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With an appendix by Sug Woo Shin

Abstract

Generalizing previous results of Deligne–Serre and Taylor, Galois representations are attached to cuspidal automorphic representations of unitary groups whose Archimedean component is a holomorphic limit of discrete series. The main ingredient is a construction of congruences, using the Hasse invariant, that is independent of q-expansions.

Contents

1. Introduction

This paper is concerned with associating Galois representations to cuspidal automorphic representations of unitary groups whose Archimedean component is a *holomorphic limit of discrete series*. The main idea of this work is to treat congruences between automorphic forms without q-expansions (see § 6). This allows us to construct Galois representations in a much wider setting than that of Taylor's previous result [Tay91]. For instance, our method applies to compact Shimura varieties, where q-expansions do not exist.

An improvement by Shin (see Theorem A.1) of the base change result of Labesse [Lab11, Corollary 5.3] that is used appears in an appendix. See Remark 1.4.2 for an explanation of why Shin's improvement is needed.

Section 1 is organized as follows: § 1.1 recounts the history of associating Galois representations to automorphic representations. Section 1.2 states the main result of this paper, see Theorem 1.2.1. Section 1.3 first describes our strategy for proving Theorem 1.2.1 and then

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proceeds to outline the contents of the paper. Some remarks about Theorem 1.2.1 and possible improvements/generalizations can be found in § 1.4.

1.1 History

In the 1950s Eichler [Eic54] and Shimura [Shi58] showed that, to every classical modular form of weight two which is cuspidal and an eigenform for the action of the Hecke operators, there corresponds a two-dimensional Galois representation, in the sense that the Hecke eigenvalue at a prime is equal to the trace of a Frobenius element at that prime. Following this work, a leading theme in number theory since the 1960s has been the association of Galois representations to automorphic representations. This theme evolved in several stages until reaching its current state.

1.1.1 *Classical modular forms.* The work of Deligne and Serre in the 1960s and 1970s generalized the theorem of Eichler–Shimura to cuspidal modular eigenforms of arbitrary weight $k \in \mathbf{Z}_{\geqslant 1}$ [Del69, DS74]. Until that point, the only points of view concerning modular forms were (a) the original one, that modular forms are certain holomorphic functions on the upper half-plane $\mathbf{H}$ with an incredible amount of symmetry and (b) the Langlands philosophy that modular forms should be understood in terms of *automorphic representations* of the $\mathbf{Q}$-algebraic reductive group GL(2) (cf. [Del73]). From within either of these two perspectives, the constructions of Deligne and Deligne–Serre appear to be miraculous, because they associate objects from the heart of algebra (algebraic numbers, Galois representations) to objects that seem deeply rooted in either complex analysis (holomorphic functions on $\mathbf{H}$) or harmonic analysis (the space $L^2(\mathrm{GL}(2, \mathbf{Q})\backslash\mathrm{GL}(2, \mathbf{A}))$, with $\mathbf{A}$ the rational adeles, which is the source of automorphic representations of GL(2)).

One of the conceptual breakthroughs of Deligne and Serre, was that they introduced purely algebro-geometric descriptions of modular forms. To do this, they used two cohomology theories of algebraic varieties that had only recently been discovered: Serre's algebraic coherent sheaves and their cohomology [Ser55] and Grothendieck and his school's (ℓ-adic) étale cohomology [AGV73, Del77a]. With such descriptions in hand, the correspondence of Deligne and Serre is still difficult to establish, but at least the mystery of how analytic objects can be so closely tied to algebraic ones is solved.

1.1.2 *The Langlands program.* As the Langlands program developed, it was conjectured that, in analogy with the construction for modular forms, for every automorphic representation π of any $\mathbf{Q}$-algebraic reductive group $\mathbf{G}$, which has integral infinitesimal character,[1] there is an associated Galois representation (see [Tay04] for an expository account, [Clo88] for a precise conjecture in the case of $\mathrm{GL}(n)$ and [BG10] for a general reductive group).

The realization of automorphic forms most directly related to Galois representations is in étale cohomology, because the latter carries an action of the absolute Galois group as part of its definition (one could therefore say that the étale realization is the 'most algebraic'). The downside of the étale realization is that it directly applies only to the automorphic representations π such that the Archimedean component π_∞ is discrete series. In the GL(2) case, only modular forms of weight at least two can be realized in the étale cohomology of modular curves, which is why Deligne's construction [Del69] only works in that case.

The 'next best thing' to an étale realization seems to be one in the coherent sheaf cohomology of a PEL Shimura variety. The strength of the latter is that it captures all π with π_∞ a non-degenerate limit of discrete series in the sense of Knapp–Zuckerman [KZ82, § 1]. The limitations

[1] This is a subtle technical condition involving 'ρ-shifts', see [BG10].

of coherent cohomology are twofold: first, coherent cohomology lacks a Galois action; second, many π that are conjectured to correspond to a Galois representation do not appear in the coherent cohomology of any Shimura variety; all π with π_∞ a degenerate limit of discrete series are of this type, but there are still even 'wilder' π (e.g. the π associated to classical Maass forms with eigenvalue 1/4).

The pioneering step of Deligne–Serre was to show that, although coherent cohomology lacks a Galois action, it may be used to construct *congruences* between modular forms of weight one and modular forms of higher weight, which allowed to reduce the construction of Galois representations in weight one to that in higher weight. As soon as Deligne–Serre introduced this idea of 'changing the weight', it was natural to try to apply the same method to automorphic representations π of groups $\mathbf{G}$ such that $\mathbf{G}(\mathbf{R})$ admits a simple subgroup of rank at least two. However, it took almost 20 years for someone to do that, because the Deligne–Serre argument for reducing to discrete series used that the corresponding Galois representations have finite image and it is conjectured that the Galois representation associated to an automorphic representation π of $\mathbf{G}$, with $\mathbf{G}(\mathbf{R})$ containing a rank two simple subgroup and π_∞ a non-degenerate limit of discrete series, has infinite image.[2]

1.1.3 *Pseudorepresentations.* In 1991, Taylor [Tay91] discovered a new approach, stemming in part from the work of Wiles on 'λ-adic forms' [Wil88], using *congruences* as well as what he called *pseudorepresentations*, which allowed him to associate Galois representations to automorphic forms on the split symplectic group GSp(4), whose weight is a *holomorphic limit of discrete series*. In Taylor's case, the Galois representations no longer have finite image; it is thus no longer possible to reduce modulo more than one prime at a time. The method of pseudorepresentations offers the necessary alternative to the Deligne–Serre method of reduction modulo infinitely many primes. Instead the pseudorepresentations method involves only reduction modulo arbitrarily high powers of a single prime.

While there has been much progress in generalizing Deligne's construction for weight at least two to larger groups and π with π_∞ discrete series, as in the work of Kottwitz [Kot92], Clozel [Clo91], Harris–Taylor [HT01], Taylor–Yoshida [TY07], Shin [Shi11], Morel [Mor10] and Chenevier–Harris [CH10], no other limits of discrete series on any other group, whose real points admit a simple subgroup of rank at least two, were treated prior to this work.

1.1.4 *Non-holomorphic limits of discrete series.* No Galois representation has ever been attached to an automorphic representation π such that π_∞ is a limit of discrete series that is not holomorphic.[3] Still, in a series of papers [Car88, Car00, Car05, CK07], Carayol has embarked on a program to attach Galois representations to automorphic representations whose Archimedean component is the *degenerate* limit of discrete series on the unitary group $\mathrm{SU}(2,1)$. Although

[2] In the case of GL(2) over a totally real field (classically known as the theory of Hilbert modular forms) the Galois representations associated to π which is holomorphic limit of discrete series at every infinite place (i.e. Hilbert modular forms of parallel weight one) do have finite image. Therefore, the Deligne–Serre method can be applied, as was done by Rogawski–Tunnel [RT83]. On the other hand, in the remaining Hilbert modular case, when π is holomorphic limit of discrete series at some but not all infinite places and holomorphic discrete series at the other infinite places, the associated Galois representations have infinite image, so Taylor's theory of pseudo-representations is already necessary and the Galois representations were constructed by Jarvis [Jar97] following Taylor's work [Tay91].

[3] If the analogue of Arthur's work [Art13] were known for unitary similitude groups, then combining it with either Taylor's result for GSp(4) or Theorem 1.2.1 of this paper would give a limited amount of examples of Galois representations associated to π such that π_∞ is a non-holomorphic, non-degenerate limit of discrete series.

such representations have no realization whatsoever in the cohomology of Shimura varieties, Carayol has shown that they do appear in the coherent cohomology of certain generalizations of Shimura varieties, termed by him *Griffiths–Schmid manifolds*. Since the pertinent Griffiths–Schmid manifolds are believed not to be algebraic varieties, the *algebraicity* of the representations studied by Carayol remains for the moment mysterious.

1.2 Main result

Throughout this paper, fix an isomorphism $\iota : \overline{\mathbf{Q}_\ell} \xrightarrow{\sim} \mathbf{C}$. Let $\mathcal{U} = (B, V, *, \langle\, ,\rangle, \tilde{h})$ be a Kottwitz datum, with associated Shimura datum $(\mathbf{G}, \mathbf{X})$, Shimura variety $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$ and reflex field $E = E(\mathbf{G}, \mathbf{X})$ (see § 3.1). Let F be the center of B and let F^+ be the fixed field of the involution $*$ determined by $\langle\, ,\rangle$. Suppose that $\mathbf{G}$ is a unitary group; equivalently F is a quadratic totally imaginary extension of F^+.

Suppose that π is a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with p-adic component π_p for every (rational) prime p. Given a (rational) prime ℓ, let $\mathcal{P}^{(\ell)}$ be the set of (rational) primes p different from ℓ such that π_p is unramified and $\mathbf{G}$ is unramified at p. Let $\mathfrak{P}^{(\ell)}$ be the set of primes of F that are split over F^+ and lie over some $p \in \mathcal{P}^{(\ell)}$.

Assume $\wp \in \mathfrak{P}^{(\ell)}$. One has a decomposition $\mathbf{G}(\mathbf{Q}_p) \cong \mathrm{GL}(n, F_\wp) \times G_{p,\mathrm{rest}}$, for some group $G_{p,\mathrm{rest}}$, where n is given by $n^2 = \dim_F \mathrm{End}_B V$. Write $\pi_p \cong \pi_\wp \otimes \pi_{p,\mathrm{rest}}$, with $\pi_\wp$ a representation of $\mathrm{GL}(n, F_\wp)$ and $\pi_{p,\mathrm{rest}}$ a representation of $G_{p,\mathrm{rest}}$.

The main theorem of this paper is as follows.

Theorem 1.2.1. *Suppose that π is a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ whose Archimedean component π_∞ is an $\mathbf{X}$-holomorphic limit of discrete series representation of $\mathbf{G}(\mathbf{R})$ (see § 2.3). Assume that ℓ is a prime (of $\mathbf{Q}$) of good reduction for $\mathcal{U}$ such that there exists a prime λ of E, above ℓ, that is split[4] in E. Then there exists a unique semisimple Galois representation*

$$R_{\ell,\iota}(\pi) : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}(n, \overline{\mathbf{Q}_\ell}) \tag{1.1}$$

satisfying the following two conditions.

Gal1. *If $p \in \mathcal{P}^{(\ell)}$ and $\wp$ is a prime of F dividing $\wp$, then $R_{\ell,\iota}(\pi)$ is unramified at $\wp$. In particular $R_{\ell,\iota}(\pi)$ is unramified at all but finitely many places.*

Gal2. *If $\wp \in \mathfrak{P}^{(\ell)}$ then there is an isomorphism of Weil–Deligne representations*

$$(R_{\ell,\iota}(\pi)|_{W_{F_\wp}})^{\mathrm{ss}} \cong \iota^{-1}\mathrm{rec}(\pi_\wp \otimes |\cdot|_\wp^{(1-n)/2}), \tag{1.2}$$

where $W_{F_\wp}$ is the Weil group of $F_\wp$, the superscript ss *denotes semi-simplification and* rec *is the local Langlands correspondence, normalized as in Harris–Taylor* [HT01].

1.3 Strategy and outline

1.3.1 *Strategy.* Let π be as in Theorem 1.2.1 and let f be a corresponding Hecke eigenform. The idea of the proof of Theorem 1.2.1 is to construct an automorphic form H^{lift}, with the following three properties.

Hasse1. The automorphic form H^{lift} is non-zero modulo ℓ.

Hasse2. The product $(H^{\mathrm{lift}})^{\ell^j} f$ is cohomological[5] for all $j \in \mathbf{Z}_{\geqslant 0}$.

Hasse3. If T is a Hecke operator that is trivial outside $\mathcal{P}^{(\ell)}$, then for all $j \in \mathbf{Z}_{\geqslant 0}$ one has

$$T((H^{\mathrm{lift}})^{\ell^j} f) \equiv (H^{\mathrm{lift}})^{\ell^j} T(f) \pmod{\ell^{j+1}}, \tag{1.3}$$

[4] That is, λ is a degree one prime.

[5] In our situation 'cohomological' is equivalent to having regular Archimedean component.

where congruence is defined in terms of an integral structure on the space of automorphic forms, not in terms of q-expansions.

Property Hasse2 ensures that the eigenforms of the same weight as $(H^{\mathrm{lift}})^{\ell^j} f$ admit Galois representations satisfying conditions Gal1 and Gal2 of Theorem 1.2.1. Using properties Hasse1 and Hasse3, one can apply Taylor's theory of pseudorepresentations to produce the desired Galois representation $R_{\ell,\iota}(\pi)$. For H^{lift} we take a lift of a power of the Hasse invariant.

The principal innovation in this work is that, in contrast to the previous works on the subject [DS74, Tay91], q-expansions are not used. We observe that the congruence Hasse3, whose formulation does not necessitate q-expansions, is sufficient for applying Taylor's pseudorepresentation method.

1.3.2 *Outline.* The paper is structured as follows. Sections 2–3 introduce notation and recall the basic notions and results about them that we need. Section 2 is concerned with objects over the complex numbers: Shimura varieties (§§ 2.1 and 2.4), holomorphic limits of discrete series (§ 2.3), equivariant vector bundles (§ 2.5) and Lie algebra cohomology (§ 2.6). Section 3 describes integral models of some of the objects studied in § 2. Sections 3.1–3.5 recall the rational and integral models of PEL Shimura varieties as constructed by Kottwitz [Kot92]. Sections 3.6–3.8 explain the integral theory, via the Hodge bundle and Schur functors, of the vector bundles previously constructed in § 2.5.

Section 4 centers on the Hasse invariant. The Hasse invariant of an abelian variety is constructed in § 4.1. The rest of § 4 is concerned with the basic properties of the Hasse invariant which give properties Hasse1–Hasse3. In § 4.2, it is shown that the Hasse invariant is compatible with isogenies and base change. The compatibility with isogenies shows that the Hasse invariant gives rise to a well-defined mod ℓ automorphic form. The compatibility with base change is the crucial ingredient for property Hasse3, as is seen later in Theorem 6.2.1. Next, § 4.3 recalls a theorem of Wedhorn on the ordinary locus of the special fiber of the Shimura variety in play and uses it to deduce property Hasse1 under the assumptions of Theorem 1.2.1. Finally, § 4.4 describes the ampleness of the determinant of the Hodge bundle on the minimal compactification and why this shows that some power of the Hasse invariant lifts to characteristic zero.

Section 5 is concerned with explicit formulas over the real and complex numbers regarding the objects of §§ 2–3. Sections 5.1–5.3 give an explicit description of the real points of the unitary groups appearing in Theorem 1.2.1 and of their roots and weights. This allows us to explicitly parameterize the holomorphic limits of discrete series of these unitary groups in § 5.4, which is one ingredient for establishing property Hasse2. Section 5.5 gives a dictionary over the complex numbers between the vector bundles constructed in § 2.5 and in § 3.8. In particular, we calculate the weight of the Hasse invariant and property Hasse2 follows.

Section 6 contains our construction of congruences and the proof of Theorem 1.2.1. Section 6.1 recalls the definitions of the relevant Hecke algebras and the action of Hecke operators on spaces of automorphic forms. In § 6.2 we establish property Hasse3. The congruence Hasse3 is then translated into the language of Hecke algebras in § 6.3. Finally Theorem 1.2.1 is deduced from properties Hasse1–Hasse3 and Taylor's method of pseudorepresentations in § 6.4.

The appendix by Sug Woo Shin extends previous work of Labesse [Lab11] to prove (automorphic) base change for cohomological automorphic representations of the unitary group $\mathbf{G}$ in Theorem 1.2.1. The importance of the appendix is discussed in Remark 1.4.2 below.

1.4 Remarks

Several remarks about Theorem 1.2.1 are in order.

Remark 1.4.1 (The hypothesis on ℓ in Theorem 1.2.1). The hypothesis that some prime λ above ℓ is split in E is necessary to ensure that the special fiber at λ of the (integral model of the)

Shimura variety $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$ admits an ordinary point, or equivalently, that the Hasse invariant is not identically zero (see § 4.3). We hope to remove this hypothesis in a forthcoming paper.

Remark 1.4.2 (Base change). This remark explains why we need Shin's strengthening (Theorem A.1) of Labesse's base change result [Lab11, Corollary 5.3].

Let H^{lift} denote a lift to characteristic zero of some positive power of the Hasse invariant (see § 4.4). Let f be a Hecke eigenform whose Archimedean component is an $\mathbf{X}$-holomorphic limit of discrete series. Our method of associating a Galois representation to f requires knowing that, for sufficiently large positive integers j, every Hecke eigenform g of the same weight as the product $(H^{\mathrm{lift}})^{\ell^j} f$ admits a Galois representation satisfying the local-global compatibility described in Theorem 1.2.1. In turn, to establish the latter for a general unitary group requires base-changing to $\mathrm{GL}(n)$, so as to be able to apply the work of Shin [Shi11].

Suppose that ψ is an automorphic representation of $\mathbf{G}$ which we need to base change to $\mathrm{GL}(n)$. Let χ_ψ be the infinitesimal character of ψ, identified with a dominant weight by the Harish-Chandra isomorphism. The base change result of Labesse requires a certain hypothesis (see condition $(*)$ following [Lab11, Remark 5.2]) which in general is only satisfied if $\chi_\psi - \rho$ is regular, where ρ is the half-sum of the positive roots. Now the infinitesimal character χ_g of the Archimedean component of g is regular, but it is possible that $\chi_g - \rho$ is singular. Hence, Corollary 5.3 of *loc. cit.*does not necessarily apply. However, since χ_g is regular, Shin's improvement (Theorem A.1) of Labesse's result does apply.

Remark 1.4.3 (Compatibility at unramified primes outside $\mathfrak{P}^{(\ell)}$). The simplest strengthening of the local–global compatibility condition Gal2 in Theorem 1.2.1 would be an analogous statement for any prime $\wp$ of F lying above $p \in \mathcal{P}^{(\ell)}$ with $\wp$ not split over F^+. It is possible that in fact the proof of Theorem 1.2.1 gives such a statement, but we have not checked this.

Remark 1.4.4 (Compatibility at $\ell = p$). The general conjectures of the Langlands program (cf. [Tay04]) predict that the Galois representation $R_{\ell,\iota}(\pi)$ and the automorphic representation π are also compatible at the places $\mathcal{L}$ of F dividing ℓ. For example, it is expected that, for every place $\mathcal{L}$ dividing ℓ, the local Galois representation $R_{\ell,\iota}(\pi)|_{\mathrm{Gal}(\overline{F_{\mathcal{L}}}/F_{\mathcal{L}})}$ is de Rham. Moreover, it is conjectured to be crystalline if and only if π_ℓ is unramified. We believe that it is an interesting question to determine whether these conjectured improvements of Theorem 1.2.1 can be attained by the techniques of this paper.

Remark 1.4.5 (Motivic origin). Given Remark 1.4.4 and condition Gal1 of Theorem 1.2.1, the Fontaine–Mazur conjecture [FM95] (see also [Tay04]) asserts that $R_{\ell,\iota}(\pi)$ is *pure*[6] and in fact *motivic* (i.e. arises as a sub-quotient of the étale cohomology of a variety). However, even the purity of $R_{\ell,\iota}(\pi)$ seems to require significant new ideas.

2. Archimedean theory I: generalities

2.1 Deligne's axioms

We recall Deligne's axioms for a Shimura variety following his original papers (see, in particular, [Del77b, § 2.1.1], but also [Del71, § 1.5]).

Let $\mathbf{G}_m$ denote the multiplicative group scheme. Let $\mathbf{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m$ be the Deligne torus, restriction of scalars from $\mathbf{C}$ to $\mathbf{R}$ of $\mathbf{G}_m$. Recall that, among the basic properties of $\mathbf{S}$ we have the isomorphisms of real and complex points $\mathbf{S}(\mathbf{R}) \cong \mathbf{C}^\times$ and $\mathbf{S}(\mathbf{C}) \cong \mathbf{C}^\times \times \mathbf{C}^\times$.

[6] See [Tay04, pp. 79–80] for a definition of purity.

Let $\mathbf{G}$ be a Zariski-connected, reductive $\mathbf{Q}$-algebraic group with adjoint group (respectively Lie algebra, complexified Lie algebra) $\mathbf{G}^{\mathrm{ad}}$ (respectively $\mathfrak{g}$, $\mathfrak{g}_{\mathbf{C}}$). As usual, denote the adjoint action of $\mathbf{G}$ on $\mathfrak{g}$ by $\mathrm{Ad}: \mathbf{G} \longrightarrow \mathrm{GL}(\mathfrak{g})$. Let $\mathbf{X}$ be a $\mathbf{G}(\mathbf{R})$-conjugacy class of homomorphisms $h: \mathbf{S}(\mathbf{R}) \longrightarrow \mathbf{G}(\mathbf{R})$.

A *Shimura datum* is a pair $(\mathbf{G}, \mathbf{X})$ which satisfies the following three conditions.

Del1. For all $h \in \mathbf{X}$, the composite $\mathrm{Ad} \circ h$ is a Hodge structure on $\mathfrak{g}$ of weight 0 and type

$$\{(0,0),(-1,1),(1,-1)\}. \tag{2.1}$$

Del2. For all $h \in \mathbf{X}$, $\mathrm{Ad}(h(i))$ is a Cartan involution of $\mathbf{G}(\mathbf{R})^{\mathrm{ad}}$.

Del3. If $\mathbf{H}$ is a $\mathbf{Q}$-factor of the adjoint group $\mathbf{G}^{\mathrm{ad}}$, then $\mathbf{H}(\mathbf{R})$ is not compact.

Given $h \in \mathbf{X}$, write $\mathrm{Hdg}(h) = \mathrm{Ad} \circ h$. Let

$$\mathfrak{g}_{\mathbf{C}} = H^{0,0}(h) \oplus H^{-1,1}(h) \oplus H^{1,-1}(h), \tag{2.2}$$

be the Hodge decomposition of $\mathrm{Hdg}(h)$. Then the Hodge filtration $F^{\bullet}(h)$ of $\mathrm{Hdg}(h)$ is

$$\mathbf{F}^0(h) \supset \mathbf{F}^1(h) \supset \mathbf{F}^2(h), \tag{2.3}$$

with $\mathbf{F}^0(h) = H^{0,0} + H^{1,-1}$, $\mathbf{F}^1 = H^{1,-1}$ and $\mathbf{F}^2 = \{0\}$.

Remark 2.1.1 (Sign choice). Throughout the paper, concerning Hodge structures, we adopt the sign choice made by Deligne in [Del77b, § 1.1.1] that $H^{p',q'}$ correspond to the character $z \mapsto z^{-p'}\bar{z}^{-q'}$. Note that this choice is the opposite of that in [Del70, § 1]. See [Del77b, Remark 1.1.6] for justifications of this choice. In particular, a *complex structure* amounts to a Hodge structure of type $\{(-1,0),(0,-1)\}$.

2.2 Some structure theory

Put $G = \mathbf{G}(\mathbf{R})$ and fix $h \in \mathbf{X}$. The group G acts on $\mathbf{X}$ by conjugation; let $\mathcal{K}_\infty$ be the stabilizer of h. Since $\mathcal{K}_\infty$ is the centralizer of a torus it is Zariski connected. By conditions Del1 and Del2, $\mathcal{K}_\infty/Z(G)$ is a maximal compact subgroup of $\mathbf{G}(\mathbf{R})^{\mathrm{ad}}$. Thus, $\mathcal{K}_\infty$ is a maximal Zariski connected, compact modulo center subgroup of G.

Let $\mathcal{H}_\infty$ be a maximal torus in $\mathcal{K}_\infty$ containing $h(\mathbf{S}(\mathbf{R}))$. Then $\mathcal{H}_\infty$ is also a maximal torus in G and $\mathcal{H}_\infty/Z(G)$ is compact. Let $\Delta = \Delta(\mathcal{H}_\infty, G)$ (respectively $\Delta_c = \Delta(\mathcal{H}_\infty, \mathcal{K}_\infty)$, $\Delta_n = \Delta - \Delta_c$) be the roots (respectively compact roots, non-compact roots) of $\mathcal{H}_\infty$ in G. Given a root $\alpha \in \Delta$ let $\mathfrak{g}_{\mathbf{C}}^{\alpha}$ denote the α-root space in $\mathfrak{g}_{\mathbf{C}}$.

The root space decomposition respects the Hodge decomposition $\mathrm{Hdg}(h)$ in the sense that, for all $\alpha \in \Delta$, we have $\mathfrak{g}_{\mathbf{C}}^{\alpha} \in H^{0,0}(h)$, $\mathfrak{g}_{\mathbf{C}}^{\alpha} \in H^{-1,1}(h)$ or $\mathfrak{g}_{\mathbf{C}}^{\alpha} \in H^{1,-1}(h)$. Moreover, $H^{0,0}$ is the direct sum of its center (as Lie algebra) and the compact root spaces, while $H^{-1,1} \oplus H^{1,-1}$ is the direct sum of the non-compact root spaces. We define, once and for all, that a non-compact root $\alpha \in \Delta_n$ is positive if $\mathfrak{g}_{\mathbf{C}}^{\alpha} \subset H^{-1,1}$ and denote the set of non-compact positive roots by Δ_n^+. Since $H^{-2,2} = \{0\}$, the sum of two non-compact positive roots is not a root. Furthermore, since $[H^{0,0}, H^{-1,1}] \subset H^{-1,1}$, if the sum of a positive non-compact root and an arbitrary compact root is a root, it is necessarily positive. It follows that any choice of positive compact roots Δ_c^+ is compatible with Δ_n^+, meaning that forming $\Delta^+ = \Delta_c^+ \cup \Delta_n^+$ yields a system of positive roots for G.

Put $\mathfrak{q} = \mathbf{F}^0(h)$. Let Q be the stabilizer of $\mathbf{F}^{\bullet}(h)$ in $\mathbf{G}(\mathbf{C})$. Then Q is a parabolic subgroup of $\mathbf{G}(\mathbf{C})$ whose Lie algebra is $\mathfrak{q}$. Thus, the quotient $\mathbf{G}(\mathbf{C})/Q$ has a natural complex structure (it is a flag variety) with complex Lie algebra $H^{-1,1}$. Furthermore, the map $\mathrm{Hdg}(h) \longmapsto \mathbf{F}^{\bullet}(h)$ gives a smooth embedding $\mathbf{X} \hookrightarrow \mathbf{G}(\mathbf{C})/Q$. Henceforth, we consider $\mathbf{X}$ as a complex manifold with the complex structure induced from that of $\mathbf{G}(\mathbf{C})/Q$.

2.3 Holomorphic limits of discrete series I

Fix a choice Δ_c^+ and the corresponding system of positive roots Δ^+. Let $\mathbf{X}^*(\mathcal{H}_\infty)$ be the group of algebraic characters of $\mathcal{H}_\infty$ and let $\mathbf{\Lambda} \subset \mathfrak{h}_{\infty,\mathbf{C}}$ denote the differentials of the elements of $\mathbf{X}^*(\mathcal{H}_\infty)$. The exponential mapping gives an isomorphism $\exp : \mathbf{\Lambda} \xrightarrow{\sim} \mathbf{X}^*(\mathcal{H}_\infty)$. Let $\mathrm{Dom} = \mathrm{Dom}(\mathcal{H}_\infty, G)$ be the dominant Weyl chamber corresponding to Δ^+ i.e. given $\eta \in \mathbf{i}\mathfrak{h}_\infty^*$, one has $\eta \in \mathrm{Dom}$ if and only if $\langle \eta, \alpha \rangle \geqslant 0$ for all $\alpha \in \Delta^+$, where $\langle \, , \rangle$ is deduced from the Killing form. Put $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, $\rho_n = \frac{1}{2} \sum_{\alpha \in \Delta_n^+} \alpha$ and $\rho_c = \frac{1}{2} \sum_{\alpha \in \Delta_c^+} \alpha$.

Given a subset of roots $\Gamma \subset \Delta$ and an element $\lambda \in i\mathfrak{h}_\infty^*$, we say λ is Γ-regular if $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Gamma$. If λ is not Γ regular, we say it is Γ-singular.

An $\mathbf{X}$-holomorphic limit of discrete series[7] is a representation $\pi(\lambda, \mathrm{Dom})$ whose Harish-Chandra parameter in the sense of [KZ82, § 1] is (λ, Dom) with

$$\lambda \in \mathrm{Dom} \cap (\mathbf{\Lambda} + \rho) \tag{2.4}$$

an element that is Δ_n-singular and Δ_c-regular.[8] A more explicit description of $\mathbf{X}$-holomorphic limits of discrete series of unitary groups will be given in § 5.4.

Suppose $\eta \in \mathbf{\Lambda}$ is a Δ_c^+-dominant weight for $\mathcal{K}_\infty$. Let V_η denote the irreducible $\mathcal{K}_\infty$-module of highest weight η and denote by $V_\eta^\vee$ its dual. Applying[9] [Har88, Theorem 3.4] to an $\mathbf{X}$-holomorphic limit of discrete series $\pi(\lambda, \mathrm{Dom})$, one has[10]

$$\dim H^0(\mathfrak{q}, \mathcal{K}_\infty; \pi(\lambda, \mathrm{Dom}) \otimes V^\vee_{\lambda+\rho_n-\rho_c}) = 1. \tag{2.5}$$

2.4 Shimura varieties over C

Suppose that $(\mathbf{G}, \mathbf{X})$ is a Shimura datum. Let $\mathbf{G}(\mathbf{Q})$ act on $\mathbf{X}$ by conjugation and on $\mathbf{G}(\mathbf{A}_f)$ by left multiplication via the diagonal embedding $\mathbf{G}(\mathbf{Q}) \hookrightarrow \mathbf{G}(\mathbf{A}_f)$. Given these two actions, for every open compact subgroup $\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)$, one has the Shimura variety $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_\mathcal{K}$ of level $\mathcal{K}$, whose complex points are given as the quotient

$$\mathrm{Sh}(\mathbf{G}, \mathbf{X})_\mathcal{K}(\mathbf{C}) = \mathbf{G}(\mathbf{Q}) \backslash (\mathbf{X} \times \mathbf{G}(\mathbf{A}_f)/\mathcal{K}). \tag{2.6}$$

As the level $\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)$ varies the Shimura varieties $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_\mathcal{K}$ form a projective system of quasi-projective varieties, equipped with an action of $\mathbf{G}(\mathbf{A}_f)$ given by right multiplication.

2.5 Equivariant vector bundles via the flag variety

This section is based on [CF90, ch. 6, § 4, especially p. 222]. There is a pair (VB, FI) of mutually inverse functors

$$\begin{tikzcd} \underline{\mathrm{Rep}}^{\mathrm{alg}}(Q) \arrow[r, bend left, "\mathrm{VB}_Q"] & \underline{\mathrm{EVect}}(G_\mathbf{C}/Q) \arrow[l, bend left, "\mathrm{FI}_Q"] \end{tikzcd} \tag{2.7}$$

---

[7] Vogan has pointed out to us that some authors use the terminology 'relative discrete series' (respectively 'relative limit of discrete series') for representations that are not square integrable but only square integrable modulo center (respectively limits thereof).

[8] If C is a general Weyl chamber, then the condition on a Harish-Chandra parameter (λ, C), that λ is Δ_n-singular and Δ_c-regular entails that the associated representation $\pi(\lambda, C)$ is a non-degenerate limit of discrete series. However, when $C = \mathrm{Dom}$ all limits of discrete series are holomorphic.

[9] Part (i) of [Har88, Theorem 3.4] is not correct since it does not take into account the possible disconnectedness of G in the classical topology. When G is disconnected in the classical topology a given non-degenerate (in particular holomorphic) limit of discrete series will have $(\mathfrak{q}, \mathcal{K}_\infty)$-cohomology in more than one degree. However, part (ii) of [Har88, Theorem 3.4] is true and we only use part(ii).

[10] Note the placement of the dual.

which give an equivalence of categories between the category $\underline{\mathrm{Rep}}^{\mathrm{alg}}(Q)$ of finite-dimensional algebraic representations of Q and the category $\underline{\mathrm{EVect}}(G_{\mathbf{C}}/Q)$ of $G_{\mathbf{C}}$-equivariant vector bundles on $G_{\mathbf{C}}/Q$. Let $(\sigma, E_\sigma) \in \underline{\mathrm{Rep}}^{\mathrm{alg}}(Q)$ be a finite-dimensional algebraic representation of Q. The vector bundle functor VB_Q is defined by

$$\mathrm{VB}_Q((\sigma, E_\sigma)) = G \times_{Q,\sigma} E_\sigma = \left\{ (G \times E_\sigma) \Big/ \begin{array}{c} (gx, \sigma^{-1}(x)v) \sim (g, v) \\ \text{for all } g \in G, x \in Q, v \in E_\sigma \end{array} \right\}. \tag{2.8}$$

In the other direction the 'fiber at the identity' functor FI_Q associates to a $G_{\mathbf{C}}$-equivariant vector bundle $\mathcal{V}_Q$ on $G_{\mathbf{C}}/Q$ its fiber $\mathcal{V}_{Q,e}$ at the identity coset $eQ \in G_{\mathbf{C}}/Q$. Since $\mathcal{V}_Q$ is $G_{\mathbf{C}}$-equivariant, $G_{\mathbf{C}}$ acts on $\mathcal{V}_Q$ by bundle maps; the restriction of this action to Q preserves the fiber $\mathcal{V}_{Q,e}$, so it induces a Q-module structure on $\mathcal{V}_{Q,e}$.

Since $\mathcal{K}_{\infty,\mathbf{C}}$ is the Levi factor of Q, one may extend and lift V_η to a representation $\tilde{V}_\eta$ of Q by first complexifying and then setting $\tilde{V}_\eta$ to be trivial on the unipotent radical of Q. Restricting $\mathrm{VB}_Q(\tilde{V}_\eta)$ to $\mathbf{X}$ gives a G-equivariant vector bundle on $\mathbf{X}$ which will be denoted by $\mathcal{V}_\eta$. The fiber at the identity of $\mathcal{V}_\eta$ is V_η. Since $\mathcal{V}_\eta$ is $\mathbf{G}(\mathbf{Q})$-equivariant, it descends to a $\mathbf{G}(\mathbf{A}_f)$-equivariant vector bundle on the Shimura variety $\mathrm{Sh}(\mathbf{G}, \mathbf{X})(\mathbf{C})$.

2.6 Lie algebra cohomology

Define the cohomology $\bar{H}^0(\mathrm{Sh}(\mathbf{G}, \mathbf{X}), \mathcal{V}_\eta)$ as in [Har88, § 2] and [Tay91, § 3.2], where the bar indicates that one is considering the image of the cohomology of the subcanonical extension of $\mathcal{V}_\eta$ in the cohomology of the canonical extension. Combining (2.5) with [Har88, Propositions 2.7.2 and 3.2.2] gives the following result.

Theorem 2.6.1. *Suppose that π is an automorphic representation of $\mathbf{G}$. Write $\pi = \pi_\infty \otimes \pi_f$. Assume that π_∞ is an $\mathbf{X}$-holomorphic limit of discrete series with Harish-Chandra parameter (λ, Dom). Then there is a $\mathbf{G}(\mathbf{A}_f)$ equivariant embedding*

$$\pi_f \hookrightarrow \bar{H}^0(\mathrm{Sh}(\mathbf{G}, \mathbf{X}), \mathcal{V}^\vee_{\lambda+\rho_n-\rho_c}). \tag{2.9}$$

3. Rational and integral theory

3.1 Kottwitz data

By a Kottwitz datum, we mean a quintuple

$$\mathcal{U} = (B, V, *, \langle\ ,\ \rangle, \tilde{h}) \tag{3.1}$$

satisfying the following six conditions:

KD1. B is a finite dimensional simple $\mathbf{Q}$-algebra;

KD2. V is a finitely generated left B-module;

KD3. $*$ is a positive involution on B, in the sense that $\mathrm{tr}_{B(\mathbf{R})/\mathbf{R}}(xx^*) > 0$ for all $x \in B(\mathbf{R}) - \{0\}$;

KD4. $\langle\cdot,\cdot\rangle : V \times V \longrightarrow \mathbf{Q}$ is a non-degenerate symplectic form such that $\langle bu, v\rangle = \langle u, b^*v\rangle$ for all $b \in B$ and $u, v \in V$;

KD5. we have that

$$\tilde{h} : \mathbf{C} \longrightarrow \mathrm{End}_B V \otimes \mathbf{R}$$

is a homomorphism of $\mathbf{R}$-algebras with involution, where $\mathbf{C}$ is viewed as an $\mathbf{R}$-algebra with involution given by complex conjugation;

KD6. the symmetric bilinear form

$$\begin{array}{ccc} V(\mathbf{R}) \times V(\mathbf{R}) & \longrightarrow & \mathbf{R} \\ (u, v) & \longmapsto & \langle u, \tilde{h}(i)v \rangle \end{array}$$

is positive definite.

Let $\mathcal{U}$ be a Kottwitz datum. Associated to $\mathcal{U}$, one has the $\mathbf{Q}$-algebraic group $\mathbf{G} = \mathbf{G}(\mathcal{U})$, whose points in a $\mathbf{Q}$-algebra R are given by

$$\mathbf{G}(R) = \{T \in \mathrm{End}_B(V) \otimes_{\mathbf{Q}} R \mid TT^* \in R^\times\}. \tag{3.2}$$

Let h be the inverse of the restriction of $\tilde{h}$ to $\mathbf{S}(\mathbf{R})$. Then the image of h is contained in $\mathbf{G}(\mathbf{R})$. Let $\mathbf{X}$ be the set of homomorphisms $\mathbf{S}(\mathbf{R}) \longrightarrow \mathbf{G}(\mathbf{R})$ that are $\mathbf{G}(\mathbf{R})$-conjugate to h. Then $(\mathbf{G}, \mathbf{X})$ is a Shimura datum. We call the Shimura varieties associated to Kottwitz data *Shimura varieties of Kottwitz type.*

There is a direct sum decomposition of $B(\mathbf{C})$ modules $V(\mathbf{C}) = V_+ \oplus V_-$, where $\tilde{h}$ acts on V^+ (respectively V_-) by z (respectively $\bar{z}$). The reflex field $E = E(\mathbf{G}, \mathbf{X})$ of the Shimura datum $(\mathbf{G}, \mathbf{X})$ is the field of definition of the $B(\mathbf{C})$ module V_+. It is a number field.

Let F be the center of B and let F^+ be the maximal totally real subfield of F. Put $d = [F^+ : \mathbf{Q}]$, $r = (\dim_F B)^{1/2}$ and $n = (\dim_F \mathrm{End}_B V)^{1/2}$. We say that $\mathcal{U}$ (respectively $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$) is a *unitary* Kottwitz datum (respectively a *unitary* Shimura variety of Kottwitz type) if $F \neq F^+$ (in which case F is a quadratic totally imaginary extension of F^+).

3.2 Moduli interpretation I: characteristic zero

Let $\mathcal{U}$ be a Kottwitz datum, $(\mathbf{G}, \mathbf{X})$ (respectively $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$) the corresponding Shimura datum (respectively Shimura variety) and $\mathcal{K}$ a *neat* [Har88, pp. 51–52] open compact subgroup of $\mathbf{G}(\mathbf{A}_f)$. In this section, we give a moduli interpretation of $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_{\mathcal{K}}$ over the reflex field $E = E(\mathbf{G}, \mathbf{X})$ of $(\mathbf{G}, \mathbf{X})$ following Kottwitz [Kot92, § 5]. In fact, Kottwitz gave a moduli problem that is slightly different from the one presented in this section, which has the advantage of providing an integral model of $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$. Kottwitz's integral moduli problem will be described in § 3.4.

Let $\underline{\mathrm{Sch}}_{/E}$ be the category of locally Noetherian schemes over Spec E. Suppose $T \in \underline{\mathrm{Sch}}_{/E}$ is connected. By a *$\mathcal{U}$-enriched abelian scheme* we mean a quadruple $(A, \lambda, i, \bar{\eta})$ where:

Ab1. A is an abelian scheme over T;

Ab2. $\lambda : A \longrightarrow A^\vee$ is a polarization;

Ab3. $i : B \longrightarrow \mathrm{End}(A) \otimes \mathbf{Q}$ is a homomorphism of algebras with involution, the involution on $\mathrm{End}(A) \otimes \mathbf{Q}$ being the Rosati involution associated to the polarization λ;

Ab4. (A, λ, i) satisfies the Kottwitz determinant condition (see [Kot92, p. 390]);

Ab5. $\bar{\eta}$ is a level structure of type $\mathcal{K}$ on A in the sense of *loc. cit.*

Next one defines an equivalence on the set of $\mathcal{U}$-enriched abelian schemes over T. Suppose that $(A_1, \lambda_1, i_1, \bar{\eta}_1)$ and $(A_2, \lambda_2, i_2, \bar{\eta}_2)$ are two $\mathcal{U}$-enriched abelian schemes over T. Say that $(A_1, \lambda_1, i_1, \bar{\eta}_1)$ is equivalent to $(A_2, \lambda_2, i_2, \bar{\eta}_2)$, written $(A_1, \lambda_1, i_1, \bar{\eta}_1) \sim (A_2, \lambda_2, i_2, \bar{\eta}_2)$, if there exists an isogeny $\varphi : A_1 \longrightarrow A_2$ such that:

EquivAb1. $\varphi_* \lambda_1 = c\lambda_2$ for some $c \in \mathbf{Q}^\times$;

EquivAb2. $\varphi \circ i_1(b) = i_2(b) \circ \varphi$ for all $b \in B$;

EquivAb3. $\varphi_* \bar{\eta}_1 = \bar{\eta}_2$.

Let $\underline{\mathrm{SETS}}$ be the category of sets. Consider the functor

$$\mathcal{F}_{\mathcal{U}} : \underline{\mathrm{Sch}}_{/E} \longrightarrow \underline{\mathrm{SETS}} \tag{3.3}$$

defined, for connected $T \in \underline{\mathrm{Sch}}_{/E}$, by

$$\mathcal{F}_{\mathcal{U}}(T) = \left\{ (A, \lambda, i, \bar{\eta}) \;\middle|\; \begin{matrix} (A, \lambda, i, \bar{\eta}) \text{ is a } \mathcal{U}\text{-enriched} \\ \text{abelian scheme over } T \end{matrix} \right\} \Big/ \sim \tag{3.4}$$

and extended to all $T \in \underline{\mathrm{Sch}}_{/E}$ by $\mathcal{F}_{\mathcal{U}}(T_1 \cup T_2) = \mathcal{F}_{\mathcal{U}}(T_1) \times \mathcal{F}_{\mathcal{U}}(T_2)$ for disjoint, connected $T_1, T_2 \in \underline{\mathrm{Sch}}_{/E}$. Since $\mathcal{K}$ is neat, $\mathcal{F}_{\mathcal{U}}$ is representable by a smooth, quasi-projective scheme $S_{\mathcal{K},E} \in \underline{\mathrm{Sch}}_{/E}$ (see [Kot92, p. 391]). Moreover, $S_{\mathcal{K},E} \otimes \mathbf{C} \cong \mathrm{Sh}(\mathbf{G}, \mathbf{X})_{\mathcal{K}}(\mathbf{C})$ as $\mathbf{C}$-schemes (see [Kot92, § 8]).

3.3 Integral Kottwitz data

Assume that ℓ is a (rational) prime and let $\mathbf{Z}_{(\ell)}$ denote the localization of $\mathbf{Z}$ at ℓ.

By an (ℓ-)integral Kottwitz datum (IKD), we mean a triple

$$\mathcal{U}^{(\ell)} = (\mathcal{U}, \mathcal{O}_{B,\ell}, L), \tag{3.5}$$

where $\mathcal{U} = (B, V, *, \langle\, , \rangle, \tilde{h})$ is a Kottwitz datum and in addition the following four conditions hold:

IKD1. $B(\mathbf{Q}_\ell)$ is a product of matrix algebras over unramified extensions of $\mathbf{Q}_\ell$;

IKD2. $\mathcal{O}_{B,\ell}$ is a $\mathbf{Z}_{(\ell)}$-order in B;

IKD3. the ℓ-adic completion $\widehat{\mathcal{O}_{B,\ell}}$ of $\mathcal{O}_{B,\ell}$ is a maximal order in $B(\mathbf{Q}_\ell)$;

IKD4. there exists a lattice L in $V(\mathbf{Q}_\ell)$ that is self-dual with respect to $\langle\, , \rangle$ and is preserved by $\mathcal{O}_{B,\ell}$.

If $\mathcal{U}$ is a Kottwitz datum and ℓ is a prime, we say that ℓ is a *prime of good reduction* for $\mathcal{U}$ if there exists $\mathcal{O}_{B,\ell}$ and L such that the triple $(\mathcal{U}, \mathcal{O}_{B,\ell}, L)$ is an ℓ-IKD.

An ℓ-IKD furnishes a reductive $\mathbf{Z}_\ell$-model of $\mathbf{G}$, which we continue to denote by $\mathbf{G}$, whose points in a $\mathbf{Z}_\ell$-algebra R are give by

$$\mathbf{G}(R) = \{T \in \mathrm{End}_{\mathcal{O}_{B,\ell}}(L) \otimes R \mid TT^* \in R^\times\}. \tag{3.6}$$

In particular, if ℓ is a prime of good reduction for $\mathcal{U}$, then $\mathbf{G}$ is unramified at ℓ.

3.4 Moduli interpretation II: integrality

Let $\mathcal{O}_E$ be the ring of integers of E and put $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(\ell)} = \mathcal{O}_{E,\ell}$. In this section, the moduli problem of § 3.2 is modified so as to be defined over $\mathcal{O}_{E,\ell}$ and thus give a model of the Shimura variety over $\mathcal{O}_{E,\ell}$.

Suppose that $\mathcal{K}^\ell$ is a neat, open compact subgroup of $\mathbf{G}(\mathbf{A}_f^\ell)$. Let $\underline{\mathrm{Sch}}_{/\mathcal{O}_{E,\ell}}$ denote the category of locally Noetherian schemes over Spec $\mathcal{O}_{E,\ell}$.

Given a connected $C_\ell \in \underline{\mathrm{Sch}}_{/\mathcal{O}_{E,\ell}}$, an *$\ell$-integral $\mathcal{U}^{(\ell)}$-enriched abelian scheme* is a quadruple $(A, \lambda_\ell, i_\ell, \bar{\eta}_\ell)$ satisfying:

IAb1. A is an abelian scheme over C_ℓ;

IAb2. $\lambda_\ell : A \longrightarrow A^\vee$ is a prime-to-ℓ polarization;

IAb3. $i_\ell : \mathcal{O}_{B,\ell} \longrightarrow \mathrm{End}(A) \otimes \mathbf{Z}_{(\ell)}$ is a homomorphism of algebras with involution;

IAb4. $(A, \lambda_\ell, i_\ell)$ satisfies the Kottwitz determinant condition (see [Kot92, p. 390]);

IAb5. $\bar{\eta}_\ell$ is a level structure of type $\mathcal{K}^\ell$ in the sense of *loc. cit.*

Note that, in particular, the base change of an ℓ-integral $\mathcal{U}^{(\ell)}$-enriched abelian scheme to Spec E is a $\mathcal{U}$-enriched abelian scheme.

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Two ℓ-integral $\mathcal{U}^{(\ell)}$-enriched abelian schemes $(A_1, \lambda_{\ell,1}, i_{\ell,1}, \bar{\eta}_{\ell,1})$ and $(A_2, \lambda_{\ell,2}, i_{\ell,2}, \bar{\eta}_{\ell,2})$ are defined to be ℓ-equivalent, denoted by $\sim_\ell$, if there exists a prime-to-ℓ isogeny $\varphi_\ell : A_1 \longrightarrow A_2$ such that:

IEquivAb1. $(\varphi_\ell)_* \lambda_{\ell,1} = u\lambda_{\ell,2}$ for some $u \in \mathbf{Z}_{(\ell)}^\times$;

IEquivAb2. $\varphi_\ell \circ i_{\ell,1}(b) = i_{\ell,2}(b) \circ \varphi_\ell$ for all $b \in \mathcal{O}_B$;

IEquivAb3. $(\varphi_\ell)_* \bar{\eta}_{\ell,1} = \bar{\eta}_{\ell,2}$.

Consider the functor

$$\mathcal{F}_{\mathcal{U}^{(\ell)}} : \underline{\mathrm{Sch}}_{/\mathcal{O}_{E,\ell}} \longrightarrow \underline{\mathrm{SETS}} \tag{3.7}$$

defined, for connected $C_\ell \in \underline{\mathrm{Sch}}_{/\mathcal{O}_{E,\ell}}$, by

$$\mathcal{F}_{\mathcal{U}^{(\ell)}}(C_\ell) = \left\{ (A, \lambda_\ell, i_\ell, \bar{\eta}_\ell) \;\middle|\; \begin{array}{c} (A, \lambda_\ell, i_\ell, \bar{\eta}_\ell) \text{ is an } \ell\text{-integral} \\ \mathcal{U}^{(\ell)}\text{-enriched abelian scheme over } C_\ell \end{array} \right\} \Big/ \sim_\ell \tag{3.8}$$

and extended to all $C_\ell \in \underline{\mathrm{Sch}}_{/\mathcal{O}_{E,\ell}}$ in the same way as for $\mathcal{F}_{\mathcal{U}}$. Again, the neatness of $\mathcal{K}^\ell$ implies that $\mathcal{F}_{\mathcal{U}^{(\ell)}}$ is representable by a smooth, quasi-projective scheme $S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}} \in \underline{\mathrm{Sch}}_{/\mathcal{O}_{E,\ell}}$ (see [Kot92, p. 391]). Furthermore $S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}} \otimes E \cong S_{\mathcal{K}_+^{(\ell)}, E}$, where $\mathcal{K}_+^{(\ell)} = \mathbf{G}(\mathbf{Z}_\ell)\mathcal{K}^{(\ell)}$.

Given a level subgroup $\mathcal{K} = \mathcal{K}_{(\ell)}\mathcal{K}^{(\ell)}$ with $K_{(\ell)} \subset \mathbf{G}(\mathbf{Z}_\ell)$, define $S_{\mathcal{K}, \mathcal{O}_{E,\ell}}$ to be the normalization of $S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}$ in $S_{\mathcal{K},E}$. By definition $S_{\mathcal{K}, \mathcal{O}_{E,\ell}}$ is normal and $S_{\mathcal{K}, \mathcal{O}_{E,\ell}} \otimes E \cong S_{\mathcal{K},E}$.

3.5 Action of $\mathbf{G}(\mathbf{A}_f^\ell)$

We now define, as in [Kot92, § 6, p. 392], an action of $\mathbf{G}(\mathbf{A}_f^\ell)$ on the system $\{S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}\}$, as $\mathcal{K}^{(\ell)}$ ranges over open compact subgroups of $\mathbf{G}(\mathbf{A}_f^\ell)$.

Given $g \in \mathbf{G}(\mathbf{A}_f^\ell)$, one has an isomorphism, still denoted by g, between the integral models of the Shimura varieties of levels $\mathcal{K}^{(\ell)}$ and its conjugate $g^{-1}\mathcal{K}^{(\ell)}g$,

$$g : S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}} \longrightarrow S_{g^{-1}\mathcal{K}^{(\ell)}g, \mathcal{O}_{E,\ell}}, \tag{3.9}$$

defined by

$$(A, \lambda_\ell, i_\ell, \bar{\eta}_\ell) \longmapsto (A, \lambda_\ell, i_\ell, \overline{\eta_\ell g}) \tag{3.10}$$

for all ℓ-integral $\mathcal{U}^{(\ell)}$-enriched abelian schemes $(A, \lambda_\ell, i_\ell, \bar{\eta}_\ell)$.

Extending scalars to $\mathbf{C}$ one recovers the action defined in § 2.4.

3.6 The Hodge bundle

Let $\mathcal{U}^{(\ell)}$ be an ℓ-IKD as in § 3.4. Let $[\mathcal{A}, \underline{\lambda}_\ell, \underline{i}_\ell, \underline{\bar{\eta}}_\ell]$ be the universal ℓ-equivalence class of ℓ-integral $\mathcal{U}^{(\ell)}$-enriched abelian schemes over $S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}$. Suppose that $(\mathcal{A}_1, \lambda_{\ell,1}, i_{\ell,1}, \bar{\eta}_{\ell,1})$ and $(\mathcal{A}_2, \lambda_{\ell,2}, i_{\ell,2}, \bar{\eta}_{\ell,2})$ are two representatives of the class $[\mathcal{A}, \underline{\lambda}_\ell, \underline{i}_\ell, \underline{\bar{\eta}}_\ell]$. Since the level $\mathcal{K}^{(\ell)}$ is assumed neat, there exists a unique prime-to-ℓ isogeny $\varphi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ satisfying conditions IEquivAb1–IEquivAb3 of § 3.4 (i.e. compatible with the ℓ-integral $\mathcal{U}^{(\ell)}$ structure). For $j \in \{1, 2\}$, let $e_j : S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}} \longrightarrow \mathcal{A}_j$ be the identity section of the $S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}$-abelian scheme $\mathcal{A}_j$. Let $\Omega_{\mathcal{A}_j/S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}}$ denote the sheaf of relative differentials on $\mathcal{A}_j$ over $S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}$. Since φ is a separable isogeny, $\varphi^*\Omega_{\mathcal{A}_2/S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}} \cong \Omega_{\mathcal{A}_1/S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}}$ and also $e_2 = \varphi \circ e_1$. Hence there is a *canonical isomorphism* of sheaves

$$e_2^*\Omega_{\mathcal{A}_2/S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}} = (\varphi \circ e_1)^*\Omega_{\mathcal{A}_2/S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}} = e_1^*(\varphi^*\Omega_{\mathcal{A}_2/S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}}) \cong e_1^*\Omega_{\mathcal{A}_1/S_{\mathcal{O}_{E,\ell}}} \tag{3.11}$$

on $S_{\mathcal{K}^{(\ell)}, \mathcal{O}_{E,\ell}}$.

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Let $\Omega_{\mathcal{K}^{(\ell)}}$ be the isomorphism class of the locally free sheaf $e_1^*\Omega_{\mathcal{A}_1/S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}}$ on $S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$. In view of (3.11), $\Omega_{\mathcal{K}^{(\ell)}}$ depends only on the equivalence class $[\mathcal{A},\underline{\lambda}_\ell,\underline{i}_\ell,\underline{\bar{\eta}}_\ell]$. Put $\omega_{\mathcal{K}^{(\ell)}} = \det \Omega_{\mathcal{K}^{(\ell)}}$.

Suppose $\mathcal{K} = \mathcal{K}_{(\ell)}\mathcal{K}^{(\ell)}$ is a level subgroup with $K_{(\ell)} \subset \mathbf{G}(\mathbf{Z}_\ell)$. Then define $\Omega_{\mathcal{K}}$ (respectively $\omega_{\mathcal{K}}$) as the inverse image of $\Omega_{\mathcal{K}^{(\ell)}}$ (respectively $\omega_{\mathcal{K}^{(\ell)}}$) along $\mathrm{Id} : S_{\mathcal{K},\mathcal{O}_{E,\ell}} \longrightarrow S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$.

We write $\Omega \otimes \mathbf{C}$ (respectively $\omega \otimes \mathbf{C}$) for the corresponding $\mathbf{G}(\mathbf{A}_f)$ equivariant vector bundle on $\mathrm{Sh}(\mathbf{G},\mathbf{X})(\mathbf{C})$.

3.7 Decomposition of the Hodge bundle

Suppose henceforth that $\mathcal{U}$ (respectively $\mathcal{U}^{(\ell)}$) is a unitary Kottwitz datum (respectively an ℓ-integral unitary Kottwitz datum).

The endomorphism action in the moduli problems of our Shimura varieties, described in §§ 3.2, 3.4, entails that, over a finite extension of the reflex field E, one has a decomposition of the Hodge bundle $\Omega_{\mathcal{K}^{(\ell)}}$. Replacing E with a finite extension if necessary, we may assume that (i) E contains all embeddings of F into $\mathbf{C}$ and (ii) B splits over E.

Let $\tau_1,\ldots,\tau_d$ denote the real places of F^+. For every i, $1 \leqslant i \leqslant d$, let τ_i^+ and τ_i^- denote the two complex places of F lying above τ_i. On the one hand, in view of (§ 3.4, condition IAb3), $\mathcal{O}_{B,\ell}$ acts on $\Omega_{\mathcal{K}^{(\ell)}}$. On the other hand, $\mathcal{O}_{E,\ell}$ also acts on $\Omega_{\mathcal{K}^{(\ell)}}$, since the Shimura variety $S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$ is defined over $\mathcal{O}_{E,\ell}$. Thus, $\Omega_{\mathcal{K}^{(\ell)}}$ admits an action of $\mathcal{O}_{B,\ell} \otimes \mathcal{O}_{E,\ell}$. By our assumptions on E, one has

$$\mathcal{O}_{B,\ell} \otimes \mathcal{O}_{E,\ell} = \bigoplus_{i=1}^{d} M_r(\mathcal{O}_{E,\ell})_{\tau_{i+}} \oplus M_r(\mathcal{O}_{E,\ell})_{\tau_{i-}} \quad \text{with } (\mathcal{O}_{E,\ell})_{\tau_i^\pm} \cong \mathcal{O}_{E,\ell}. \tag{3.12}$$

Using the Morita equivalence $N \mapsto N^{\oplus r}$ between the category of $\mathcal{O}_{E,\ell}$-modules and that of $M_r(\mathcal{O}_{E,\ell})$-modules, let $\Omega_{\mathcal{K}^{(\ell)},\tau_i^+}$ (respectively $\Omega_{\mathcal{K}^{(\ell)},\tau_i^-}$) be the sub-bundle of $\Omega_{\mathcal{K}^{(\ell)}}$ such that $\mathcal{O}_{B,\ell} \otimes \mathcal{O}_{E,\ell}$ acts on $\Omega^{\oplus r}_{\mathcal{K}^{(\ell)},\tau_i^+}$ (respectively $\Omega^{\oplus r}_{\mathcal{K}^{(\ell)},\tau_i^-}$) via the direct summand $(\mathcal{O}_{E,\ell})_{\tau_i^+}$, (respectively $(\mathcal{O}_{E,\ell})_{\tau_i^-}$). Then we have the decomposition

$$\Omega_{\mathcal{K}^{(\ell)}} = \bigoplus_{i=1}^{d} (\Omega^{\oplus r}_{\mathcal{K}^{(\ell)},\tau_i^+} \oplus \Omega^{\oplus r}_{\mathcal{K}^{(\ell)},\tau_i^-}). \tag{3.13}$$

Put $p_i = \operatorname{rank} \Omega_{\mathcal{K}^{(\ell)},\tau_i^+}$ and $q_i = \operatorname{rank} \Omega_{\mathcal{K}^{(\ell)},\tau_i^-}$ for all i, $1 \leqslant i \leqslant d$.

Let $\omega_{\mathcal{K}^{(\ell)},\tau_i^+}$ (respectively $\omega_{\mathcal{K}^{(\ell)},\tau_i^-}$) be the determinant of $\Omega_{\mathcal{K}^{(\ell)},\tau_i^+}$ (respectively $\Omega_{\mathcal{K}^{(\ell)},\tau_i^-}$). Taking determinants in (3.13) yields

$$\omega_{\mathcal{K}^{(\ell)}} = \bigotimes_{i=1}^{d} (\omega^{\otimes r}_{\mathcal{K}^{(\ell)},\tau_i^+} \otimes \omega^{\otimes r}_{\mathcal{K}^{(\ell)},\tau_i^-}). \tag{3.14}$$

As was done for $\Omega_{\mathcal{K}^{(\ell)}}$ in § 3.6, we define the level $\mathcal{K} = \mathcal{K}_{(\ell)}\mathcal{K}^{(\ell)}$ versions $\Omega_{\mathcal{K},\tau_i^+}$ etc. as the pull-backs of $\Omega_{\mathcal{K}^{(\ell)},\tau_i^+}$ etc., and denote by $\Omega_{\tau_i^+} \otimes \mathbf{C}$ etc. the corresponding $\mathbf{G}(\mathbf{A}_f)$ equivariant bundles on $\mathrm{Sh}(\mathbf{G},\mathbf{X})(\mathbf{C})$.

3.8 Schur functors

In characteristic zero, Schur functors of degree r may be defined as images of r-fold tensor products under the action of primitive idempotents in the group ring of the symmetric group on r letters [FH91, pp. 75–77, 231–232]. However, this approach fails in characteristic $\ell > 0$.

There are at least three ways to construct Schur functors integrally. The one we will use is the construction of Carter and Lusztig [CL74, § 3.2], which is based on Kostant's $\mathbf{Z}$-form of the universal enveloping algebra. An alternative[11] would be to use the limited analogue of the Borel–Weil–Bott theorem in positive characteristic given by Kempf's vanishing theorem (see [Jan87, ch. II.4, p. 227]). A third more elementary and combinatorial approach[12] is given in Fulton's book on Young tableaux (see [Ful97, ch. 8, pp. 108–112])

Let $\mathfrak{M}$ be a free module of rank s over a commutative ring $\mathfrak{R}$ (with unit). For every positive integer s, put

$$\mathbf{Z}^s_{\mathrm{Dom}} = \{(a_1, \ldots, a_s) \in \mathbf{Z}^s \mid a_1 \geqslant \cdots \geqslant a_s\},$$

and define $\mathbf{Z}^0_{\mathrm{Dom}} = \{0\}$. Given $\nu \in \mathbf{Z}^s_{\mathrm{Dom}}$, Carter and Lusztig define a certain free submodule $\mathfrak{M}^\nu$ of $\mathfrak{M}^{\otimes s}$ (see [CL74, § 3.2, p. 211, Equations 28–29])[13]. The definition of $\mathfrak{M}^\nu$ is functorial (p. 213 of *loc. cit.*), so we may define a Schur functor $\mathbf{S}_{\nu,\mathfrak{R}}$, from the category $\underline{\mathrm{Mod}}^{s-\mathrm{free}}_{\mathfrak{R}}$ of rank s, free $\mathfrak{R}$-modules to the category $\underline{\mathrm{Mod}}^{\mathrm{free}}_{\mathfrak{R}}$ of all free $\mathfrak{R}$-modules, by setting $\mathbf{S}_{\nu,\mathfrak{R}}(\mathfrak{M}) = \mathfrak{M}^\nu$.

Since $\mathbf{S}_{\nu,\mathfrak{R}}$ is a functor, if $\mathfrak{M}$ admits an $\mathfrak{R}$-linear action of a group $\mathcal{G}$, then there is an induced action of $\mathcal{G}$ on the module $\mathbf{S}_{\nu,\mathfrak{R}}(\mathfrak{M})$. The prototypical example of such a group action, which is the only one we shall use, is the standard action of $\mathcal{G} = \mathrm{GL}(\mathfrak{M})$ on $\mathfrak{M}$.

We will need two properties of the Carter–Lusztig construction.

Theorem 3.8.1 (Carter–Lusztig [CL74, p. 220, Corollary]). (1) *For all ν, the Schur functor $\mathbf{S}_{\nu,\mathfrak{R}}$ is compatible with base change: suppose that $\mathfrak{R}$ and $\mathfrak{R}'$ are two commutative rings and $\phi : \mathfrak{R} \longrightarrow \mathfrak{R}'$ is a ring homomorphism. Then for every $\mathfrak{M} \in \underline{\mathrm{Mod}}^{s-\mathrm{free}}_{\mathfrak{R}}$, there is a canonical isomorphism of $(\mathrm{GL}(\mathfrak{M}) \otimes_{\mathfrak{R},\phi} \mathfrak{R}')$-modules:*

$$\mathbf{S}_{\nu,\mathfrak{R}}(\mathfrak{M}) \otimes_{\mathfrak{R},\phi} \mathfrak{R}' \xrightarrow{\sim} \mathbf{S}_{\nu,\mathfrak{R}'}(\mathfrak{M} \otimes_{\mathfrak{R},\phi} \mathfrak{R}'). \tag{3.15}$$

(2) *Suppose that $\mathfrak{R}$ is a field of characteristic zero. Then $\mathbf{S}_{\nu,\mathfrak{R}}(\mathfrak{M})$ is the irreducible representation of $\mathrm{GL}(\mathfrak{M})$ of highest weight ν.*

The compatibility of the Schur functors with base change implies the following corollary.

Corollary 3.8.2. *Given a scheme $\mathfrak{T}$, the Schur functor construction sheafifies to give a functor $\mathbf{S}_{\nu,\mathfrak{T}}$, from the category $\mathrm{VB}^s_{\mathfrak{T}}$ of rank s vector bundles on $\mathfrak{T}$ to the category $\mathrm{VB}_{\mathfrak{T}}$ of all finite-rank vector bundles on $\mathfrak{T}$.*

For all i, $1 \leqslant i \leqslant d$, let $\eta^+_i \in \mathbf{Z}^{p_i}_{\mathrm{Dom}}$ (respectively $\eta^-_i \in \mathbf{Z}^{q_i}_{\mathrm{Dom}}$) and put

$$\eta = (\eta^+_1, \eta^-_1, \ldots, \eta^+_d, \eta^-_d). \tag{3.16}$$

Define

$$\mathbf{S}^{\mathrm{fund}}_\eta(\Omega_{\mathcal{K}}) = \bigoplus_{i=1}^{d} (\mathbf{S}_{\eta^+_i, S_{\mathcal{K},\mathcal{O}_{E,\ell}}}(\Omega^\vee_{\mathcal{K},\tau^+_i}) \oplus \mathbf{S}_{\eta^-_i, S_{\mathcal{K},\mathcal{O}_{E,\ell}}}(\Omega_{\mathcal{K},\tau^-_i})). \tag{3.17}$$

For every η, $\mathbf{S}^{\mathrm{fund}}_\eta(\Omega_{\mathcal{K}})$ is a vector bundle on $S_{\mathcal{K},\mathcal{O}_{E,\ell}}$.

Let R be an $\mathcal{O}_{E,\ell}$-algebra. Define

$$M_\eta(\mathcal{K}, R) = \bar{H}^0(S_{\mathcal{K},\mathcal{O}_{E,\ell}} \otimes R, \mathbf{S}^{\mathrm{fund}}_\eta(\Omega_{\mathcal{K}}) \otimes R). \tag{3.18}$$

[11] This was suggested to us by Deligne.
[12] Thanks are due to Fulton for pointing out to us this approach.
[13] What we call ν (respectively $\mathfrak{M}^\nu$) is denoted μ (respectively V^μ) in *loc. cit.*

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4. The Hasse invariant

4.1 The Hasse invariant of an abelian variety

Let T be a scheme over $\mathbf{F}_\ell$. Let A be an abelian scheme over T admitting a prime-to-ℓ polarization, with structure map π_A and identity section e_A. Let g be the relative dimension of A over T. Let $\Omega_{A/T}$ be the sheaf of relative differentials on A. Put $\omega_A = \bigwedge^g e_A^* \Omega_{A/T}$. Then ω_A is a line bundle on T. The Hasse invariant $H(A) = H(A/T)$ of A (over T) will be an element of $H^0(T, \omega_A^{\otimes(\ell-1)})$.

The key to constructing the Hasse invariant $H(A)$ is the (relative) Frobenius isogeny $\mathrm{Frob}_{A/T} : A \to A^{(\ell)}$. Let $\mathbf{F}_T : T \longrightarrow T$ denote the absolute Frobenius morphism. As a map of schemes, $\mathbf{F}_T$ is the identity on the underlying topological space and $\mathbf{F}_T$ acts on the sheaf of regular functions $\mathcal{O}_T$ on T by $f \mapsto f^\ell$ for every local section f of $\mathcal{O}_T$. Thus if $\psi : T' \to T$ is a map of $\mathbf{F}_\ell$-schemes, the following diagram commutes.

$$\begin{tikzcd} T' \arrow[r, "\mathbf{F}_{T'}"] \arrow[d, "\psi"'] & T' \arrow[d, "\psi"] \\ T \arrow[r, "\mathbf{F}_T"'] & T \end{tikzcd} \tag{4.1}$$

The abelian scheme $A^{(\ell)}$ is defined as the fiber product of A and T over T along the maps π_A and $\mathbf{F}_T$. Denote the projection from $A^{(\ell)}$ to A (respectively T) by $W_{A/T}$ (respectively $\pi_{A^{(\ell)}}$). Thus, we have a fiber square.

$$\begin{tikzcd} A^{(\ell)} \arrow[r, "W_{A/T}"] \arrow[d, "\pi_{A^{(\ell)}}"'] & A \arrow[d, "\pi_A"] \\ T \arrow[r, "\mathbf{F}_T"'] & T \end{tikzcd} \tag{4.2}$$

Now A maps to A by $\mathbf{F}_A$ and to T by π_A. Moreover, the commutativity of (4.1) with $T' = A$ and $\psi = \pi_A$ entails that $\pi_A \circ \mathbf{F}_A = \mathbf{F}_T \circ \pi_A$. By the universal property of fiber products, there is an isogeny $\mathrm{Frob}_{A/T} : A \longrightarrow A^{(\ell)}$ of abelian schemes over T rendering the following diagram commutative.

$$\begin{tikzcd} A \arrow[dr, dotted, "\mathrm{Frob}_{A/T}"] \arrow[drr, bend left, "\mathbf{F}_A"] \arrow[ddr, bend right, "\pi_A"'] & & \\ & A^{(\ell)} \arrow[r, "W_{A/T}"] \arrow[d, "\pi_{A^{(\ell)}}"'] & A \arrow[d, "\pi_A"] \\ & T \arrow[r, "\mathbf{F}_T"'] & T \end{tikzcd} \tag{4.3}$$

The multiplication by ℓ map $[\ell] : A \longrightarrow A$ induces the zero map on differentials. Therefore there exists a unique factorization

$$\begin{tikzcd} A \arrow[rr, "{[\ell]}"] \arrow[dr, "\mathrm{Frob}_{A/T}"'] & & A \\ & A^{(\ell)} \arrow[ur] & \end{tikzcd} \tag{4.4}$$

(cf. [Ein80, Lemma 1.4]). The resulting isogeny $A^{(\ell)} \longrightarrow A$ in (4.4) is called Verschiebung and denoted $\mathrm{Ver}_{A/T} : A^{(\ell)} \longrightarrow A$. Like the (relative) Frobenius, the Verschiebung is also a map of

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T-schemes. By definition $\mathrm{Ver}_{A/T} \circ \mathrm{Frob}_{A/T} = [\ell]$, from which it follows that also $\mathrm{Frob}_{A/T} \circ \mathrm{Ver}_{A/T} = [\ell]$.

Pull-back of differentials from A to $A^{(\ell)}$ via Verschiebung gives a map of sheaves on $A^{(\ell)}$:

$$\mathrm{Ver}^*_{A/T} : \mathrm{Ver}^*_{A/T}\Omega_{A/T} \longrightarrow \Omega_{A^{(\ell)}/T}. \tag{4.5}$$

Now pull back the map $\mathrm{Ver}^*_{A/T}$ by means of $e^*_{A^{(\ell)}}$. This in turn yields a map of sheaves on T:

$$e^*_{A^{(\ell)}}(\mathrm{Ver}^*_{A/T}) : e^*_A\Omega_{A/T} \longrightarrow e^*_{A^{(\ell)}}\Omega_{A^{(\ell)}/T}, \tag{4.6}$$

since $e_A = \mathrm{Ver}_{A/T} \circ e_{A^{(\ell)}}$, which implies $e^*_{A^{(\ell)}}\mathrm{Ver}^*_{A/T}\Omega_{A/T} = (\mathrm{Ver}_{A/T} \circ e_{A^{(\ell)}})^*\Omega_{A/T} = e^*_A\Omega_{A/T}$. Taking top exterior powers in (4.6) gives a map of line bundles on T:

$$\omega_A \longrightarrow \omega_{A^{(\ell)}}. \tag{4.7}$$

One has $\omega_{A^{(\ell)}} = \omega_A^{(\ell)}$ and since ω_A is a line bundle, $\omega_A^{(\ell)} = \omega_A^{\otimes \ell}$. Hence, (4.7) becomes

$$h(A) : \omega_A \longrightarrow \omega_A^{\otimes \ell}. \tag{4.8}$$

We call the map $h(A)$ the *Hasse invariant map* of A/T. It induces a map on global sections, to which we give the same name:

$$h(A) : H^0(T, \omega_A) \longrightarrow H^0(T, \omega_A^{\otimes \ell}). \tag{4.9}$$

Since

$$\mathrm{Hom}(\omega_A, \omega_A^{\otimes \ell}) = \omega_A^{\otimes \ell} \otimes \omega_A^\vee = \omega_A^{\otimes \ell} \otimes \omega_A^{-1} = \omega_A^{\otimes(\ell-1)}, \tag{4.10}$$

the Hasse invariant map $h(A)$ corresponds to an element $H(A) \in H^0(T, \omega_A^{\otimes(\ell-1)})$, which is the Hasse invariant of A/T.

4.2 Compatibility with isogenies and base change

Suppose $\varphi : A \longrightarrow B$ is a separable isogeny of T-abelian schemes. There is an induced map of sheaves of relative differentials

$$\varphi^* : \varphi^*\Omega_{B/T} \longrightarrow \Omega_{A/T}, \tag{4.11}$$

which is in fact an isomorphism of sheaves on A, since φ is separable. Pulling back the map φ^* of (4.11) by e_A gives a map of sheaves on T,

$$e^*_A(\varphi^*) : e^*_B\Omega_{B/T} \longrightarrow e^*_A\Omega_{A/T}, \tag{4.12}$$

since $\varphi \circ e_A = e_B$. Denote the top exterior power of the map in (4.12) by $\overline{\varphi^*}$:

$$\overline{\varphi^*} : \omega_B \longrightarrow \omega_A. \tag{4.13}$$

Theorem 4.2.1. *One has*

$$\overline{\varphi^*}H(B) = H(A). \tag{4.14}$$

Proof. The proof is in three steps: we first show (Lemma 4.2.2) that Frobenius is compatible with isogenies, then deduce (Lemma 4.2.3) that Verschiebung is also compatible with isogenies and lastly we show how compatibility of Verschiebung with isogenies implies the theorem.

Given $\varphi : A \longrightarrow B$ one has an induced isogeny $\varphi^{(\ell)} : A^{(\ell)} \longrightarrow B^{(\ell)}$, the base change of φ along $\mathbf{F}_T$. That is, $\varphi^{(\ell)}$ is the map obtained from the universal property of fiber products applied to

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the fiber square defining $B^{(\ell)}$ and the pair of maps $(\pi_{A^{(\ell)}}, \varphi \circ W_{A/T})$ in the diagram

$$
\begin{tikzcd}
A^{(\ell)} \arrow[r, "W_{A/T}"] \arrow[dr, dotted, "\varphi^{(\ell)}"'] \arrow[ddr, bend right, "\pi_{A^{(\ell)}}"'] & A \arrow[dr, "\varphi"] \arrow[ddr, bend left, "\pi_A"] & \\
& B^{(\ell)} \arrow[r, "W_{B/T}"] \arrow[d, "\pi_{B^{(\ell)}}"'] & B \arrow[d, "\pi_B"] \\
& T \arrow[r, "\mathbf{F}_T"'] & T
\end{tikzcd}
\tag{4.15}
$$

because $\pi_B \circ \varphi = \pi_A$ and $\pi_A \circ W_{A/T} = \mathbf{F}_T \circ \pi_{A^{(\ell)}}$, which gives $\mathbf{F}_T \circ \pi_{A^{(\ell)}} = \pi_B \circ (\varphi \circ W_{A/T})$.

Lemma 4.2.2 (Relative). *Frobenius is compatible with isogenies. More precisely, the following diagram commutes.*

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{Frob}_{A/T}"] \arrow[d, "\varphi"'] & A^{(\ell)} \arrow[d, "\varphi^{(\ell)}"] \\
B \arrow[r, "\mathrm{Frob}_{B/T}"'] & B^{(\ell)}
\end{tikzcd}
\tag{4.16}
$$

Proof. The square (4.16) whose commutativity we need to prove is the top left-hand piece of the following larger diagram.

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{Frob}_{A/T}"] \arrow[d, "\varphi"'] & A^{(\ell)} \arrow[r, "W_{A/T}"] \arrow[d, "\varphi^{(\ell)}"] & A \arrow[d, "\varphi"] \\
B \arrow[r, "\mathrm{Frob}_{B/T}"'] \arrow[dr, "\pi_B"'] & B^{(\ell)} \arrow[r, "W_{B/T}"'] \arrow[d, "\pi_{B^{(\ell)}}"] & B \arrow[d, "\pi_B"] \\
& T \arrow[r, "\mathbf{F}_T"'] & T
\end{tikzcd}
\tag{4.17}
$$

Since $\pi_B \circ \varphi = \pi_A$ and $W_{A/T} \circ \mathrm{Frob}_{A/T} = \mathbf{F}_A$, the commutativity (4.1) with $T' = A$ and $\psi = \pi_A$ entails

$$\mathbf{F}_T \circ \pi_B \circ \varphi = \pi_B \circ \varphi \circ W_{A/T} \circ \mathrm{Frob}_{A/T}. \tag{4.18}$$

In other words, starting at the top left corner of (4.17) and proceeding along the boundary in either direction to the bottom right corner gives the same result. Since the bottom right square is the fiber square defining $B^{(\ell)}$, the universal property of fiber products implies that there is a unique map $\chi : A \to B^{(\ell)}$ such that

$$\pi_B \circ \varphi = \pi_{B^{(\ell)}} \circ \chi \tag{4.19}$$

and

$$\varphi \circ W_{A/T} \circ \mathrm{Frob}_{A/T} = W_{B/T} \circ \chi. \tag{4.20}$$

We claim that both $\mathrm{Frob}_{B/T} \circ \varphi$ and $\varphi^{(\ell)} \circ \mathrm{Frob}_{A/T}$ satisfy (4.19) and (4.20). Indeed, $\mathrm{Frob}_{B/T} \circ \varphi$ satisfies (4.19) because $\mathrm{Frob}_{B/T}$ is a map of T-schemes and it satisfies (4.20) by the commutativity of (4.1) with $\psi = \varphi$. Since $\mathrm{Frob}_{A/T}$, φ and $\varphi^{(\ell)}$ are all maps of T-schemes, (4.19) holds with $\chi = \varphi^{(\ell)} \circ \mathrm{Frob}_{A/T}$. By definition of $\varphi^{(\ell)}$, the top right-hand square in (4.17) commutes, which says that (4.20) holds with $\chi = \varphi^{(\ell)} \circ \mathrm{Frob}_{A/T}$. By uniqueness, $\mathrm{Frob}_{B/T} \circ \varphi = \varphi^{(\ell)} \circ \mathrm{Frob}_{A/T}$, i.e. (4.16) commutes. $\square$

Lemma 4.2.3. *Verschiebung is compatible with isogenies. More precisely, the following diagram commutes.*

$$\begin{tikzcd} A^{(\ell)} \arrow[r, "\mathrm{Ver}_{A/T}"] \arrow[d, "\varphi^{(\ell)}"'] & A \arrow[d, "\varphi"] \\ B^{(\ell)} \arrow[r, "\mathrm{Ver}_{B/T}"'] & B \end{tikzcd} \tag{4.21}$$

Proof. The proof amounts to placing the Frobenius and Verschiebung squares (4.16) and (4.21) together. Thus, consider the following rectangle.

$$\begin{tikzcd} A \arrow[r, "\mathrm{Frob}_{A/T}"] \arrow[d, "\varphi"'] \arrow[rr, bend left, "{[\ell]}"] & A^{(\ell)} \arrow[r, "\mathrm{Ver}_{A/T}"] \arrow[d, "\varphi^{(\ell)}"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\mathrm{Frob}_{B/T}"'] \arrow[rr, bend right, "{[\ell]}"'] & B^{(\ell)} \arrow[r, "\mathrm{Ver}_{B/T}"'] & B \end{tikzcd} \tag{4.22}$$

Since an isogeny is a homomorphism, it commutes with multiplication by ℓ, so $[\ell] \circ \varphi = \varphi \circ [\ell]$, which means that the outer (curved) rectangle in (4.22) commutes. The left-hand Frobenius square commutes by Lemma 4.2.2. Therefore,

$$\varphi \circ \mathrm{Ver}_{A/T} \circ \mathrm{Frob}_{A/T} = \mathrm{Ver}_{B/T} \circ \varphi^{(\ell)} \circ \mathrm{Frob}_{A/T}. \tag{4.23}$$

Since $\mathrm{Frob}_{A/T}$ is an isogeny, it is faithfully flat, so we may cancel it from both sides of (4.23) (cf. [MvdG12, Lemma 5.4]). The result is $\varphi \circ \mathrm{Ver}_{A/T} = \mathrm{Ver}_{B/T} \circ \varphi^{(\ell)}$ i.e. the right-hand side Verschiebung square also commutes. $\square$

Next we claim that the following diagram commutes.

$$\begin{tikzcd} e_B^*\Omega_{B/T} \arrow[r, "e_A^*(\varphi^*)"] \arrow[d, "e^*_{B^{(\ell)}}(\mathrm{Ver}^*_{B/T})"'] & e_A^*\Omega_{A/T} \arrow[d, "e^*_{A^{(\ell)}}(\mathrm{Ver}^*_{A/T})"] \\ e^*_{B^{(\ell)}}\Omega_{B^{(\ell)}/T} \arrow[r, "e^*_{A^{(\ell)}}((\varphi^{(\ell)})^*)"'] & e^*_{A^{(\ell)}}\Omega_{A^{(\ell)}/T} \end{tikzcd} \tag{4.24}$$

By Lemma 4.2.3, $\mathrm{Ver}_{B/T} \circ \varphi^{(\ell)} = \varphi \circ \mathrm{Ver}_{A/T}$. But also $(\mathrm{Ver}_{B/T} \circ \varphi^{(\ell)})^* = (\varphi^{(\ell)})^* \circ \mathrm{Ver}^*_{B/T}$ and $(\varphi \circ \mathrm{Ver}_{A/T})^* = \mathrm{Ver}^*_{A/T} \circ \varphi^*$. Hence,

$$e^*_{A^{(\ell)}}((\varphi^{(\ell)})^* \circ \mathrm{Ver}^*_{B/T}) = e^*_{A^{(\ell)}}(\mathrm{Ver}^*_{A/T} \circ \varphi^*). \tag{4.25}$$

The pull-back of a composite of maps of sheaves is the composite of the pull-backs, so

$$e^*_{A^{(\ell)}}((\varphi^{(\ell)})^* \circ \mathrm{Ver}^*_{B/T}) = e^*_{A^{(\ell)}}((\varphi^{(\ell)})^*) \circ (e^*_{B^{(\ell)}}(\mathrm{Ver}^*_{B/T})) \tag{4.26}$$

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and

$$e^*_{A^{(\ell)}}(\mathrm{Ver}^*_{A/T}\circ\varphi^*) = (e^*_{A^{(\ell)}}(\mathrm{Ver}^*_{A/T}))\circ(e^*_A(\varphi^*)). \tag{4.27}$$

Therefore,

$$e^*_{A^{(\ell)}}((\varphi^{(\ell)})^*)\circ(e^*_{B^{(\ell)}}(\mathrm{Ver}^*_{B/T})) = (e^*_{A^{(\ell)}}(\mathrm{Ver}_{A/T}))\circ(e^*_A(\varphi^*)), \tag{4.28}$$

which says exactly that (4.24) commutes. This proves the claim.

Since the top exterior power of a product is the product of the top exterior powers, applying top exterior powers throughout in (4.24) preserves commutativity, whence the following diagram also commutes.

$$\begin{tikzcd} \omega_B \arrow[r, "\overline{\varphi^*}"] \arrow[d, "h(B)"'] & \omega_A \arrow[d, "h(A)"] \\ \omega_B^{\otimes\ell} \arrow[r, "\overline{(\varphi^{(\ell)})^*}"'] & \omega_A^{\otimes\ell} \end{tikzcd} \tag{4.29}$$

The commutativity of (4.29) shows that Hasse invariant maps are compatible with isogenies. Since the Hasse invariant is directly defined in terms of the Hasse invariant map, compatibility of Hasse invariant maps implies compatibility of Hasse invariants. □

Next we note that the Hasse invariant is compatible with base change.

THEOREM 4.2.4. *Suppose $\xi: T_1 \longrightarrow T_2$ is a finite map of Spec $\mathbf{F}_\ell$-schemes and B/T_2 is an abelian T_2-scheme. Consider the T_1-abelian scheme A that is the base change of B along ξ, i.e. $A = T_1 \times_{T_2,(\xi,\pi_B)} B$. Then*

$$\xi^* H(B/T_2) = H(A/T_1). \tag{4.30}$$

Proof. The theorem is proved analogously to the compatibility of the Hasse invariant with isogenies (Theorem 4.2.1): it is well known that the relative Frobenius isogeny is compatible with base change, from which it follows that Verschiebung, and hence also the Hasse invariant, are compatible with base change. □

Finally we have the following important remarks.

Remark 4.2.5. Suppose that $\mathcal{C}$ is a prime-to-ℓ isogeny class of T-abelian schemes. In view of the compatibility given in Theorem 4.2.1, we shall henceforth refer simply to *the Hasse invariant* $H(\mathcal{C}) \in H^0(T, \omega^{\otimes(\ell-1)})$ and omit reference to representatives of $\mathcal{C}$.

Remark 4.2.6. Let λ be a prime of E above ℓ and let $\mathcal{S}_{\mathcal{K}^{(\ell)},\lambda}$ denote the special fiber of $S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$ at λ. Let $\mathcal{A}_\lambda$ be the universal prime-to-ℓ isogeny class of $\mathcal{U}^{(\ell)}$-enriched abelian schemes. Then we denote the Hasse invariant $H(\mathcal{A}_\lambda)$ by $H_{\mathcal{K}^{(\ell)},\lambda}$.

4.3 Non-vanishing

Keep the notation of Remark 4.2.6.

Suppose that A/T is as in § 4.1. We say that A/T is *ordinary* if, for every geometric point t of T with residue field $\mathbf{k}(t)$, the fiber A_t of A at t satisfies

$$|A_t[\ell^e](\mathbf{k}(t))| = \ell^{e(\dim A_t)} \quad \text{for all } e \in \mathbf{Z}^+, \tag{4.31}$$

where $A_t[\ell^e]$ denotes the subgroup scheme of ℓ^e torsion.

The property of being ordinary is preserved under isogenies. Therefore, the notion of an ordinary point in $\mathcal{S}_{\mathcal{K}^{(\ell)},\lambda}$ is meaningful. Let $\mathcal{S}^{\mathrm{ord}}_{\mathcal{K}^{(\ell)},\lambda}$ denote the ordinary locus in $\mathcal{S}_{\mathcal{K}^{(\ell)},\lambda}$.

THEOREM 4.3.1 (Wedhorn [Wed99, p. 584, (1.6.3)]). *Suppose, as in Theorem 1.2.1, that* $\mathbf{G}$ *is unramified at* ℓ. *Then the ordinary locus* $\mathcal{S}^{\mathrm{ord}}_{\mathcal{K}^{(\ell)},\lambda}$ *is open and dense in* $\mathcal{S}_{\mathcal{K}^{(\ell)},\lambda}$ *if and only if* $E_\lambda = \mathbf{Q}_\ell$ *(i.e.* λ *is split in* E*).*

The relationship between the Hasse invariant and the ordinary locus is given by the following lemma.

LEMMA 4.3.2. *Let* $D(\mathcal{K}^{(\ell)}, \lambda)$ *be the divisor cut out by the Hasse invariant* $H_{\mathcal{K}^{(\ell)},\lambda}$ *and let* $D^{\mathrm{red}}(\mathcal{K}^{(\ell)}, \lambda)$ *be the associated reduced scheme. Then* $D^{\mathrm{red}}(\mathcal{K}^{(\ell)}, \lambda)$ *is the complement of the ordinary locus* $\mathcal{S}^{\mathrm{ord}}_{\mathcal{K}^{(\ell)},\lambda}$ *in* $\mathcal{S}_{\mathcal{K}^{(\ell)},\lambda}$.

Proof. It suffices to show that if A is an abelian variety over an algebraically closed field $\mathbf{k}$ of characteristic ℓ, then $H(A) \neq 0$ if and only if A is ordinary.

We show that A is ordinary if and only if the Verschiebung isogeny $\mathrm{Ver}_{A/\mathbf{k}} : A^{(\ell)} \longrightarrow A$ is separable. Our argument is taken from [Sil86, ch. 5, § 3, p. 138, proof of 3.1, part (a)], which treats the case of elliptic curves, but the argument is valid in general. Let $\deg_s$ denote the separable degree of an isogeny. (An isogeny is a finite map, so it induces a finite extension of function fields and $\deg_s$ may be defined to be the separable degree of this field extension.) Since $\mathrm{Frob}_{A/\mathbf{k}} \circ \mathrm{Ver}_{A/\mathbf{k}} = [\ell]$, for every integer $e \geqslant 1$ one has $\mathrm{Frob}^e_{A/\mathbf{k}} \circ \mathrm{Ver}^e_{A/\mathbf{k}} = [\ell^e]$. The separable degree is additive with respect to composition of isogenies. Hence, $\deg_s(\mathrm{Frob}^e_{A/\mathbf{k}}) + \deg_s(\mathrm{Ver}^e_{A/\mathbf{k}}) = \deg_s([\ell^e])$. However, $\mathrm{Frob}_{A/\mathbf{k}}$, hence also $\mathrm{Frob}^e_{A/\mathbf{k}}$, is purely inseparable, so $\deg_s(\mathrm{Frob}^e_{A/\mathbf{k}}) = 0$ and we conclude that $\deg_s(\mathrm{Ver}^e_{A/\mathbf{k}}) = \deg_s([\ell^e])$.

Over an algebraically closed field, the separable degree of an isogeny equals the cardinality of its kernel. Therefore, $\deg_s \mathrm{Ver}^e_\ell = |A[\ell^e]|$. By our definition of ordinary, A is ordinary if and only if $|A[\ell^e]| = \ell^{e \dim A}$, hence if and only if $\deg_s \mathrm{Ver}^e_{A/\mathbf{k}} = \ell^{e \dim A} = \deg \mathrm{Ver}^e_{A/\mathbf{k}}$. This completes the argument, since an isogeny is separable if and only if its degree and separable degree coincide.

On the other hand, an isogeny is separable if and only if the induced map on top degree differential forms is non-zero. This proves that $H(A) \neq 0$ if and only if A is ordinary, since, by definition, the Hasse invariant map $h(A)$ is the pull-back by $e_{A^{(\ell)}}$ of the map on top degree differentials induced from $\mathrm{Ver}_{A/\mathbf{k}} : A^{(\ell)} \longrightarrow A$. □

Combining Theorem 4.3.1 with Lemma 4.3.2, one has the following corollary.

COROLLARY 4.3.3. *The Hasse invariant* $H_{\mathcal{K}^{(\ell)},\lambda}$ *is a non-zero* mod ℓ *automorphic form if and only if* λ *is split in* E.

4.4 Lifting to characteristic zero

The following lemma shows that some power of the Hasse invariant lifts to characteristic zero.

LEMMA 4.4.1. *There is a positive integer* a *and an automorphic form*

$$\tilde{H}^a_{\mathcal{K}^{(\ell)},\lambda} \in H^0(S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes a(\ell-1)}_{\mathcal{K}^{(\ell)}}) \tag{4.32}$$

whose image in $H^0(S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes a(\ell-1)}_{\mathcal{K}^{(\ell)}}) \otimes \mathcal{O}_{E,\ell}/\lambda$ *is* $H^a_{\mathcal{K}^{(\ell)},\lambda}$.

Proof. Since the corollary is well known when $\dim_{\mathcal{O}_{E,\ell}} S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}} = 1$, we may assume $\dim_{\mathcal{O}_{E,\ell}} S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}} > 1$.

Let $S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$ (respectively $\mathcal{S}^{\min}_{\mathcal{K}^{(\ell)},\lambda}$) denote the Baily–Borel minimal compactification of $S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$ (respectively its fiber at λ) constructed by Lan in [Lan08, § 7.2, see especially Theorem 7.2.4.1]. By Theorem7.2.4.1, no. 2 of *loc. cit.*the line bundle $\omega_{\mathcal{K}^{(\ell)}}$ (introduced in § 3.6)

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extends to an ample line bundle $\omega_{\mathcal{K}^{(\ell)},\min}$ on $S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$. Given this fact, the lemma is a consequence of a standard cohomological argument coupled with the Koecher principle, as we now recall.

Consider the short exact sequence of sheaves on $S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$ arising from the multiplication by λ map $[\lambda]$ on $S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$:

$$0 \longrightarrow \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \xrightarrow{[\lambda]} \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \longrightarrow \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \otimes \mathcal{O}_{E,\ell}/\lambda \longrightarrow 0. \tag{4.33}$$

It induces a long exact sequence of cohomology groups, which begins as follows:

$$\begin{aligned} 0 \longrightarrow H^0(S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min}) \longrightarrow H^0(S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min}) \\ \longrightarrow H^0(S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \otimes \mathcal{O}_{E,\ell}/\lambda) \longrightarrow H^1(S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min}) \longrightarrow \cdots . \end{aligned} \tag{4.34}$$

Since $\omega_{\mathcal{K}^{(\ell)},\min}$ is an ample line bundle and $S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$ is a projective scheme over $\mathcal{O}_{E,\ell}$, which is Noetherian, *Serre vanishing* (cf. [Har77, ch. 3, Theorem 5.2]) implies that there exists a positive integer k_0 such that, for all integers $k \geqslant k_0$ one has

$$H^1(S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min}) = 0. \tag{4.35}$$

Therefore,

$$H^0(S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min}) \otimes \mathcal{O}_{E,\ell}/\lambda = H^0(S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \otimes \mathcal{O}_{E,\ell}/\lambda). \tag{4.36}$$

By [Lan08, Proposition 7.2.4.3], the special fiber $\mathcal{S}^{\min}_{\mathcal{K}^{(\ell)},\lambda}$ is normal. Hence, the *Koecher principle* implies that

$$H^0(\mathcal{S}^{\min}_{\mathcal{K}^{(\ell)},\lambda}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \otimes \mathcal{O}_{E,\ell}/\lambda) = H^0(\mathcal{S}_{\mathcal{K}^{(\ell)},\lambda}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)}} \otimes \mathcal{O}_{E,\ell}/\lambda). \tag{4.37}$$

Choose $k \geqslant k_0$ divisible by $\ell - 1$, say $k = a(\ell - 1)$ with $a \in \mathbf{Z}_{>0}$. Now $H^a_{\mathcal{K}^{(\ell)},\lambda}$ is an element of $H^0(\mathcal{S}_{\mathcal{K}^{(\ell)},\lambda}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)}} \otimes \mathcal{O}_{E,\ell}/\lambda)$, hence also of $H^0(\mathcal{S}^{\min}_{\mathcal{K}^{(\ell)},\lambda}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \otimes \mathcal{O}_{E,\ell}/\lambda)$. Since the sheaf $\omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \otimes \mathcal{O}_{E,\ell}/\lambda$ on $S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$ is supported on the fiber $\mathcal{S}^{\min}_{\mathcal{K}^{(\ell)},\lambda}$, one has $H^0(\mathcal{S}^{\min}_{\mathcal{K}^{(\ell)},\lambda}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \otimes \mathcal{O}_{E,\ell}/\lambda) = H^0(S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \otimes \mathcal{O}_{E,\ell}/\lambda)$. Thus, $H^a_{\mathcal{K}^{(\ell)},\lambda} \in H^0(S^{\min}_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)},\min} \otimes \mathcal{O}_{E,\ell}/\lambda)$ so it is the image of some $\tilde{H}^a_{\mathcal{K}^{(\ell)},\lambda}$ in $H^0(S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega^{\otimes k}_{\mathcal{K}^{(\ell)}})$, as desired. □

5. Archimedean theory II: explicit formulas

5.1 Classical unitary groups over R

Let p_0 and q_0 be non-negative integers, not both zero, with $n = p_0 + q_0$. Define a Hermitian form

$$\mathcal{B}_{p_0,q_0} : \mathbf{C}^n \times \mathbf{C}^n \longrightarrow \mathbf{C} \tag{5.1}$$

of signature (p_0, q_0) by

$$\mathcal{B}_{p_0,q_0}(z, w) = z_1\bar{w}_1 + \cdots + z_{p_0}\bar{w}_{p_0} - z_{p_0+1}\bar{w}_{p_0+1} - \cdots - z_{p_0+q_0}\bar{w}_{p_0+q_0} \tag{5.2}$$

for all $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbf{C}^n$. Let $\mathrm{GU}(p_0, q_0)$ be the unitary similitude group of the form $\mathcal{B}_{p_0,q_0}$:

$$\mathrm{GU}(p_0, q_0) = \left\{ \gamma \in \mathrm{GL}(n, \mathbf{C}) \;\middle|\; \begin{array}{l} \text{there exists } \mu_{p_0,q_0}(\gamma) \in \mathbf{R}^\times \text{ such that} \\ \mathcal{B}(\gamma z, \gamma w) = \mu_{p_0,q_0}(\gamma)\mathcal{B}(z, w) \text{ for all } z, w \in \mathbf{C}^n \end{array} \right\}. \tag{5.3}$$

The function $\mu_{p_0,q_0} : \mathrm{GU}(p_0, q_0) \longrightarrow \mathbf{R}^\times$ is the multiplier character. Put $U(p_0, q_0) = \ker \mu_{p_0,q_0}$ and $U(p_0) = U(p_0, 0)$. Define

$$G(U(p_0) \times U(q_0)) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GU}(p_0, q_0) \;\middle|\; B = C = 0 \right\}. \tag{5.4}$$

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Then $G(U(p_0) \times U(q_0))$ is a maximal Zariski-connected, compact modulo center subgroup of $\mathrm{GU}(p_0, q_0)$.

5.2 The real group associated to a unitary Kottwitz datum

Let $\mathcal{U}$ be a unitary Kottwitz datum (§ 3.1), let $\mathbf{G} = \mathbf{G}(\mathcal{U})$ be the associated unitary $\mathbf{Q}$-algebraic group and put $G = \mathbf{G}(\mathbf{R})$. Then

$$G \cong \left\{ \gamma = (\gamma_1, \dots \gamma_d) \in \prod_{i=1}^{d} \mathrm{GU}(p_i, q_i) \,\middle|\, \mu_{p_1,q_1}(\gamma_1) = \cdots = \mu_{p_d,q_d}(\gamma_d) \right\}, \tag{5.5}$$

where the non-negative pairs of integers (p_i, q_i) are as in § 3.7. We henceforth identify G with the right-hand side of (5.5).

Given $\gamma \in G$, define $\mu_G(\gamma) = \mu_{p_1,q_1}(\gamma_1)$ and set $G^{\mu=1} = \ker \mu_G$. Then

$$G^{\mu=1} = \prod_{i=1}^{d} U(p_i, q_i). \tag{5.6}$$

For $z \in \mathbf{C}^\times$, let z_{p_0,q_0} denote the diagonal $n \times n$ matrix with the first p_0 diagonal entries z and the last q_0 diagonal entries $\bar{z}$. By [Kot92, § 4] we may assume that $h : \mathbf{S}(\mathbf{R}) \longrightarrow G$ is given by

$$h(z) = (z_{p_1,q_1}, \dots, z_{p_d,q_d}). \tag{5.7}$$

Then

$$\mathcal{K}_\infty = \left\{ \gamma = (\gamma_1, \dots, \gamma_d) \in G \,\middle|\, \gamma_i = r \begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix}, \begin{array}{l} \text{for some } A_i \in U(p_i), \\ D_i \in U(q_i), r \in \mathbf{R}^\times \end{array} \right\}. \tag{5.8}$$

For $\mathcal{H}_\infty$ we may take the subgroup of $\mathcal{K}_\infty$ consisting of γ such that γ_i is diagonal for all i, $1 \leqslant i \leqslant d$.

5.3 Roots and weights

Consider the embedding $G \hookrightarrow \mathrm{GL}(2n, \mathbf{R})^d$ induced from the following standard embedding.

$$\begin{array}{rcl} \mathrm{GL}(n, \mathbf{C}) & \hookrightarrow & \mathrm{GL}(2n, \mathbf{R}) \\ X + iY & \mapsto & \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \end{array}. \tag{5.9}$$

Write $\mathrm{diag}(a_1, \dots, a_n)$ for the diagonal $n \times n$ matrix with $a_1, \dots, a_n$ along the diagonal. Then we can identify $\mathfrak{h}_{\infty,\mathbf{C}}$ with those $b = (b_1, \dots, b_d) \in \mathfrak{gl}(2n, \mathbf{R})^d$ such that, in blocks,

$$b_i = \begin{pmatrix} \mathrm{diag}(t, \dots, t) & -\mathrm{diag}(a_{i1}, \dots, a_{in}) \\ \mathrm{diag}(a_{i1}, \dots, a_{in}) & \mathrm{diag}(t, \dots, t) \end{pmatrix}, \tag{5.10}$$

for some $a_{ij}, t \in \mathbf{C}$. If R is a commutative ring, let $M_{d\times n}(R)$ denote the R-module of d times n matrices with entries in R. Given $(\lambda_{ij}) \in M_{d\times n}(\mathbf{C})$ and $c \in \mathbf{C}$, define a functional

$$((\lambda_{ij}), c) : \mathfrak{h}_{\infty,\mathbf{C}} \longrightarrow \mathbf{C} \tag{5.11}$$

by

$$b \mapsto ct + \sum_{i=1,j=1}^{i=d,j=n} \mathbf{i}\lambda_{ij} a_{ij}. \tag{5.12}$$

This allows us to identify $\mathfrak{h}^*_{\infty,\mathbf{C}}$ with $M_{d\times n}(\mathbf{C})\oplus\mathbf{C}$. Under this identification, one has

$$\boldsymbol{\Lambda}=\left\{((\lambda_{ij}),c)\in M_{d\times n}(\mathbf{Z})\oplus\mathbf{Z}\;\middle|\;\sum_{i=1,j=1}^{i=d,j=n}\lambda_{ij}\equiv c\pmod 2\right\},\tag{5.13}$$

where $\boldsymbol{\Lambda}$ is the weight space, as defined in § 2.3.

Let E_{ij} denote the $d\times n$ matrix with (i,j) entry equal to 1 and all other entries 0. The set of roots Δ is

$$\Delta=\{(E_{ij}-E_{ik},0)\mid 1\leqslant i\leqslant d, 1\leqslant j,k\leqslant n, j\neq k\},\tag{5.14}$$

and the subset of compact roots is

$$\Delta_c=\{(E_{ij}-E_{ik},0)\in\Delta\mid 1\leqslant j,k\leqslant p_i \text{ or } p_{i+1}\leqslant j,k\leqslant n\}.\tag{5.15}$$

The choice of positive non-compact roots Δ_n^+ in § 2.2 translates to

$$\Delta_n^+=\{(E_{ij}-E_{ik},0)\in\Delta\mid 1\leqslant j\leqslant p_i, p_{i+1}\leqslant k\leqslant n\}.\tag{5.16}$$

According to § 2.2 we are free to choose any system of positive compact roots; we choose the one for which the set of all positive roots is

$$\Delta^+=\{(E_{ij}-E_{ik},0)\in\Delta\mid j<k\}.\tag{5.17}$$

Then

$$\mathrm{Dom}=\{((\lambda_{ij}),c)\in M_{d\times n}(\mathbf{R})\oplus\mathbf{C}\mid \lambda_{i1}\geqslant\cdots\geqslant\lambda_{in} \text{ for all } i, 1\leqslant i\leqslant d\}\tag{5.18}$$

and $\rho=(\rho_{ij},0)$ where ρ_{ij} is the matrix with every row equal to

$$\left(\frac{n-1}{2},\frac{n-3}{2},\dots,\frac{-(n-3)}{2},\frac{-(n-1)}{2}\right).\tag{5.19}$$

In particular, ρ lies in $\boldsymbol{\Lambda}$ if and only if n is odd.

5.4 Holomorphic limits of discrete series II

Combining §§ 2.3 and 5.3 we have the following explicit description of the Harish-Chandra parameters of $\mathbf{X}$-holomorphic limits of discrete series: suppose $\lambda\in\boldsymbol{\Lambda}+\rho\cap\mathrm{Dom}$; the latter set is explicitly determined by means of (5.18) and (5.19). Write $\lambda=((\lambda_{ij}),c)$. Then $\pi(\lambda,\mathrm{Dom})$ is an $\mathbf{X}$-holomorphic limit of discrete series if and only if:

HLDS1. there exists an i, $1\leqslant i\leqslant d$ such that $\lambda_{ip_i}=\lambda_{ip_i+1}$;

HLDS2. if $\lambda_{ij}=\lambda_{ij+1}$ for some i,j, then $j=p_i$.

5.5 Vector bundle dictionary

Let $\mathcal{U}$ be a unitary Kottwitz datum with associated Shimura variety $\mathrm{Sh}(\mathbf{G},\mathbf{X})$ as defined in § 3.1. Two constructions of $\mathbf{G}(\mathbf{A}_f)$-equivariant vector bundles on $\mathrm{Sh}(\mathbf{G},\mathbf{X})(\mathbf{C})$ have been given: first in § 2.5 and second in § 3.8. It is now explained how the two are linked over $\mathbf{C}$.

Theorem 5.5.1. *Let $\Omega_{\tau_i^+}\otimes\mathbf{C}$ (respectively $\Omega_{\tau_i^-}\otimes\mathbf{C}$) be the vector bundle on $\mathrm{Sh}(\mathbf{G},\mathbf{X})(\mathbf{C})$ defined in § 3.7. Under the identification (5.13), one has*

$$\Omega_{\tau_i^+}\otimes\mathbf{C}\cong\mathcal{V}_{(-E_{ip_i},-1)}\quad\text{and}\quad\Omega_{\tau_i^-}\otimes\mathbf{C}\cong\mathcal{V}_{(E_{ip_i+1},-1)}.\tag{5.20}$$

Before embarking on the proof of Theorem 5.5.1, we note the following three immediate corollaries.

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Corollary 5.5.2. *Put*

$$\epsilon^+_{\tau_i} = \Big(-\sum_{j=1}^{p_i} E_{ij}, -p_i\Big) \quad \text{and} \quad \epsilon^-_{\tau_i} = \Big(\sum_{j=p_i+1}^{n} E_{ij}, -q_i\Big). \tag{5.21}$$

Then

$$\omega_{\tau_i^+} \otimes \mathbf{C} \cong \mathcal{V}_{\epsilon_{\tau_i^+}} \quad \text{and} \quad \omega_{\tau_i^-} \otimes \mathbf{C} \cong \mathcal{V}_{\epsilon_{\tau_i^-}}. \tag{5.22}$$

Proof. Apply determinants to (5.20). □

Corollary 5.5.3. *Define* $\epsilon = r\sum_{i=1}^{d}(\epsilon_{\tau_i^+} + \epsilon_{\tau_i^-})$. *Then*

$$\omega \otimes \mathbf{C} \cong \mathcal{V}_\epsilon. \tag{5.23}$$

Proof. Combine (3.14) with Corollary 5.5.2. □

Corollary 5.5.4. *Suppose that* $H^{\mathrm{lift}}_{\mathcal{K}^{(\ell)},\lambda}$ *is a lift of* $H^{a}_{\mathcal{K}^{(\ell)},\lambda}$. *Then the weight of* $H^{\mathrm{lift}}_{\mathcal{K}^{(\ell)},\lambda}$ *is* $a(\ell-1)\epsilon$.

Proof. This follows from Corollary 5.5.3 and the fact that $H^{\mathrm{lift}}_{\mathcal{K}^{(\ell)},\lambda} \in H^0(S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}, \omega_{\mathcal{K}^{(\ell)}}^{\otimes(\ell-1)})$. □

Proof of Theorem 5.5.1. Let $\mathcal{V}_{\mathbf{C},\mathbf{X}}$ be the holomorphically trivial, G-equivariant vector bundle on $\mathbf{X}$ whose fiber at every point is a copy of $V(\mathbf{C})$, where V pertains to the unitary Kottwitz datum $\mathcal{U}$. Let $\mathcal{V}_{\mathbf{C},\mathrm{Sh}(\mathbf{G},\mathbf{X})}$ denote the corresponding $\mathbf{G}(\mathbf{A}_f)$-equivariant vector bundle on $\mathrm{Sh}(\mathbf{G},\mathbf{X})(\mathbf{C})$. On the other hand, let $H_1(\mathcal{A}/\mathrm{Sh}(\mathbf{G},\mathbf{X}))$ be the $\mathbf{G}(\mathbf{A}_f)$-equivariant vector bundle on $\mathrm{Sh}(\mathbf{G},\mathbf{X})(\mathbf{C})$ whose fiber at a point x is $H_1(A_x,\mathbf{C})$, where A_x is any representative of the isogeny class parameterized by x.

Since V is a left B-module, the vector bundle $\mathcal{V}_{\mathrm{Sh}(\mathbf{G},\mathbf{X})}$ admits an action of $B(\mathbf{C})$. By (§ 3.2, property Ab3) $B(\mathbf{C})$ also acts on $H_1(\mathcal{A}/\mathrm{Sh}(\mathbf{G},\mathbf{X}))$. It follows from [Kot92, § 8] that $\mathcal{V}_{\mathbf{C},\mathrm{Sh}(\mathbf{G},\mathbf{X})} \cong H_1(\mathcal{A}/\mathrm{Sh}(\mathbf{G},\mathbf{X}))$ as $\mathbf{G}(\mathbf{A}_f)\times B(\mathbf{C})$-equivariant vector bundles.

Furthermore, each of $\mathcal{V}_{\mathbf{C},\mathrm{Sh}(\mathbf{G},\mathbf{X})}$ and $H_1(\mathcal{A}/\mathrm{Sh}(\mathbf{G},\mathbf{X}))$ has a natural holomorphic quotient bundle, namely the $H^{-1,0}$ of the corresponding weight -1 variations of Hodge structure: let $\mathcal{V}_{\mathbf{R},\mathbf{X}}$ be the G-equivariant vector bundle on $\mathbf{X}$ whose fiber at $h \in \mathbf{X}$ is $V(\mathbf{R})$ with the complex structured induced from h. Let $\mathcal{V}_{\mathbf{R},\mathrm{Sh}(\mathbf{G},\mathbf{X})}$ denote the descended vector bundle on $\mathrm{Sh}(\mathbf{G},\mathbf{X})(\mathbf{C})$. On the other hand, one has the vector bundle $\mathrm{Lie}\,(\mathcal{A}/\mathrm{Sh}(\mathbf{G},\mathbf{X}))$ whose fiber at x is $H_1(A_x,\mathbf{R}) \cong \mathrm{Lie}\,A_x$. Again it follows from § 8 of *loc. cit.*that $\mathcal{V}_{\mathbf{R},\mathrm{Sh}(\mathbf{G},\mathbf{X})} \cong \mathrm{Lie}\,(\mathcal{A}/\mathrm{Sh}(\mathbf{G},\mathbf{X}))$ as $\mathbf{G}(\mathbf{A}_f)\times B(\mathbf{C})$-equivariant vector bundles. Taking duals gives

$$\mathcal{V}^\vee_{\mathbf{R},\mathrm{Sh}(\mathbf{G},\mathbf{X})} \cong \Omega \otimes \mathbf{C}. \tag{5.24}$$

Since $B(\mathbf{C})$ acts on both sides of (5.24), both sides decompose as in § 3.7 and we obtain

$$\Omega_{\tau_i^+} \otimes \mathbf{C} \cong (\mathcal{V}^\vee_{\mathbf{R},\mathrm{Sh}(\mathbf{G},\mathbf{X})})_{\tau_i^+} \quad \text{and} \quad \Omega_{\tau_i^-} \otimes \mathbf{C} \cong (\mathcal{V}^\vee_{\mathbf{R},\mathrm{Sh}(\mathbf{G},\mathbf{X})})_{\tau_i^-}. \tag{5.25}$$

In view of (5.25), the following lemma is sufficient for completing the proof of Theorem 5.5.1.

Lemma 5.5.5. *Let* $(\mathbf{e}_1,\ldots,\mathbf{e}_n)$ *be the standard (ordered) basis of the* $\mathbf{C}$*-vector space* $\mathbf{C}^n$*; let* $(\mathbf{e}_1,\ldots,\mathbf{e}_n,\mathbf{ie}_1,\ldots,\mathbf{ie}_n)$ *be an (ordered) basis of* $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{C}^n$. *Let* Std_{p_0,q_0} *be the standard representation of* $\mathrm{GU}(p_0,q_0)$ *on* $\mathbf{C}^n$. *Let* σ *be the complexification of the* $2n$*-dimensional representation of* $\mathrm{GU}(p_0,q_0)$ *on* $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{C}^n$ *gotten by composing* Std_{p_0,q_0} *with the* *embedding (5.9). Let* σ^{cpt} *(respectively* $\mathrm{Std}^{\mathrm{cpt}}_{p_0,q_0}$, $\overline{\mathrm{Std}}^{\mathrm{cpt}}_{p_0,q_0}$*) denote the restriction of* σ *(respectively* Std_{p_0,q_0}, $\overline{\mathrm{Std}}_{p_0,q_0}$*) to* $G(U(p_0)\times U(q_0))$*. Then:*

(1) *the representation* σ *decomposes as* $\sigma = \mathrm{Std}_{p_0,q_0} \oplus \overline{\mathrm{Std}}_{p_0,q_0}$*, where the bar indicates the conjugate representation;*

(2) *one has* $\mathrm{Std}^{\mathrm{cpt}}_{p_0,q_0} = V_{(\mathbf{e}_1,1)} \oplus V_{(\mathbf{e}_{p+1},1)}$ *and* $\overline{\mathrm{Std}}^{\mathrm{cpt}}_{p_0,q_0} = V_{(-\mathbf{e}_p,1)} \oplus V_{(-\mathbf{e}_n,1)}$*, where the notation* V_η *is that of § 2.3;*

(3) *consider* $h : z \mapsto z_{p_0,q_0}$ *(see § 5.2); then*

$$(\sigma\circ h)(z)v = \begin{cases} zv & \text{if } v\in \mathrm{span}_{\mathbf{R}}(\{\mathbf{e}_i+\mathbf{i}\mathbf{e}_i \mid p_0+1\leqslant i\leqslant n\}\cup\{\mathbf{e}_i-\mathbf{i}\mathbf{e}_i \mid 1\leqslant i\leqslant p_0\})\\ \bar{z}v & \text{if } v\in \mathrm{span}_{\mathbf{R}}(\{\mathbf{e}_i+\mathbf{i}\mathbf{e}_i \mid 1\leqslant i\leqslant p_0\}\cup\{\mathbf{e}_i-\mathbf{i}\mathbf{e}_i \mid p_0+1\leqslant i\leqslant n\}). \end{cases}$$

Proof. Suppose $Z\in \mathrm{GU}(p_0,q_0)$. Write $Z = X+\mathbf{i}Y$. Let

$$U = \left\{ v\in \mathbf{C}^{2n} \;\middle|\; v = \begin{pmatrix} u\\ \mathbf{i}u\end{pmatrix} \text{ for some } u\in\mathbf{C}^n\right\}. \tag{5.26}$$

The computations

$$\begin{pmatrix} X & -Y\\ Y & X\end{pmatrix}\begin{pmatrix} u\\ \mathbf{i}u\end{pmatrix} = (X-\mathbf{i}Y)\begin{pmatrix} u\\ \mathbf{i}u\end{pmatrix} \tag{5.27}$$

and

$$\begin{pmatrix} X & -Y\\ Y & X\end{pmatrix}\begin{pmatrix} u\\ -\mathbf{i}u\end{pmatrix} = (X+\mathbf{i}Y)\begin{pmatrix} u\\ -\mathbf{i}u\end{pmatrix} \tag{5.28}$$

show that $\mathbf{C}^{2n} = U\oplus\bar{U}$ as representations of $\mathrm{GU}(p_0,q_0)$ and that $U\cong\overline{\mathrm{Std}}_{p_0,q_0}$ (respectively $\bar{U}\cong \mathrm{Std}_{p_0,q_0}$). This proves part (1).

Suppose now that $Z\in G(U(p_0)\times U(q_0))$. Let

$$W_{p_0} = \{u=(u_1,\ldots,u_n)\in\mathbf{C}^n \mid u_{p_0+1}=\cdots=u_n=0\} \tag{5.29}$$

and

$$W_{q_0} = \{u=(u_1,\ldots,u_n)\in\mathbf{C}^n \mid u_1=\cdots=u_{p_0}=0\}. \tag{5.30}$$

Writing $Z = \left(\begin{smallmatrix} A & 0\\ 0 & D\end{smallmatrix}\right)$, one sees that $G(U(p_0)\times U(q_0))$ preserves the decomposition $\mathbf{C}^n = W_{p_0}\oplus W_{q_0}$. The set of weights of $G(U(p_0)\times U(q_0))$ acting on W_{p_0} (respectively W_{q_0}) is $\{(\mathbf{e}_i,1)\mid 1\leqslant i\leqslant p_0\}$ (respectively $\{(\mathbf{e}_i,1)\mid p_0+1\leqslant i\leqslant n\}$). Since each of these two sets contains a unique dominant weight, the representations W_{p_0} and W_{q_0} are irreducible, each with highest weight the unique dominant weight. This proves part (2).

Finally part (3) follows by direct computation. □

Having proved Lemma 5.5.5, the proof of Theorem 5.5.1 is complete. □

6. Congruences and Galois representations

6.1 Hecke algebras

Suppose that π is the cuspidal automorphic representation of Theorem 1.2.1. Let $\mathcal{P}^{(\ell)}$ be as in § 1.2 and let $p\in\mathcal{P}^{(\ell)}$. The local (spherical) Hecke algebra $\mathcal{H}_p(\mathbf{G},\mathbf{Z}_\ell)$ at p of $\mathbf{G}$, with $\mathbf{Z}_\ell$ coefficients, is defined as follows.

(i) The underlying vector space is

$$\mathcal{H}_p(\mathbf{G},\mathbf{Z}_\ell) = \left\{ f : \mathbf{G}(\mathbf{Z}_p)\backslash\mathbf{G}(\mathbf{Q}_p)/\mathbf{G}(\mathbf{Z}_p)\longrightarrow\mathbf{Z}_\ell \;\middle|\; \begin{matrix} f \text{ is locally constant}\\ \text{with compact support}\end{matrix}\right\}. \tag{6.1}$$

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(ii) The algebra structure is given by convolution with respect to the unique Haar measure which assigns $\mathbf{G}(\mathbf{Z}_p)$ volume one. As g ranges over $\mathbf{G}(\mathbf{Q}_p)$, the characteristic functions of

$$\mathbf{G}(\mathbf{Z}_p)g\mathbf{G}(\mathbf{Z}_p) \in \mathbf{G}(\mathbf{Z}_p)\backslash\mathbf{G}(\mathbf{Q}_p)/\mathbf{G}(\mathbf{Z}_p) \tag{6.2}$$

span $\mathcal{H}_p(\mathbf{G}, \mathbf{Z}_\ell)$.

Let $\mathbf{A}_{\mathcal{P}^{(\ell)}}$ be the adeles that are trivial away from $\mathcal{P}^{(\ell)}$. Put

$$\mathcal{K}_{\mathcal{P}^{(\ell)}} = \prod_{p \in \mathcal{P}^{(\ell)}} \mathbf{G}(\mathbf{Z}_p) \tag{6.3}$$

Then $\mathcal{K}_{\mathcal{P}^{(\ell)}}$ is an open compact subgroup of $\mathbf{G}(\mathbf{A}_{\mathcal{P}^{(\ell)}})$. Analogous to (6.1), we have a global Hecke algebra $\mathcal{H}_{\mathcal{P}^{(\ell)}}(\mathbf{G}, \mathbf{Z}_\ell)$ at $\mathcal{P}^{(\ell)}$, with values in $\mathbf{Z}_\ell$ defined in the same way, using the double coset space $\mathcal{K}_{\mathcal{P}^{(\ell)}}\backslash\mathbf{G}(\mathbf{A}_{\mathcal{P}^{(\ell)}})/\mathcal{K}_{\mathcal{P}^{(\ell)}}$ instead of $\mathbf{G}(\mathbf{Z}_p)\backslash\mathbf{G}(\mathbf{Q}_p)/\mathbf{G}(\mathbf{Z}_p)$ and the Haar measure given as the product of the Haar measures at each $p \in \mathcal{P}^{(\ell)}$. When there is no possibility of confusion, we shall often write simply $\mathcal{H}$ in place of $\mathcal{H}_{\mathcal{P}^{(\ell)}}(\mathbf{G}, \mathbf{Z}_\ell)$.

Given $g \in \mathbf{G}(\mathbf{Q}_p)$, define the Hecke operator $\mathcal{T}_{\mathcal{K},g}$ associated to the double coset $\mathbf{G}(\mathbf{Z}_p)g\mathbf{G}(\mathbf{Z}_p) \in \mathbf{G}(\mathbf{Z}_p)\backslash\mathbf{G}(\mathbf{Q}_p)/\mathbf{G}(\mathbf{Z}_p)$ by the following diagram (notation as in (3.18)).

$$\begin{tikzcd} & M_\eta(g\mathcal{K}g^{-1} \cap \mathcal{K}, R) \arrow[d, "\mathrm{tr}\ g"] \\ M_\eta(\mathcal{K}, R) \arrow[ur, "\mathrm{Id}^*"] \arrow[r, "\mathcal{T}_{\mathcal{K},g}"'] & M_\eta(\mathcal{K}, R) \end{tikzcd} \tag{6.4}$$

Given a Δ_c^+ dominant weight $\eta \in \mathbf{\Lambda}$, the Hecke algebra of weight η, denoted $\mathcal{H}_\eta$, is the $\mathbf{Z}_\ell$-span of $\mathcal{T}_{\mathcal{K},g}$, over all $g \in \mathbf{G}(\mathbf{Q}_p)$, in $\mathrm{End}_R(M_\eta(\mathcal{K}, R))$. The algebra $\mathcal{H}_\eta$ is a free $\mathbf{Z}_\ell$-module of finite rank. The assignment $\mathbf{G}(\mathbf{Z}_p)g\mathbf{G}(\mathbf{Z}_p) \longmapsto \mathcal{T}_{\mathcal{K}^{(\ell)},g}$ extends to a surjective algebra homomorphism

$$\mathcal{H} \xrightarrow{\mathrm{wt}_\eta} \mathcal{H}_\eta. \tag{6.5}$$

As observed in [Tay91, p. 304], a consequence of [Har88, Proposition 2.7.2 and Formula 3.0.1] is the following theorem.

Theorem 6.1.1. *The module $\bar{H}^0(\mathrm{Sh}(\mathbf{G}, \mathbf{X})(\mathbf{C}), \mathbf{S}_\eta^{\mathrm{fund}}(\Omega) \otimes \mathbf{C})$ of the complexified Hecke algebra $\mathcal{H} \otimes_\iota \mathbf{C}$ is semisimple.*

6.2 Congruences I: automorphic forms

Continue to suppose that $\mathcal{K} = \mathcal{K}_{(\ell)}\mathcal{K}^{(\ell)}$ with $\mathcal{K}_{(\ell)} \subset \mathbf{G}(\mathbf{Z}_\ell)$ and corresponding projection $\mathrm{Id} : S_{\mathcal{K},\mathcal{O}_{E,\ell}} \longrightarrow S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$. Our construction of congruences is the following theorem.

Theorem 6.2.1. *Suppose that $H^a_{\mathcal{K}^{(\ell)},\lambda}$ ($a \in \mathbf{Z}_{\geqslant 1}$) is a power of the Hasse invariant $H_{\mathcal{K}^{(\ell)},\lambda}$ that lifts to characteristic zero, let $H^{\mathrm{lift}}_{\mathcal{K}^{(\ell)},\lambda}$ denote a lift and put $H^{\mathrm{lift}}_{\mathcal{K},\lambda} = \mathrm{Id}^* H^{\mathrm{lift}}_{\mathcal{K}^{(\ell)},\lambda}$. Let $\eta \in \mathbf{\Lambda}$ be a Δ_c^+-dominant weight and suppose $f \in M_\eta(\mathcal{K}, R)$ is non-zero modulo λ. Then, for all $j \in \mathbf{Z}_{\geqslant 1}$, the product*

$$(H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes f \in M_{\eta + a\epsilon(\ell-1)\ell^{j-1}}(\mathcal{K}, R) \tag{6.6}$$

is non-zero modulo λ and satisfies

$$T((H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes f) \equiv (H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes T(f) \pmod{\lambda^{j+1}}, \tag{6.7}$$

for all $T \in \mathcal{H}$.

Remark 6.2.2. It is important to note that the product $(H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes f$ is *not* necessarily an eigenform, even if f is.

The key step in proving Theorem 6.2.1 is the following lemma, which recasts the compatibility of the Hasse invariant with base change (Theorem 4.2.4) in terms of degeneracy maps among Shimura varieties of different level.

Lemma 6.2.3. *Suppose that $\mathcal{K}_1^{(\ell)}, \mathcal{K}_2^{(\ell)}$ are open compact subgroups of $\mathbf{G}(\mathbf{A}_f^{\ell})$ and $g_1, g_2 \in \mathbf{G}(\mathbf{A}_f^{\ell})$ satisfy $\mathcal{K}_1^{(\ell)} \subset (g_1 \mathcal{K}_2^{(\ell)} g_1^{-1} \cap g_2 \mathcal{K}_2^{(\ell)} g_2^{-1})$. Consider the following two degeneracy maps.*

$$\begin{tikzcd} & S_{\mathcal{K}_1^{(\ell)}} \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ S_{\mathcal{K}_2^{(\ell)}} & & S_{\mathcal{K}_2^{(\ell)}} \end{tikzcd} \tag{6.8}$$

One has

$$g_1^* H_{\mathcal{K}_2^{(\ell)},\lambda} = g_2^* H_{\mathcal{K}_2^{(\ell)},\lambda}. \tag{6.9}$$

Proof. By Theorem 4.2.4, both pull-backs $g_1^* H_{\mathcal{K}_2^{(\ell)},\lambda}$ and $g_2^* H_{\mathcal{K}_2^{(\ell)},\lambda}$ are equal to $H_{\mathcal{K}_1^{(\ell)},\lambda}$; hence, they are equal to each other. $\square$

Proof of Theorem 6.2.1. First we show the product in (6.6) is non-zero modulo λ. Given our assumption that λ is split in E, Corollary 4.3.3 implies that $(H^{\mathrm{lift}}_{\mathcal{K}^{(\ell)},\lambda})$ is non-zero modulo λ. Since the map $\mathrm{Id} : S_{\mathcal{K},\mathcal{O}_{E,\ell}} \longrightarrow S_{\mathcal{K}^{(\ell)},\mathcal{O}_{E,\ell}}$ is separable, the pull-back $(H^{\mathrm{lift}}_{\mathcal{K},\lambda})$ is also non-zero modulo λ. It is assumed that f is also non-zero modulo λ. Since the product of two sections that are each non-zero modulo λ is non-zero modulo λ, we conclude that

$$(H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes f \neq 0 \in M_{\eta + a\epsilon(\ell-1)\ell^{j-1}}(\mathcal{K}, R \otimes_{\mathcal{O}_{E,\ell}} \mathcal{O}_{E,\ell}/\lambda). \tag{6.10}$$

Now consider (6.7). It suffices to prove it for $\mathcal{T} = \mathcal{T}_{\mathcal{K},g}$. By definition,

$$\mathcal{T}_{\mathcal{K},g}((H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes f) = ((\mathrm{tr}\, g) \circ \mathrm{Id}^*)((H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes f). \tag{6.11}$$

Since the pull-back of a tensor product is the tensor product of the pull-backs,

$$((\mathrm{tr}\, g) \circ \mathrm{Id}^*)((H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes f) = (\mathrm{tr}\, g)((\mathrm{Id}^*((H^{\mathrm{lift}}_{\mathcal{K},\lambda})))^{\ell^j} \otimes \mathrm{Id}^*(f)). \tag{6.12}$$

Lemma 6.2.3 shows that

$$\mathrm{Id}^*(H^{\mathrm{lift}}_{\mathcal{K}^{(\ell)},\lambda}) \equiv g^*(H^{\mathrm{lift}}_{\mathcal{K}^{(\ell)},\lambda}) \pmod{\lambda}; \tag{6.13}$$

hence, also

$$\mathrm{Id}^*(H^{\mathrm{lift}}_{\mathcal{K},\lambda}) \equiv g^*(H^{\mathrm{lift}}_{\mathcal{K},\lambda}) \pmod{\lambda}. \tag{6.14}$$

Therefore,

$$(\mathrm{Id}^*(H^{\mathrm{lift}}_{\mathcal{K},\lambda}))^{\ell^j} \equiv (g^*(H^{\mathrm{lift}}_{\mathcal{K},\lambda}))^{\ell^j} \pmod{\lambda^{j+1}}. \tag{6.15}$$

Substituting (6.15) into (6.12) and using $(g^*(H^{\mathrm{lift}}_{\mathcal{K},\lambda}))^{\ell^j} = g^*((H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j})$ yields

$$((\mathrm{tr}\, g) \circ \mathrm{Id}^*)((H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes f) \equiv (\mathrm{tr}\, g)(g^*((H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j}) \otimes \mathrm{Id}^*(f)) \pmod{\lambda^{j+1}}. \tag{6.16}$$

The projection formula entails

$$(\mathrm{tr}\, g)(g^*((H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j}) \otimes \mathrm{Id}^* f) = (H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes ((\mathrm{tr}\, g) \circ \mathrm{Id}^*) f. \tag{6.17}$$

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By definition of $\mathcal{T}_{\mathcal{K},g}$, one has $(H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes ((\mathrm{tr}\, g) \circ \mathrm{Id}^*) f = (H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes \mathcal{T}_{\mathcal{K},g}(f)$, so the proof of (6.7) is complete. □

6.3 Congruences II: Hecke algebras

The purpose of this section is to translate the main congruence result Theorem 6.2.1 into a statement about maps of Hecke algebras, see Corollary 6.3.1. The point is that Corollary 6.3.1 applies more naturally to Taylor's pseudorepresentation method than Theorem 6.2.1.

Let $\eta \in \mathbf{\Lambda}$ be a Δ_c^+-dominant weight and fix a level $\mathcal{K} = \mathcal{K}_{(\ell)} K^{(\ell)}$ with $K_{(\ell)} \subset \mathbf{G}(\mathbf{Z}_\ell)$. Let $L = L(\eta, \mathcal{K})$ be a finite extension of $\mathbf{Q}_\ell$ with ring of integers $\mathcal{O}_L$ such that $M_\eta(\mathcal{K}, \mathcal{O}_L)$ admits a basis of Hecke eigenforms.

For every Hecke eigenform $f \in M_\eta(\mathcal{K}, \mathcal{O}_L)$, let

$$\theta_f : \mathcal{H}_\eta \longrightarrow \mathcal{O}_L \tag{6.18}$$

be the 'eigenvalue at f' homomorphism of $\mathcal{O}_L$-algebras given by $\theta_f(T) = T(f)/f$ for all $T \in \mathcal{H}_\eta$. Let $\overline{\theta_f} : \mathcal{H}_\eta \longrightarrow \mathcal{O}_L/\lambda^{j+1}\mathcal{O}_L$ denote the reduction of θ_f modulo λ^{j+1}.

Corollary 6.3.1. *If $\xi_j = \eta + a\epsilon(\ell - 1)\ell^j$, then the composition $\overline{\theta_f} \circ \mathrm{wt}_\eta : \mathcal{H} \longrightarrow \mathcal{O}_L/\lambda^{j+1}\mathcal{O}_L$ factors through $wt_{\xi_j} : \mathcal{H} \longrightarrow \mathcal{H}_{\xi_j}$. In other words, there exists a map $\overline{\theta_{f,j}} : \mathcal{H}_{\xi_j} \longrightarrow \mathcal{O}_L/\lambda^{j+1}\mathcal{O}_L$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathcal{H} \arrow[r, two heads, "\mathrm{wt}_\eta"] \arrow[d, two heads, "\mathrm{wt}_{\xi_j}"'] & \mathcal{H}_\eta \arrow[d, "\overline{\theta_f}"] \\ \mathcal{H}_{\xi_j} \arrow[r, dotted, "\overline{\theta_{f,j}}"'] & \mathcal{O}_L/\lambda^{j+1}\mathcal{O}_L \end{tikzcd} \tag{6.19}$$

Proof. The claimed factorization is equivalent to $\ker \mathrm{wt}_{\xi_j} \subset \ker(\overline{\theta_f} \circ \mathrm{wt}_\eta)$, so we proceed to show the inclusion of kernels. Suppose $T \in \ker \mathrm{wt}_{\xi_j}$. Since $(H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} f \in M_{\xi_j}(\mathcal{K}, \mathcal{O}_L)$, one has

$$T((H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} f) = 0. \tag{6.20}$$

By our congruence result (Theorem 6.2.1, (6.7)), equation (6.20) entails

$$(H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} T(f) \equiv 0 \pmod{\lambda^{j+1}}. \tag{6.21}$$

Since λ does not divide $H^{\mathrm{lift}}_{\mathcal{K},\lambda}$, one has that λ^{j+1} divides $T(f)$. Hence, $\overline{\theta_f}(T) = 0$ in $\mathcal{O}_L/\lambda^{j+1}\mathcal{O}_L$. □

6.4 Galois representations

All of the necessary ingredients for proving Theorem 1.2.1 have been assembled. We now conclude the proof Theorem 1.2.1, following [Tay91, § 1.3, Example 2].

Proof of Theorem 1.2.1. By the 'patching lemma' (over quadratic extensions, cf. the proof of Theorem VII.1.9 in [HT01]) we may assume without loss of generality that:

(1) F is the composite of F^+ and an imaginary quadratic field;

(2) all primes of F^+ above ℓ split in F.

Let π be as in the statement of Theorem 1.2.1. Let (β, Dom) be the Harish-Chandra parameter of π_∞. There exists an open compact subgroup $\mathcal{K} = \mathcal{K}_{(\ell)}\mathcal{K}^{(\ell)}$, with $K_{(\ell)} \subset \mathbf{G}(\mathbf{Z}_\ell)$ and $\mathcal{K}^{(\ell)} \subset \mathbf{G}(\mathbf{A}_f^\ell)$, such that $\pi^{\mathcal{K}} \neq \{0\}$. Let $f_\pi \in \pi^{\mathcal{K}}$ be a corresponding Hecke eigenform. By Theorem 2.6.1, $f_\pi \in M_\eta(\mathcal{K}, \overline{\mathbf{Q}_\ell})$, where η is the highest weight of $V^\vee_{\beta+\rho_n-\rho_c}$.

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As in § 6.3, put $\xi_j = \eta + a\epsilon(\ell-1)\ell^j$ for all $j \in \mathbf{Z}_{\geqslant 1}$. Then $(H^{\mathrm{lift}}_{\mathcal{K},\lambda})^{\ell^j} \otimes f_\pi \in M_{\xi_j}(\mathcal{K}, \overline{\mathbf{Q}_\ell})$. Let ψ be an automorphic representation generated by an eigenform in $M_{\xi_j}(\mathcal{K}, \overline{\mathbf{Q}_\ell})$. By the Casselman–Osborne theorem (cf. [Har88, Proposition 3.1.4]), the infinitesimal character of the Archimedean component ψ_∞ is identified with $\beta + a\epsilon(\ell-1)\ell^j$ (modulo the Weyl group). By condition HLDS2 (see § 5.4), $\beta + a\epsilon(\ell-1)\ell^j$ is regular. Hence, ψ is cohomological in the sense of the appendix (see § A.1).

This means the base change result of Labesse [Lab11, Corollary 5.3], as strengthened by Shin in Theorem A.1, Shin's construction of Galois representations [Shi11, Theorem 1.2], as slightly extended by Chenevier–Harris [CH10], and the determination of the residual spectrum by Moeglin–Waldspurger [MW89] may be applied to the eigenforms in $M_{\xi_j}(\mathcal{K}, \overline{\mathbf{Q}_\ell})$. The result is a Galois representation

$$\rho_j : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}(n, \mathcal{H}_{\xi_j} \otimes \overline{\mathbf{Q}_\ell}) \tag{6.22}$$

whose trace is contained in the Hecke algebra $\mathcal{H}_{\xi_j}$.

Hence, [Tay91, Lemma 1, no. 2] applied to $\mathcal{H}_{\xi_j} \hookrightarrow \mathcal{H}_{\xi_j} \otimes \overline{\mathbf{Q}_\ell}$, gives a pseudorepresentation

$$r_j : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathcal{H}_{\xi_j}. \tag{6.23}$$

Composing with the map $\overline{\theta_{f_\pi,j}}$, constructed in Corollary 6.3.1, gives a pseudorepresentation $s_j = \overline{\theta_{f_\pi,j}} \circ r_j$ with values in $\mathcal{O}_L/\lambda^{j+1}\mathcal{O}_L$. Moreover, the pseudorepresentations s_j are compatible in j in the sense that $s_{j+1} \equiv s_j \pmod{\lambda^{j+1}}$. Hence, the system $\{s_j\}$ gives a pseudorepresentation

$$s : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathcal{O}_L \hookrightarrow \overline{\mathbf{Q}_\ell} \tag{6.24}$$

into an algebraically closed field of characteristic zero. By [Tay91, Theorem1, no. 2], s is the trace of a Galois representation

$$\rho_s : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}(n, \overline{\mathbf{Q}_\ell}) \tag{6.25}$$

which satisfies Theorem 1.2.1. This proves the existence of $R_{\ell,i}(\pi)$.

As for uniqueness, the set of primes of F which are split (i.e. of degree one) has density one in the set of all primes of F (cf. [Ser68, ch. I, § 2.2]). Since only finitely many primes are not in $\mathcal{P}^{(\ell)}$ (see § 1.2), the set of primes $\wp$ of F which are split and which lie over some prime in $\mathcal{P}^{(\ell)}$ also has density one. If $\wp$ is such a prime, then $\wp \in \mathfrak{P}^{(\ell)}$. Hence, $\mathfrak{P}^{(\ell)}$ has density one, so $R_{\ell,\iota}(\pi)$ is unique by the Cebotarev density theorem. □

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Appendix. On the cohomological base change for unitary similitude groups

Sug Woo Shin

A.1 Introduction

This appendix[14] is devoted to the proof of Theorem A.1 on the automorphic base change for unitary similitude groups. The relationship with other results in the literature is explained between Theorem A.1 and Remark A.2. We wish to thank Wushi Goldring and Sophie Morel for their valuable comments on this appendix.

Let F be a complex multiplication (CM) field and F^+ its maximal totally real subfield so that $[F:F^+]=2$. Let $n\geqslant 1$. Let G^1 be a unitary group over F^+ associated to a Hermitian form on n-dimensional F-vector space. Let G be the associated unitary similitude group over $\mathbb{Q}$ with multipliers in $\mathbb{Q}^\times$ so that the multiplier map $G\to\mathbb{G}_m$ has kernel $\mathrm{Res}_{F^+/\mathbb{Q}}G^1$. We assume that

- F contains an imaginary quadratic subfield E (so that $F=EF^+$).

However, we do not assume that G is quasi-split at all finite places, nor do we impose any condition on $G(\mathbb{R})$. Let (ξ,V) be an irreducible algebraic representation of G over $\mathbb{C}$. Let π be a discrete automorphic representation of $G(\mathbb{A})$ such that π_∞ is ξ-cohomological. The latter means that there exists $j\geqslant 0$ such that

$$H^j(\mathfrak{g},K,\pi_\infty\otimes\xi)\neq 0. \tag{A.1}$$

Let S_{ram} be the set of finite places v of $\mathbb{Q}$ such that either G or π is ramified at v. Let $\mathbf{G}:=\mathrm{Res}_{E/\mathbb{Q}}G\times_{\mathbb{Q}}E$. There is an L-embedding

$$BC:{}^LG=\widehat{G}\rtimes\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to{}^L\mathbf{G}\simeq(\widehat{G}\times\widehat{G})\rtimes\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \tag{A.2}$$

given by $g\rtimes\sigma\mapsto(g,g)\rtimes\sigma$. The corresponding functoriality is usually referred to as (automorphic) base change. Although the global base change is expected to exist unconditionally,

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[Lab11, Corollary 5.3] and [Mor10, Proposition 8.5.3] seem to be the best results available so far. Our modest goal is to make a small improvement on their work so that Goldring's result applies without unnecessary restriction.

The global base change is believed to be compatible with local base change, which can be constructed explicitly and unconditionally at almost all places. There are two cases to consider.

- At finite places outside S_{ram}: according to the unramified Langlands correspondence, (A.2) induces
$$BC^{S_{\mathrm{ram}},\infty} : \mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{A}^{S_{\mathrm{ram}},\infty})) \to \mathrm{Irr}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S_{\mathrm{ram}},\infty}))$$
as well as a $\mathbb{C}$-algebra morphism $BC^* : \mathscr{H}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S_{\mathrm{ram}},\infty})) \to \mathscr{H}^{\mathrm{ur}}(G(\mathbb{A}^{S_{\mathrm{ram}},\infty}))$ such that
$$\mathrm{tr}\, \pi(BC^*\phi) = \mathrm{tr}\, BC^{S_{\mathrm{ram}},\infty}(\pi)(\phi), \quad \forall \pi \in \mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{A}^{S_{\mathrm{ram}},\infty})),\ \phi \in \mathscr{H}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S_{\mathrm{ram}},\infty})). \tag{A.3}$$
- At a finite place v split in E: using the isomorphism $\mathbf{G}(\mathbb{Q}_v) \simeq G(\mathbb{Q}_v) \times G(\mathbb{Q}_v)$, define
$$BC_v : \mathrm{Irr}(G(\mathbb{Q}_v)) \to \mathrm{Irr}(\mathbf{G}(\mathbb{Q}_v))$$
by $BC_v(\pi) := \pi \otimes \pi$. There is a corresponding algebra morphism $BC^* : \mathscr{H}(\mathbf{G}(\mathbb{Q}_v)) \to \mathscr{H}(G(\mathbb{Q}_v))$ such that
$$\mathrm{tr}\, \pi(BC^*\phi) = \mathrm{tr}\, BC_v(\pi)(\phi), \quad \forall \pi \in \mathrm{Irr}(G(\mathbb{Q}_v)),\ \phi \in \mathscr{H}(\mathbf{G}(\mathbb{Q}_v))). \tag{A.4}$$

The main theorem of this appendix is as follows. Let $\chi_{(\cdot)}$ signify the central character of a representation.

Theorem A.1. *For π and S_{ram} as above, there exists an automorphic representation $\Pi = \psi \otimes \Pi^1$ of $\mathbf{G}(\mathbb{A}) \simeq \mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_n(\mathbb{A}_F)$ such that:*

(i) $\Pi^{S_{\mathrm{ram}},\infty} \simeq BC^{S_{\mathrm{ram}},\infty}(\pi^{S_{\mathrm{ram}},\infty})$;

(ii) $\Pi_v \simeq BC_v(\pi_v)$ *for any place* $v \in S_{\mathrm{ram}}$ *which splits in* E;

(iii) *the infinitesimal character of* Π_∞ *is the same as that of* $(\xi \otimes \xi)^\vee$ *of* $\mathbf{G}(\mathbb{C}) \simeq G(\mathbb{C}) \times G(\mathbb{C})$;

(iv) $\chi_{\Pi^1}|_{\mathbb{A}_E^\times} = \psi^c/\psi$ *and* $(\Pi^1)^\vee \simeq \Pi^1 \circ c$;

(v) Π^1 *is isomorphic to an isobaric sum* $\Pi_1 \boxplus \cdots \boxplus \Pi_r$ *for some* $r \geqslant 1$ *and discrete representations* Π_i *such that* $\Pi_i^\vee \simeq \Pi_i \circ c$.

The theorem is due to Labesse [Lab11, Corollary 5.3] if the following two conditions hold:

- ξ has regular highest weight or $G(\mathbb{R})$ is compact modulo center;
- $[F^+ : \mathbb{Q}] \geqslant 2$.

(Labesse makes it clear in his footnote 1 that the second condition can be removed with additional work. He works with unitary groups rather than their similitude groups but it should not be difficult to carry over his results.) In the other cases his method does not apply as his condition $(*)$ in Corollary 5.3 is hardly satisfied. The failure of $(*)$ causes the trouble that the coefficients in a certain sum are no longer non-negative and have alternating signs. His argument relies on the non-vanishing of that sum, which is not obvious when there are alternating signs.

The purity of weight for intersection cohomology is what enables us to get around the above difficulty coming from alternating signs. This strategy (already used by [CL99, Theorem A.4.2, Proposition A.4.3], based on a result of Kottwitz for a simpler Shimura variety) was adopted in [Mor10, Corollary 8.5.3], which led to the proof of Theorem A.1 modulo the fact that it proves parts (i) and (ii) outside an unspecified finite set of finite primes unless G is quasi-split over $\mathbb{Q}$. (Strictly speaking, Morel works in the setting $F^+ = \mathbb{Q}$. However, her method yields a similar result without that assumption. On the other hand, see [Mor10, Remark 8.5.4] for a case when

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the unspecified set can be specified.) Thus, our contribution may be seen as getting rid of the unspecified set from the picture. For this it could have sufficed to supply necessary changes and complements to Morel's proof. However, for the reader's convenience and completeness of the argument, we decided to rewrite the proof, sketchy as it may sometimes be. No originality is claimed on our part.

Remark A.2. Eventually the above theorem should be a consequence of the most general base change result for G, which would follow from a full stabilization of the (twisted) invariant trace formula for G and $\mathbf{G}$ and their endoscopic groups, with all the complicated terms. As such a general result would have to await some years to come,[15] we find it reasonable to prove here a simple case, namely Theorem A.1, especially when it has an immediate arithmetic application.

A.2 Proof of Theorem A.1

We will freely adopt the notation and terminology of [Mor10]. (The reader may refer to the index at the end of that book.) Occasionally we adopt a few things from [Shi11] as well. The symbols ξ and V will be used interchangeably. (The former is used in [Shi11] while the latter is used in [Mor10].) One notable difference from [Mor10] is our selection of notation for groups, which is as follows:

- G is a unitary similitude group as above, and H denotes its elliptic endoscopic group;
- $\mathbf{G} := \mathrm{Res}_{E/\mathbb{Q}} G \times_{\mathbb{Q}} E$, $\mathbf{H} := \mathrm{Res}_{E/\mathbb{Q}} H \times_{\mathbb{Q}} E$.

(Compare this with the two different uses of $\mathbf{G}$ and $\mathbf{H}$ in [Mor10]. For instance, see §§ 2.3 and 8.4 in that book.)

Choose a Hecke character $\omega : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ whose restriction to $\mathbb{A}^\times/\mathbb{Q}^\times$ is the quadratic character associated to the extension $E/\mathbb{Q}$ via class field theory. Let $\mathrm{Ram}(\omega)$ be the set of finite primes v such that ω is ramified at a place dividing v. We may and will arrange that every prime v in $\mathrm{Ram}(\omega)$ splits in E. For each elliptic endoscopic group H of G, one uses ω to fix an L-embedding $\eta : {}^L H \to {}^L G$ as in [Shi11, § 3.2]. Then η is unramified outside $S_{\mathrm{ram}} \cup \mathrm{Ram}(\omega)$.

Let p be a prime outside $S_{\mathrm{ram}} \cup \mathrm{Ram}(\omega)$ which splits in E. Let $\wp$ be a prime of F dividing p. Put $S := S_{\mathrm{ram}} \cup \mathrm{Ram}(\omega) \cup \{p\}$. As G is unramified at p, it has a smooth reductive integral model over $\mathbb{Z}_p$. Choose a place λ of F not dividing p. For $i \geqslant 0$, define a λ-adic vector space

$$H^i(\mathrm{Sh}, V) := \varinjlim_{K} H^i(M^K(G, \mathcal{X})^*_{\overline{\mathbb{Q}}}, IC^K V_{\overline{\mathbb{Q}}}),$$

where $K = K^p G(\mathbb{Z}_p)$, and K^p runs over all sufficiently small open compact subgroups K^p of $G(\mathbb{A}^{p,\infty})$. Define

$$W_\lambda^+ := \sum_{2 \mid i} H^i(\mathrm{Sh}, V), \quad W_\lambda^- := \sum_{2 \nmid i} H^i(\mathrm{Sh}, V), \quad W_\lambda = W_\lambda^+ - W_\lambda^-,$$

which are considered in the Grothendieck group of $\mathcal{H}(G(\mathbb{A}^{p,\infty})) \times \mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p)) \times \mathrm{Frob}_\wp^{\mathbb{Z}}$-modules (cf. [Mor10, Remark 6.3.3]). In view of Zucker's conjecture (proved by Looijenga, Saper-Stern and Looijenga-Rapoport) and the Matsushima–Borel–Casselman's formula, (A.1) implies that $H^j(\mathrm{Sh}, V) \neq 0$. In particular,

$$W_\lambda^+ \neq 0 \quad \text{or} \quad W_\lambda^- \neq 0. \tag{A.5}$$

[15] At the time of press Chung Pang Mok released a paper extending Arthur's endoscopic classification for automorphic representations to quasi-split unitary groups. This represents a significant step.

It is primarily due to Beilinson, Deligne and Pink that $IC^K V_{\overline{\mathbb{Q}}}$ is pure of some weight and the following is satisfied (cf. [Mor10, pp. 112–113]; $IC^K V_{\overline{\mathbb{Q}}}$ is pure of weight zero there due to the assumption that V is pure of weight zero, but we do not impose it here).

Lemma A.3. *There exists some integer $a \in \mathbb{Z}$ such that for every $i \geqslant 0$, every eigenvalue α of $\mathrm{Frob}_\wp$ on $H^i(\mathrm{Sh}, V)$ is a Weil $i+a$ number.*

Corollary 6.3.2, Remark 6.3.3 and Proposition 8.3.1 of [Mor10] state that the following holds.

Proposition A.4. *Let $f^\infty = f^{p,\infty}\mathbf{1}_{G(\mathbb{Z}_p)}$ with $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}^{p,\infty}))$.*

(i) *One can construct a function $f^H = (f^H)^{p,\infty} f^{H,(j)} f^H_\xi \in C_c^\infty(H(\mathbb{A}), \xi_H^{-1})$ for each elliptic endoscopic triple (H, s, η_0) such that for every sufficiently large integer $j > 0$,*

$$\mathrm{tr}\,(\Phi_\wp^j f^\infty \mid W_\lambda) = \sum_{(H,s,\eta_0)\in\mathcal{E}(G)} \iota(G,H) ST^H(f^{H,(j)}).$$

(ii) *Suppose that $f^H \in C_c^\infty(H(\mathbb{A}), \xi_H^{-1})$ and $\phi^{\mathbf{H}} \in C_c^\infty(\mathbf{H}^0(\mathbb{A}), \xi_{\mathbf{H}}^{-1})$ are associated in the sense of [Lab99, 3.2] and that f^H_∞ and $\phi^{\mathbf{H}}_\infty$ are as in [Mor10, Proposition 8.3.1]. Then there is a constant $c \in \mathbb{R}^\times$ (independent of $\phi^{\mathbf{H}}$ and f^H) such that*

$$T^{\mathbf{H}}(\phi^{\mathbf{H}}) = c \cdot ST^H(f^H).$$

Now we are ready to start the proof. In the notation of diagram of [Shi11, (4.18)] (exception: η is used instead of $\widetilde{\eta}$ to conform to the notation of [Mor10]), we have commutative diagrams

$$\begin{tikzcd} \mathscr{H}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S,\infty})) \arrow[r, "\widetilde{\zeta}^*"] \arrow[d, "BC^*"'] & \mathscr{H}^{\mathrm{ur}}(\mathbf{H}(\mathbb{A}^{S,\infty})) \arrow[d, "BC^*"] \\ \mathscr{H}^{\mathrm{ur}}(G(\mathbb{A}^{S,\infty})) \arrow[r, "\eta^*"'] & \mathscr{H}^{\mathrm{ur}}(H(\mathbb{A}^{S,\infty})) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Irr}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S,\infty})) & \mathrm{Irr}^{\mathrm{ur}}(\mathbf{H}(\mathbb{A}^{S,\infty})) \arrow[l, "\widetilde{\zeta}_*"'] \\ \mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{A}^{S,\infty})) \arrow[u, "BC"] & \mathrm{Irr}^{\mathrm{ur}}(H(\mathbb{A}^{S,\infty})) \arrow[l, "\eta_*"] \arrow[u, "BC"'] \end{tikzcd} \tag{A.6}$$

and similarly over $\mathbb{A}^{S,p,\infty}$. Choose any $\phi^{S,p,\infty} \in \mathcal{H}^{\mathrm{ur}}(\mathbf{G}(\mathbb{A}^{S,p,\infty}))$. Put $(\phi^H)^{S,p,\infty} := \widetilde{\zeta}^*(\phi^{S,p,\infty})$, $f^{S,p,\infty} := BC^*(\phi^{S,p,\infty})$ and $(f^H)^{S,p,\infty} := \eta^*(f^{S,p,\infty})$. Take ϕ_p, ϕ_p^H, f_p and f_p^H to be the unit elements in the corresponding unramified Hecke algebras. At S, choose f_S and let f_S^H be its transfer. Make a hypothesis, depending on f_S, that there exists ϕ_S (respectively $\phi_S^{\mathbf{H}}$) whose BC transfer is f_S (respectively f_S^H). (This assumption will be satisfied by our later choice of f_S.) Since p splits in E, one can find a function $\phi_p^{\mathbf{H},(j)}$ such that $f_p^{H,(j)}$ and $\phi_p^{\mathbf{H},(j)}$ are associated in the sense of Labesse. At infinity, by construction [Kot88, § 7] (see also [Mor10, 6.2]), f_ξ^H is a finite linear combination of Euler–Poincaré functions. Hence, there exists $\phi_\xi^{\mathbf{H}}$ such that f_ξ^H and $\phi_\xi^{\mathbf{H}}$ are associated [Mor10, Corollary 8.1.11].

Applying (A.3) at finite places away from S one obtains

$$\mathrm{tr}\,(\Phi_\wp^j f^\infty \mid W_\lambda) = \mathrm{tr}\,(\Phi_\wp^j f_S \phi^{S,\infty} \mid BC^{S,\infty}(W_\lambda)).$$

On the other hand the spectral expansion of $T^{\mathbf{H}}(\phi^{\mathbf{H}})$ can be put in the form (cf. [Mor10, Proposition 8.2.3] or [Art88, Theorem 7.1])

$$T^{\mathbf{H}}(\phi^{\mathbf{H}}) = \sum_{\Pi_{\mathbf{H}}} a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S, \xi)\mathrm{tr}\,\Pi_{\mathbf{H}}^{S,p,\infty}((\phi^{\mathbf{H}})^{S,p,\infty}) \tag{A.7}$$

where $\Pi_{\mathbf{H}}$ runs over automorphic representations of $\mathbf{H}(\mathbb{A})$ which are θ-stable and θ-discrete (but not necessarily discrete). Here we wrote $a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S, j, \xi)$ for

$$a_{\mathrm{disc}}(\Pi_{\mathbf{H}}) \cdot \mathrm{tr}\,(\Pi_{\mathbf{H},p}(\phi_p^{\mathbf{H},(j)}) A_{\Pi_{\mathbf{H},p}}) \cdot \mathrm{tr}\,(\Pi_{\mathbf{H},S}(\phi_S^{\mathbf{H}}) A_{\Pi_{\mathbf{H},S}}) \cdot \mathrm{tr}\,(\Pi_{\mathbf{H},\infty}(\phi_\xi^{\mathbf{H}}) A_{\Pi_{\mathbf{H},\infty}}). \tag{A.8}$$

Note that an intertwining operator for θ is not needed in the expression $\mathrm{tr}\,\Pi^{S,\infty}_{\mathbf{H}}((\phi^{\mathbf{H}})^{S,\infty})$ of (A.7) because it does not matter for unramified representations up to sign (due to a normalization of the intertwining operator). (See the paragraph above (4.5) in [Shi11].)

We may use (A.6) to rewrite (A.7) as

$$T^{\mathbf{H}}(\phi^{\mathbf{H}}) = \sum_{\Pi_{\mathbf{H}}} a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S, j, \xi) \cdot \mathrm{tr}\,\widetilde{\zeta}_*(\Pi^{S,p,\infty}_{\mathbf{H}})(\phi^{S,p,\infty}).$$

Hence, Proposition A.4 tells us that $\mathrm{tr}\,(\Phi^j_\wp f_S \phi^{S,\infty} \mid BC^{S,\infty}(W_\lambda))$ equals

$$c^{-1} \sum_{(H,s,\eta_0)} \sum_{\Pi_{\mathbf{H}}} \iota(G,H) a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S, j, \xi) \cdot \mathrm{tr}\,\widetilde{\zeta}_*(\Pi^{S,p,\infty}_{\mathbf{H}})(\phi^{S,p,\infty}).$$

When the functions at $S \cup \{p, \infty\}$ are fixed, there are only finitely many terms contributing to both sides of the formula as the choice of $\phi^{S,p,\infty}$ varies (and the other functions outside $S \cup \{p,\infty\}$ vary accordingly). By using the linear independence of $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}^{S,p,\infty}))$ modules, we deduce

$$\mathrm{tr}\,(\Phi^j_\wp f_S \mid W_\lambda\{\Pi^{S,p,\infty}\}) = \sum_{(H,s,\eta_0)} \sum_{\substack{\Pi_{\mathbf{H}} \\ \widetilde{\zeta}_*(\Pi^{S,p,\infty}_{\mathbf{H}}) \simeq BC(\pi^{S,p,\infty})}} \iota(G,H) a^{\mathbf{H}}_{\Pi_{\mathbf{H}}}(f_S, j, \xi). \tag{A.9}$$

Claim. *The left-hand side of (A.9) does not vanish for some $j \gg 0$ and f_S. Moreover, this holds for f_S such that the following holds: for every H, any endoscopic transfer f_S^H of f_S is in the image of the BC transfer from $\mathbf{H}$ to H. (Namely f_S^H is a BC transfer of some $\phi_S^{\mathbf{H}}$.)*

Proof of claim. For the first assertion it suffices to show that

$$\mathrm{tr}\,(f_S \mid W_\lambda\{\Pi^{S,\infty}\}) = \mathrm{tr}\,(f_S \mid W^+_\lambda\{\Pi^{S,\infty}\}) - \mathrm{tr}\,(f_S \mid (W^-_\lambda\{\Pi^{S,\infty}\}) \in \mathrm{Groth}(\mathrm{Frob}^{\mathbb{Z}}_\wp) \otimes_{\mathbb{Z}} \mathbb{C}$$

is non-trivial. Thanks to purity of weight, it is enough to show that $\mathrm{tr}\,(f_S \mid W^?_\lambda\{\Pi^{S,\infty}\}) \neq 0$ for either $? = +$ or $? = -$. Take $f_S = \mathbf{1}_{K_S}$ for an open compact subgroup $K_S \subset G(\mathbb{Q}_S)$. Since π is automorphic and cohomological, Matsushima-type formula for L^2-cohomology (see [Art96, § 2] for instance) implies that $H^j(\mathrm{Sh}, V)$ contains π as a $G(\mathbb{A}^\infty)$-submodule where j is as in (A.1). Hence, $\mathrm{tr}\,(f_S \mid W^?_\lambda\{\Pi^{S,\infty}\}) \neq 0$ for $? = +$ (respectively $? = -$) when j is even (respectively odd), if K_S is small enough such that π_S has a non-zero K_S-fixed vector.

It remains to take care of the second requirement of the claim. This is satisfied if K_S is sufficiently small by [Mor10, Lemma 8.4.1.(i)]. □

The claim implies that the right-hand side of (A.9) is non-zero. In particular there exists a θ-stable and θ-discrete automorphic representation $\Pi_{\mathbf{H}}$ such that $\widetilde{\zeta}_*(\Pi^{S,p,\infty}_{\mathbf{H}}) \simeq BC(\pi^{S,p,\infty})$. Hence, $\Pi := \widetilde{\zeta}_*(\Pi_{\mathbf{H}})$, defined to be a character twist of $\text{n-ind}^{\mathbf{G}(\mathbb{A})}_{\mathbf{H}(\mathbb{A})} \Pi_{\mathbf{H}}$ (see [Shi11, § 4.4] for the precise definition), is automorphic and satisfies part (iv) of the theorem, which amounts to the θ-stable property of Π. *A fortiori* assertion (v) follows easily from the construction of Π and the fact that Π_H is θ-stable and θ-discrete. Moreover,

$$\Pi^{S,p,\infty} \simeq BC^{S,p,\infty}(\pi^{S,p,\infty}). \tag{A.10}$$

The character identities at $v \in S$ obtained from (A.9) have the form

$$\mathrm{tr}\,(\phi_S \mid a\pi_S + \cdots) = \sum_H \sum_{i \in I_H} b_i \mathrm{tr}\,(\widetilde{\zeta}_*(\Pi^i_{\mathbf{H}})(f_S) A_{\widetilde{\zeta}_*(\Pi^i_{\mathbf{H}})})$$

where a and b_i are non-zero complex numbers and I_H is a finite index set parametrizing $\Pi^i_{\mathbf{H}}$ such that $BC(\pi^{S,p,\infty}) = (\Pi^i_{\mathbf{H}})^{S,p,\infty}$ and the summand of (A.9) is non-zero. The base change character identity at split places (cf. [Shi11, 4.2]) shows that there exists $\Pi^i_{\mathbf{H}}$ (i.e. on the right-hand side of (A.9)) such that $BC_v(\pi_v) = \Pi_v$ for every $v \in S$ split in E. So we could have defined Π by using that $\Pi^i_{\mathbf{H}}$. Then condition (ii) holds. Moreover, the coefficient for $\Pi_{\mathbf{H}} = \Pi^i_{\mathbf{H}}$ in (A.9) being non-zero implies, in view of (A.8), that $\Pi_{\mathbf{H},p}$ is unramified at p, since $\phi_p^{\mathbf{H},(j)}$ belongs to the unramified Hecke algebra.

Recall that $S = S_{\mathrm{ram}} \cup \mathrm{Ram}(\varpi)$ and every $v \in \mathrm{Ram}(\varpi)$ splits in E. Hence, (A.10) is improved to

$$\Pi^{S_{\mathrm{ram}},p,\infty} \simeq BC^{S_{\mathrm{ram}},p,\infty}(\pi^{S_{\mathrm{ram}},p,\infty}). \tag{A.11}$$

For part (iii), one uses the trace computation of Euler–Poincaré functions and their twisted analogues at infinity. A careful book-keeping of their infinitesimal characters yields the result.

It remains to improve upon (A.11) to include the place p. The key point is that the choice of p, made at the start of the proof, was auxiliary. Choose any other prime p' outside $S_{\mathrm{ram}} \cup \mathrm{Ram}(\omega)$ which splits in E and repeat the above argument. Then we obtain Π' satisfying $(\Pi')^{S_{\mathrm{ram}},p',\infty} \simeq BC^{S_{\mathrm{ram}},p',\infty}(\pi^{S_{\mathrm{ram}},p',\infty})$ as well as parts (ii), (iii) and (iv). Applying Jacquet–Shalika's strong multiplicity one to Π and Π', we deduce that Π_p and Π'_p appear as sub-quotients of the same parabolic induction. On the other hand, Π_p and Π'_p are both unramified. Indeed, we have seen this for Π_p above, and $\Pi'_p \simeq BC(\pi_p)$ is unramified as π_p is. Therefore, $\Pi_p \simeq \Pi'_p$ since there exists at most one unramified representation in a parabolic induction. Hence $\Pi_p \simeq BC(\pi_p)$ as desired.

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