)-\binom{2p+2t}{2t}\binom{2t}{t}(r,\ p+t,\ 0).$$

To develop the expression $(r,\ p,\ t)$ in terms of analogous expressions with $t=1$, let us for brevity put

$$\{r,\ p,\ t\}\equiv(r,\ p,\ t)\times t\,!\,t\,!\,2p\,!$$

and $\qquad a^{(m)}\equiv a(a+1)(a+2)\ldots(a+m-1).$

Then (29) becomes

$$\begin{aligned}
\{r, p, t\} &= \{r, p+t-1, 1\} + (n-r-p-t+1)(r-p-t+1)\{r, p, t-1\} \\
\{r, p, t-1\} &= \{r, p+t-2, 1\} + (n-r-p-t-2)(r-p-t+2)\{r, p, t-2\} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\{r, p, 2\} &= \{r, p+1, 1\} \quad + (n-r-p-1) \quad (r-p-t-1)\{r, p, 1\}.
\end{aligned}$$

Combining these to eliminate $\{r, p, t-1\}$, etc., we get

(31)
$$\begin{aligned}
\{r, p, t\} = \{r, p+t-1, 1\} &+ (n-r-p-t+1) \quad (r-p-t+1) \quad \{r, p+t-2, 1\} \\
&+ (n-r-p-t+1)^{(2)} \ (r-p-t+1)^{(2)} \ \{r, p+t-3, 1\} \\
&+ \qquad \cdots\cdots\cdots \\
&+ (n-r-p-t+1)^{(q)} \ (r-p-t+1)^{(q)} \ \{r, p+t-q-1, 1\} \\
&+ \qquad \cdots\cdots\cdots \\
&+ (n-r-p-t+1)^{(t-1)}(r-p-t+1)^{(t-1)}\{r, p, 1\}
\end{aligned}$$

e.g., putting $r=5$, $p=1$, $t=3$, we have

$$2!3!3!\Sigma a^2bc(def-ghk)^2 = 6!\Sigma a^2bcdefg(h-k)^2 + (n-8).2.4!\Sigma a^2b^2cdef(g-h)^2 + (n-8)(n-7).2.3.2!\Sigma a^2b^2c^2de(f-g)$$

$$\therefore \qquad \Sigma a^2bc(def-ghk)^2 = 10\Sigma a^2bcdefg(h-k)^2 + \tfrac{2}{3}(n-8)\Sigma a^2b^2cdef(g-h)^2 + \tfrac{1}{6}(n-8)(n-7)\Sigma a^2b^2c^2de(f-g)^2.$$

Application to the Difference D(n, r).

Let
$$D(n, r) \equiv 2\binom{n}{r-1}\binom{n}{r+1}P_r^2 - 2\binom{n}{r}^2 P_{r-1}P_{r+1}$$

where
$$P_r \equiv \Sigma abc \ldots \text{ to } r \text{ factors,}$$

the summation extending over all possible products of r out of the n letters $a, b, c \ldots$.

Thus
$$D(n, r) = \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}\{r(n-r)P_r^2 - (r+1)(n-r+1)P_{r-1}P_{r+1}\}.$$

But
$$P_r^2 = (r, 0, 0) + \frac{2}{1}(r, 1, 0) + \frac{4 \cdot 3}{1 \cdot 2}(r, 2, 0) + \ldots + \binom{2p}{p}(r, p, 0) + \ldots + \binom{2r}{r}(r, r, 0)$$

where, as defined above,
$$(r, p, 0) \equiv \Sigma(abc \ldots \text{ to } 2p \text{ factors})\ (f^2g^2 \ldots \text{ to } r-2p \text{ factors}).$$

Again
$$P_{r-1}P_{r+1} = (r, 1, 0) + \binom{4}{1}(r, 2, 0) + \binom{6}{2}(r, 3, 0) + \ldots + \binom{2p}{p-1}(r, p, 0) + \ldots + \binom{2r}{r-1}(r, r, 0).$$

Hence
$$D(n, r) = \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}\left[r(n-r)(r, 0, 0) + \sum_{p=1}^{p=r}\left\{r(n-r)\binom{2p}{p} - (r+1)(n-r+1)\binom{2p}{p-1}\right\}(r, p, 0)\right]$$

(32)
$$= \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}\left[\sum_{p=0}^{p=r}\binom{2p}{p}\frac{r(n-r) - p(n-p) - p(p+1)}{p+1}(r, p, 0)\right].$$

But, putting $t = 1$ in (30), we have
$$(r, p, 1) = (n-r-p)(r-p)(r, p, 0) - (2p+2)(2p+1)(r, p+1, 0)$$
$$\therefore \binom{2p}{p}\frac{1}{p+1}(r, p, 1) = \binom{2p}{p}\frac{r(n-r) - p(n-p)}{p+1}(r, p, 0) - \binom{2p+2}{p+1}(p+1)(r, p+1, 0).$$

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Hence, the expression under the symbol Σ in (32) may be written

$$\binom{2p}{p}\frac{1}{p+1}(r,\ p,\ 1)+\binom{2p+2}{p+1}(p+1)(r,\ p+1,\ 0)-\binom{2p}{p}p(r,\ p,\ 0).$$

The second and third terms cancel out in the summation, and we get

$$(33)\quad \mathrm{D}(n,\ r)=\frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}\sum_{p=0}^{p=r-1}\left\{\binom{2p}{p}\frac{1}{p+1}(r,\ p,\ 1)\right\}.$$

It may be noted that the coefficient $\binom{2p}{p}\cdot\frac{1}{p+1}$ or $\frac{(2p)!}{p!\,(p+1)!}$ is an integer, as can easily be shown by the aid of the identity

$$\binom{2p}{p}\div(p+1)\equiv\binom{2p+1}{p}\div(2p+1).$$

Now the dexter of (33), if a, b, c... are all positive and not all equal to one another, is obviously positive. Hence $\mathrm{D}(n,\ r)$ is so.

Hence we have

$$\left\{\mathrm{P}_r\div\binom{n}{r}\right\}^2>\mathrm{P}_{r-1}\mathrm{P}_{r+1}\div\left\{\binom{n}{r-1}\binom{n}{r+1}\right\}$$

an inequality proved by Euler in his *Differential Calculus*, Vol. II, 313, by the aid of the Theory of Equations. Schlömilch, who reproduces Euler's proof in his *Zeitschrift für Mathematik*, Vol. III, remarks that it would be desirable to have a proof "welche von der Natur der Sache *d.h* von combinatorischen Gründen ihren Ausgang nähme."

Note added 4th April 1903.

The formula for $\mathrm{D}(n,\ r)$ given in (33) shows that it is necessarily positive for *positive* values of a, b, c.... But by Newton's Rule in the Theory of Equations (see Todhunter's *Theory of Equations*, Chap. XXVI) we know that $\mathrm{D}(n,\ r)$ is positive for all *real* values of a, b, c.... It would, then, be of interest to modify the formula (33) in such a way as to make this obvious. This we can do by means of identities of the type

$$(35)\qquad \Sigma_1\{(cd\ldots \text{ to } q \text{ factors})\Sigma_2(fg\ldots \text{ to } r-q-1 \text{ factors})\}^2$$

$$=\binom{r-1}{q}\phi(0)+\binom{r-2}{q}\phi(1)+\ldots+\binom{r-p-1}{q}\phi(p)+\ldots+\phi(r-q-1)$$

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where $\phi(p) \equiv \binom{2p}{p}\Sigma_1\{(cd\ldots \text{ to } 2p \text{ factors})\, f^2g^2\ldots \text{ to } r-p-1 \text{ factors}\}$

and Σ_1 sums all terms that can be formed from all the letters excepting a and b; while Σ_2 sums terms that can be formed from all the letters excepting a and b and the q others which have already occurred in the bracket (cd.... to q factors); and q may have any value from 0 up to $r-1$.

Denoting the sinister of (35) by G(q), and by E the coefficient of $(a-b)^2$ in the expression

$$(36)\quad D(n, r) = \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}$$

$$\Sigma\left[(a-b)^2\left\{\phi(0) + \tfrac{1}{2}\phi(1) + \tfrac{1}{3}\phi(2) + \ldots + \frac{1}{r}\phi(r-1)\right\}\right]$$

which is (33) written in a slightly different form, we have

$$(37)\quad E = \frac{1}{r}G(0) + \frac{1}{r(r-1)}G(1) + \frac{1\,.\,2}{r(r-1)(r-2)}G(2) + \ldots$$

$$+ \frac{q\,!\,(r-q-1)\,!}{r\,!}G(q) + \ldots + \frac{1}{r}G(r-1).$$

Since each G is a square, and has a positive numerical coefficient in (37), it is obvious that E is essentially positive for *real* values of $a, b, c\ldots$, and that D(n, r) is therefore essentially positive.

Rearranging (36) in terms of G(0), G(1), etc., we get

$$(38)\quad D(n, r) = \frac{2}{r(r+1)}\binom{n}{r}\binom{n}{r-1}$$

$$\Sigma\left[\frac{1}{r}\{(a-b)\Sigma_1 cd\ldots\}^2 + \frac{1}{r(r-1)}\{(a-b)c\,\Sigma_2 de\ldots\}^2 + \ldots\right.$$

$$+ \frac{q\,!\,(r-q-1)\,!}{r\,!}\{(a-b)(cd\ldots \text{ to } q \text{ factors})\ \Sigma_2 gh\ldots\}^2 + \ldots$$

$$\left. + \frac{1}{r}\{(a-b)(cd\ldots \text{ to } r-1 \text{ factors})\}^2\right]$$

where the typical term under Σ_2 contains sufficient letters to keep the expression homogeneous.

Construction connected with the Locus of a point at which two segments of a straight line subtend equal angles.

By R. F. Muirhead, M.A., B.Sc.

Let AB, CD be the two segments. (Fig. 49.)

Let two similar circle-segments be described on AB, CD, whose circumferences intersect in PP′.

It is obvious that ABCD passes through O the external centre of similitude of the two circles. Let OP meet the circles again in R and Q.

Then the triangles

OQC, ODP, OBR, OPA are all similar to one another

and the triangles

OQD, OCP, OPB, OAR " " " " " "

$\therefore \quad \angle DPC = \angle ARB = \angle APB.$

Again, $OD.OA = OB.OC$ so that O is a fixed point when the circles are varied.

Also, $OP^2 = OD.OA$ so that OP is a fixed length when the circles are varied.

Thus the locus of P, at which AB and CD subtend equal angles, includes the circumference of the circle of which O is the centre, and $\sqrt{OB.OC}$ the length of the radius. The locus also includes the rest of the line of which AB, CD are segments.

A totally different construction is given in the appendix to Todhunter's "Euclid," Article 54.

It may be remarked that if a circle be described on AD as diameter, and $\gamma C\gamma'$ $\beta B\beta'$ be double ordinates to this diameter, then the intersections of $\beta\gamma$, $\beta'\gamma'$; and of $\beta\gamma'$, $\beta'\gamma$ are the points in which AD meets the locus of P.

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On the equation to a conic circumscribing a triangle.

By R. F. Davis, M.A.

Let $\frac{L}{\alpha}+\frac{M}{\beta}+\frac{N}{\gamma}=0$ be the equation to a given conic circumscribing the triangle of reference ABC; p, q, r the focal chords parallel to BC, CA, AB respectively. (Fig. 50.)

Then it is well known that $M\gamma+N\beta=0$ is the equation to the tangent to the conic at A, so that

$$\sin TAB : \sin TAC = -\gamma : \beta = N : M.$$

Take any point P on BA produced and draw the secant PA′C′ parallel to AC.

Then $$q : r = PA' . PC' : PA . PB$$
$$= PE' . PC' : PB^2, \text{ if } BE' \text{ is parallel to } AA'.$$

In the limit when PA′C′ moves up into coincidence with AC

$$q : r = AE . AC : AB^2, \text{ where BE is parallel to tangent AT}$$
$$= AC . \sin TAB : AB \sin TAC$$
$$= \quad Nb : \quad Mc;$$
$$\therefore \quad Mcq = Nbr;$$
$$\therefore \quad M : N = \frac{b}{q} : \frac{c}{r}.$$

Thus, by symmetry,

$$L : M : N = \frac{a}{p} : \frac{b}{q} : \frac{c}{r}.$$

The equation to the circum-conic is therefore

$$\frac{a}{p\alpha}+\frac{b}{q\beta}+\frac{c}{r\gamma}=0.$$

The form just given is due to my old friend the Rev. T. J. Milne, and appears in the *Math. Gazette*.

The proof above is my own.

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On the imaginary roots of the equation $\cos x = x$.

By T. HUGH MILLER.

It has been shewn (*Proceedings of the Edin. Math. Soc.*, Vol. IX.), that this equation has only one real root, namely $x = \cdot 73908513\ldots$ and that it has an infinite number of imaginary roots of the form $A + Bi$, where A and B are given by the equations

$$A = \cos A\{e^{\tan A\sqrt{A^2 - \cos^2 A}} + e^{-\tan A\sqrt{A^2 - \cos^2 A}}.\}/2,$$
$$B = \pm \tan A\sqrt{A^2 - \cos^2 A}. \qquad \text{- - - - - (i)}$$

It is, however, only half of the possible number of values that can be assigned to A and B in (i), that satisfy the equation $A + Bi = \cos(A + Bi)$, the other half satisfying the equation $A - Bi = \cos(A + Bi)$.

To discriminate between the roots we must go back to the equations from which (i) were found.

These are
$$A = \cos A\frac{e^B + e^{-B}}{2} \qquad \text{- - - - (1)}$$
$$B = -\sin A\frac{e^B - e^{-B}}{2} \qquad \text{- - - - (2)}$$

Let the angle A be positive when it is contained between a fixed radius OC of a circle, and a movable radius OB, rotating in a left-handed way.

(1) Let A be positive.

If B is negative, $\sin A$ must be negative from equation (2), because then $e^B < e^{-B}$. Also $\cos A$ is positive by equation (1); therefore the radius OB lies in the *fourth* quadrant.

If B is positive, $\sin A$ must be negative from (2), for now $e^B > e^{-B}$. Therefore OB still lies in the fourth quadrant.

(2) Let A be negative.

Now if B be negative, $\sin A$ must be negative, but $\cos A$ is negative by equation (1). Therefore the radius OB lies in the *third* quadrant.

If B is positive, sinA and cosA must both be negative ; therefore OB lies in the *third* quadrant.

Collecting these results, we find that of all the values of A and B which satisfy equations (i) only those positive values of A can be admitted which are of the form $2\pi n - C$, where n is any positive integer except 0, and C is positive and less than $\frac{\pi}{2}$; and those negative values of A which are of the form $(2n-1)\pi + C$, where n is any negative integer, including 0, and C is less than $\frac{\pi}{2}$.

In each case there are two roots for any value of n, namely those formed by giving B the positive and the negative sign.

Thus for positive values of A

$$\cos\{(2n\pi - C) \pm Bi\} = \cos C \frac{e^{B} + e^{-B}}{2} \pm i \,.\, \sin C \,.\, \frac{e^{B} - e^{-B}}{2},$$

and a similar equation for the negative values of A.

When A and B become large, the positive values of A give as an approximate value of x

$$\left(2n\pi - \frac{\log 4n\pi}{2n\pi}\right) \pm i\left[\log 4n\pi - \left(\frac{\log 4n\pi}{2n\pi}\right)^2\right].$$

The smallest positive value of A that satisfies equations (i) with the above conditions is 5·86956, and the corresponding roots of the equation are

$$5{\cdot}86956 \pm i2{\cdot}5449.$$

The next in order are $\quad 12{\cdot}30856 \pm i3{\cdot}2349,$

$$18{\cdot}657 \quad \pm i3{\cdot}6191.$$

Corresponding to the numerically smallest value of A, is the root

$$-2{\cdot}48714 \pm i1{\cdot}8093.$$

FIGURE 51.

To interpret these results; let O be the centre of a circle of unit radius, let OA be the initial line cutting the circle in C. Draw a radius OB, so that the ratio of the area COB to half the square on the radius, shall equal A.

Next describe a rectangular hyperbola having OB for transverse axis. Take a point P on the hyperbola, joining OP, so that the ratio of the area BOP to half the square on the radius of the circle is B.

From P draw PM, PN perpendicular to the axis of the hyperbola and to the initial line respectively; draw MR perpendicular to the initial line and produce it to Q making RQ equal to NR. Join OQ; OQ is the cosine of $A + Bi$.

$$\text{For} \quad OM = \frac{e^B + e^{-B}}{2} = \cos iB,$$

$$\text{and} \quad PM = \frac{e^B - e^{-B}}{2} = \frac{\sin iB}{i},$$

$$\therefore \quad \sin iB = iPM.$$

$$\begin{aligned} \therefore \quad \cos(A + Bi) &= \cos A \cos iB - \sin A \sin iB \\ &= \cos A \,.\, OM - \sin A \,.\, iPM. \\ &= OR - i \,.\, NR. \\ &= OQ. \end{aligned}$$

To construct the diagram; the angle BOC is obviously equal to A. Let the angle POB be called ψ, then

$$\text{area} \quad POB = \tfrac{1}{2}(OC)^2 \log\sqrt{\frac{1 + \tan\psi}{1 - \tan\psi}}.$$

$$\therefore \qquad B = \log\sqrt{\frac{1 + \tan\psi}{1 - \tan\psi}};$$

$$\therefore \qquad \frac{e^B - e^{-B}}{e^B + e^{-B}} = \tan\psi.$$

The limits of ψ are $\frac{\pi}{4}$ and $-\frac{\pi}{4}$.

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