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THEORETICAL PEARLS

Enumerators of lambda terms are reducing

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Abstract

A closed λ-term **E** is called an *enumerator* if

$$\forall M \in \Lambda^0 \exists n \in \mathbb{N}\, \mathbf{E}\ulcorner n \urcorner =_\beta M.$$

Here Λ^0 is the set of closed λ-terms, . is the set of natural numbers and the $\ulcorner n \urcorner$ are the Church's numerals $\lambda fx.f^n x$. Such an **E** is called *reducing* if, moreover

$$\forall M \in \Lambda^0 \exists n \in \mathbb{N}\, \mathbf{E}\ulcorner n \urcorner \twoheadrightarrow_\beta M.$$

An ingenious recursion theoretic proof by Statman will be presented, showing that every enumerator is reducing. I do not know any direct proof.

1 Introduction

Remember that in Barendregt (1991) a simple proof of the existence of a self-interpreter $\mathbf{E} \in \Lambda^0$ was given. Such an **E** satisfies

$$\forall M \in \Lambda^0\, \mathbf{E}\ulcorner M \urcorner =_\beta M.$$

The first construction of a self-interpreter is due to Kleene (1936), and I presented another one due to P. de Bruin. Such an **E** is automatically an enumerator. Inspection of the details of the construction of **E** by Kleene (1936) or by P. de Bruin shows that these **E** are in fact reducing enumerators.

In my thesis (Barendregt, 1971) I constructed as application a *universal generator*, that is, a term reducing to terms of arbitrary complexity.

Definition 1.1
A term $U \in \Lambda$ is called a *universal generator* iff

$$\forall M \in \Lambda\, \exists N \in \Lambda [U \twoheadrightarrow_\beta N \,\&\, M \text{ subterm of } N].$$

Proposition 1.2
There exists a universal generator $U \in \Lambda^0$.

Proof
If **E** is a reducing enumerator, then one can take $U \equiv F\ulcorner 0 \urcorner$ with

$$F\ulcorner n \urcorner \twoheadrightarrow_\beta [\mathbf{E}\ulcorner n \urcorner, F\ulcorner n+1 \urcorner],$$

where $[-,-]$ is a pairing in the λ-calculus. Indeed, one then has

$$\begin{aligned} U &\twoheadrightarrow_\beta F\ulcorner 0\urcorner \\ &\twoheadrightarrow_\beta [\mathsf{E}\ulcorner 0\urcorner, F\ulcorner 1\urcorner] \\ &\twoheadrightarrow_\beta [\mathsf{E}\ulcorner 0\urcorner, [\mathsf{E}\ulcorner 1\urcorner, F\ulcorner 2\urcorner]] \\ &\dots \\ &\twoheadrightarrow_\beta [\mathsf{E}\ulcorner 0\urcorner, [\mathsf{E}\ulcorner 1\urcorner, [\mathsf{E}\ulcorner 2\urcorner, [\dots]]]] \\ &\dots . \end{aligned}$$

Since the $\mathsf{E}\ulcorner n\urcorner$ collectively reduce to all $M \in \Lambda^0$ and any $N \in \Lambda$ is a subterm of some closed term, it follows that U is a universal generator.

F can be constructed easily from E using the fixed-point combinatory and a λ-defining term for the successor function. (For example,

$$U \equiv (\lambda ab.b(aab))(\lambda ab.b(aab))(\lambda fxz.z(\mathsf{E}x)(f(\lambda bc.b(xbc))))(\lambda bc.c)$$

works.) □

A short universal generator has been constructed in Mulder (1990): in one of the propositions (*stellingen*) accompanying the thesis he constructed

$$\begin{aligned} U \equiv (&\lambda kpy.y(\lambda l.pk(p(\lambda xyz.xz(yz)) \\ &(y(\lambda fxyz.zk(\lambda ht.x(f(phy)(phy)l))(\lambda uv.p(hu)(fvyt)))kkl)))) \\ &(\lambda xy.x)(\lambda xyzf.fxy)(\lambda f.(\lambda x.xx)(\lambda x.f(xx))). \end{aligned}$$

The fact that the given enumerators are reducing brought me to the following:

Conjecture 1.3
Every enumerator is reducing.

Some vague evidence for the conjecture is this. If E has to make every $M \in \Lambda^0$ by having $\mathsf{E}\ulcorner n\urcorner =_\beta M$ for some $n \in \mathbb{N}$, then the only way to do this is to construct every $M \in \Lambda^0$ by a reduction from $\mathsf{E}\ulcorner n\urcorner$ for an appropriate n. This is plausible, since the collection

$$B_M = \{N \in \Lambda^0 \mid M =_\beta N\}$$

is undecidable. It seems easier to make the $\mathsf{E}\ulcorner n\urcorner$ reduce to all members of all B_M than to just some of them.

Of course, this intuition is far from being a proof. I explained my conjecture to Rick Statman in 1983 and in 1987 he settled it in the positive. In fact, as we will see, he proved something much more general.

2 Proof of the conjecture

If ψ is a partial recursive function, then $\psi(n)\downarrow$ means that $\psi(n)$ is defined and $\psi(n)\uparrow$ means that $\psi(n)$ is undefined. A set $A \subseteq \mathbb{N}$ is called recursively enumerable (r.e.) if for some partial recursive $\psi : \mathbb{N} \to \mathbb{N}$ one has $A = \mathrm{dom}(\psi)$, i.e. $\forall n \in \mathbb{N}[n \in A \Leftrightarrow \psi(n)\downarrow]$. In the following the reader is supposed to know some elementary properties of r.e. sets. For example, that if A and its complement are both r.e., A is recursive; moreover, that there exists a set $K \subseteq \mathbb{N}$ that is r.e. but not recursive.

Lemma 2.1
For every $M \in \Lambda$ there exists an $M_1 \in \Lambda$ in β-nf such that $M_1 \mathsf{I} \twoheadrightarrow_\beta M$. Here $\mathsf{I} \equiv \lambda x.x$.

Proof
By induction on the structure of M we define M_1 in the following table:

M	M_1
x	$\lambda z.zx$
PQ	$\lambda z.zP_1z(zQ_1z)$
$\lambda x.P$	$\lambda zx.zP_1z$

Then by induction it follows that $M_1 \mathsf{I} \twoheadrightarrow_\beta M$. □

Remember that a term $M \in \Lambda$ is *of order* 0 if for no $P \in \Lambda$ one has $M =_\beta \lambda x.P$. For example $(\lambda x.xx)(\lambda x.xx)$ is of order 0.

Lemma 2.2
(i) *For every partial recursive function ψ there is a term $F \in \Lambda^0$ such that for all $n \in \mathbb{N}$ one has*

$$\psi(n)\downarrow \Rightarrow F\ulcorner n\urcorner =_\beta \ulcorner \psi(n)\urcorner$$
$$\psi(n)\uparrow \Rightarrow F\ulcorner n\urcorner \text{ is of order } 0.$$

(ii) *let $K \subseteq \mathbb{N}$ be an r.e. set. Then for some $P_K \in \Lambda^0$ one has for all $n \in \mathbb{N}$*

$$n \in K \Rightarrow P_K\ulcorner n\urcorner \twoheadrightarrow_\beta \mathsf{I};$$
$$n \notin K \Rightarrow P_K\ulcorner n\urcorner \text{ is of order } 0.$$

Proof
(i) Inspection of the usual proof of the λ-definability of the partial recursive functions shows that in case the function is undefined on an argument the representing λ-term is of order 0 on the corresponding numerical. (One of the next 'Pearls in Theory' will be devoted to possible representations of 'undefined'.)

(ii) Let $K = \mathrm{dom}(\psi)$. Let F λ-define ψ. Then take $P_K \equiv \lambda c.Fc\mathsf{I}\mathsf{I}$. (Remember that for Church's numerals one has $\ulcorner n\urcorner \mathsf{I}\mathsf{I} =_\beta \mathsf{I}$.] □

Theorem 2.3 (Statman, 1987)
Let $\mathscr{A} \subseteq \Lambda^0$ (after coding) be an r.e. set. Suppose

$$\forall M \in \Lambda^0 \, \exists N \in \mathscr{A} \; N =_\beta M. \tag{1}$$

Then

$$\forall M \in \Lambda^0 \, \exists N \in \mathscr{A} \; N \twoheadrightarrow_\beta M. \tag{2}$$

Proof

Assume (1). Suppose towards a contradiction that (2) does not hold, i.e. for some $M_0 \in \Lambda^0$

$$\forall N \in \mathscr{A}\ N \not\twoheadrightarrow_\beta M_0.$$

Using Lemma 2.1 construct a term M_1 in β-nf such that $M_1 \mathsf{I} \twoheadrightarrow_\beta M_0$. Define a predicate R on $\mathbb{N}$ as follows:

$$R(n) \Leftrightarrow \exists N \in \mathscr{A}\ \exists Q \in \Lambda [P \ulcorner n \urcorner \twoheadrightarrow_\beta Q \ \&\ N \twoheadrightarrow_\beta Q M_1 \mathsf{I}],$$

where $P = P_K$ as in Lemma 2.2 for some non-recursive r.e. set K. Note that R is an r.e. predicate. Claim

$$R(n) \Leftrightarrow n \notin K.$$

As to ($\Rightarrow$), suppose $R(n)$, i.e. for some $N \in \mathscr{A}$ and $Q \in \mathscr{A}$ one has

$$P \ulcorner n \urcorner \twoheadrightarrow_\beta Q \quad \text{and} \quad N \twoheadrightarrow_\beta Q M_1 \mathsf{I}.$$

If $n \in K$, then $\mathsf{I} =_\beta P \ulcorner n \urcorner =_\beta Q$, so by the Church–Rosser theorem $Q \twoheadrightarrow_\beta \mathsf{I}$ and therefore $N \twoheadrightarrow_\beta \mathsf{I} M_1 \mathsf{I} \twoheadrightarrow_\beta M_0$, contracting (2). Therefore $n \notin K$ and we are done. As to ($\Leftarrow$), suppose $n \notin K$. Then $P \ulcorner n \urcorner$ is of order 0. By (1) there is an $N \in \mathscr{A}$ such that $N =_\beta P \ulcorner n \urcorner M_1 \mathsf{I}$. By the Church–Rosser theorem there is a common reduct L of N and $P \ulcorner n \urcorner M_1 \mathsf{I}$. Since $P \ulcorner n \urcorner$ is of order 0 and M_1, I are in nf one must have $L \equiv Q M_1 \mathsf{I}$ with $P \ulcorner n \urcorner \twoheadrightarrow_\beta Q$. Therefore $R(n)$.

From the claim it follows that the complement of K is r.e., hence recursive (since K is itself r.e.) contradicting the choice of K. □

From the theorem the conjecture follows immediately by taking $\mathscr{A} = \{\mathsf{E} \ulcorner n \urcorner \mid n \in \mathbb{N}\}$.

From the proved conjecture I mistakenly concluded that every self-interpreter in the λ-calculus is reducing in the sense that

$$\forall M \in \Lambda^0\ \mathsf{E} \ulcorner M \urcorner \twoheadrightarrow_\beta M.$$

But this does not follow. Do you see why? Moreover, that this is not true was pointed out to me by Peter de Bruin, who provided a counterexample. Can you construct one?

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