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THE GENERALISATION OF TUTTE'S RESULT FOR CHROMATIC TREES, BY LAGRANGIAN METHODS

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1. Introduction. A K-coloured rooted tree t is said to have *colour partition* $\mathbf{L}$ if $\mathbf{L}$ is a $K \times \infty$ matrix with elements l_{ij} equal to the number of non-root vertices of colour i and degree j. If adjacent vertices are of different colours then t is called a *chromatic tree* and $\mathbf{L}$ a *chromatic partition*. The tree has *edge partition* $\mathbf{D}$ where $\mathbf{D}$ is a $K \times K$ matrix with elements d_{ij} equal to the number of edges, directed away from the root, from a vertex of colour i to a vertex of colour j.

In this paper we consider a method for enumerating trees with respect to colour and degree information. The method makes use of elementary decompositions of trees, and the functional equations which are induced. A number of new results are obtained by this means. More specifically, we consider (Section 3) the enumeration of rooted plane K-coloured trees with given colour and edge partitions. Remarkably, the number of such trees which are planted is a multiple of the number of spanning arborescences on a graph with the edge partition as its adjacency matrix. This result is used (Section 4) to obtain the number of rooted plane K-chromatic trees with fixed chromatic partition, a number given by Tutte [**5**] for planted trees in the case $K = 2$. Finally, a new combinatorial correspondence between two sets of trees is given (Section 5) which yields the de Bruijn-van Aardenne Ehrenfest-Smith-Tutte (BEST) theorem as a special case.

We use a familiar decomposition of rooted trees. Associated with this is a system of functional equations which may be solved by a specialisation, given in Section 2, of the Lagrange theorem in many variables. The specialisation accounts for the persistence, in combinatorial enumeration, of determinants of matrices with row and column constraints (see, for example, [**6**]).

Throughout this paper we use the notation: the number of non-root vertices of colour i is $n_i = \sum_{j \geqq 1} l_{ij} (= \sum_{j=1}^{K} d_{ji})$; the sum of the out-degrees (edges are directed away from the root) of non-root vertices of colour i is $q_i = \sum_{j \geqq 1} (j-1) l_{ij}$; the number of non-root vertices is $N + 1 = n_1 + \ldots + n_K$, where K is the number of colours.

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If $\mathbf{A}$ is a $K \times K$ matrix with elements a_{ij}, we write $\mathbf{A} = [a_{ij}]_{K\times K}$ and $a_{ij} = [\mathbf{A}]_{ij}$. $\|a_{ij}\|$ denotes the determinant of $\mathbf{A}$ and $\mathrm{cof}_{s,t}[a_{ij}]$ is the (s, t)-cofactor of $\mathbf{A}$. If $\alpha, \beta \subseteq \{1, \ldots, K\}$ then $\mathbf{A}[\alpha \mid \beta]$ is the submatrix of $\mathbf{A}$ intercepted by rows with labels in α and columns with labels in β. Let $\mathbf{M} = [m_{ij}]_{K\times K}$ be a non-negative integer matrix. Then

$$\mathbf{A}^{\mathbf{M}} = \prod_{1\leqq i,j\leqq K} a_{ij}{}^{m_{ij}}, \quad \mathbf{M}! = \prod_{1\leqq i,j\leqq K} m_{ij}!, \quad \text{and} \quad \begin{bmatrix} j \\ \mathbf{i} \end{bmatrix} = j!(\mathbf{i}!)^{-1},$$

the multinomial coefficient, where $j = i_1 + \ldots + i_n$ and $\mathbf{i} = (i_1, \ldots, i_n)$. If $f(\mathbf{x})$ is a formal power series in $\mathbf{x} = (x_1, \ldots, x_n)$ then $[\mathbf{x}^{\mathbf{i}}]f(\mathbf{x})$ denotes the coefficient of $x_1{}^{i_1} \ldots x_n{}^{i_n}$ in $f(\mathbf{x})$, $[\mathbf{x}]$ denotes the operator $[x_1 \ldots x_n]$ and $Z_{\mathbf{x}}f(\mathbf{x}) = f(0)$.

2. The Lagrange theorem and a specialisation. The following theorem is the multivariate extension of the Lagrange theorem to formal power series.

THEOREM 2.1. *Let* $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_\kappa)$ *and* γ *be formal power series in the indeterminates* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_\kappa)$ *and with no terms with negative exponents. Suppose that* $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_\kappa)$ *satisfies* $\xi_i = \zeta_i\phi_i(\boldsymbol{\xi})$ *for* $i = 1, \ldots, \kappa$. *Then if* $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)$

$$[\boldsymbol{\zeta}^{\boldsymbol{\nu}}]\gamma(\boldsymbol{\xi}(\boldsymbol{\zeta})) = [\boldsymbol{\xi}^{\boldsymbol{\nu}}]\gamma(\boldsymbol{\xi})\boldsymbol{\phi}^{\boldsymbol{\nu}}(\boldsymbol{\xi})\left\|\delta_{ij} - \frac{\xi_j}{\phi_i(\boldsymbol{\xi})}\frac{\partial\phi_i(\boldsymbol{\xi})}{\partial\xi_j}\right\|.$$

Proof. See [**3**] and [**4**] for formulas which are easily shown to be equivalent to the above by simple row and column multiplication.

The following corollary is useful in allowing us to avoid the extraction of coefficients from the determinant in Theorem 2.1.

COROLLARY 2.2. *Under the conditions of Theorem* 2.1 *further suppose that* $\phi_i(\boldsymbol{\xi})$ *is independent of* ξ_j *for each* $(i, j) \in \mathscr{S} \subseteq \{1, \ldots, \kappa\}^2$. *Then*

$$[\boldsymbol{\zeta}^{\boldsymbol{\nu}}]\boldsymbol{\xi}^{\mathbf{r}} = (\nu_1 \ldots \nu_\kappa)^{-1}\sum_{\boldsymbol{\mu}}\|\delta_{ij}\nu_i - \mu_{ij}\|\prod_{i=1}^{\kappa}([\xi_1{}^{\mu_{i1}} \ldots \xi_\kappa{}^{\mu_{i\kappa}}]\phi_i{}^{\nu_i})$$

where the summation is over all non-negative integer $\kappa \times \kappa$ *matrices such that*

$$\sum_{i=1}^{\kappa}\mu_{ij} = \nu_j - r_j, \quad j = 1, \ldots, \kappa \quad \textit{and}$$

$$\mu_{ij} = 0 \textit{ for each } (i, j) \in \mathscr{S}.$$

Proof. From Theorem 2.1 we have

$$\begin{aligned}[\boldsymbol{\zeta}^{\boldsymbol{\nu}}]\boldsymbol{\xi}^{\mathbf{r}} &= [\boldsymbol{\xi}^{\boldsymbol{\nu}-\mathbf{r}}]\left\|\phi_i{}^{\nu_i}\delta_{ij} - \frac{\xi_j}{\nu_i}\frac{\partial}{\partial\xi_j}\phi_i{}^{\nu_i}\right\| \\ &= ((\boldsymbol{\nu} - \mathbf{r})!)^{-1}Z_{\boldsymbol{\xi}}\left\{\frac{\partial^{\nu_1-r_1}}{\partial\xi_1{}^{\nu_1-r_1}} \cdots \frac{\partial^{\nu_\kappa-r_\kappa}}{\partial\xi_\kappa{}^{\nu_\kappa-r_\kappa}}\left\|\phi_i{}^{\nu_i}\delta_{ij} - \frac{\xi_j}{\nu_i}\frac{\partial}{\partial\xi_j}\phi_i{}^{\nu_i}\right\|\right\}.\end{aligned}$$

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Let $D_i^{(\mu)}$ denote the partial differential operator

$$\frac{\partial^{\mu_{i1}}}{\partial \xi_1^{\mu_{i1}}} \cdots \frac{\partial^{\mu_{i\kappa}}}{\partial \xi_\kappa^{\mu_{i\kappa}}}.$$

Then by differentiating the determinant we have

$$[\boldsymbol{\zeta}^{\mathbf{v}}]\boldsymbol{\xi}^{\mathbf{r}} = \sum_{\boldsymbol{\mu}} (\boldsymbol{\mu}!)^{-1} Z_{\boldsymbol{\xi}}\left\{\left\| \delta_{ij}(D_i^{(\boldsymbol{\mu})}\phi_i^{\nu_i}) - \nu_i^{-1}\left(D_i^{(\boldsymbol{\mu})}\xi_j \frac{\partial}{\partial \xi_j}\phi_i^{\nu_i}\right)\right\|\right\}$$

where

$$\sum_{i=1}^{\kappa} \mu_{ij} = \nu_j - r_j.$$

But, by Leibnitz's theorem we have

$$\nu_i^{-1} Z_{\boldsymbol{\xi}}\left(D_i^{(\boldsymbol{\mu})}\xi_j \frac{\partial}{\partial \xi_j}\phi_i^{\nu_i}\right) = \nu_i^{-1}\mu_{ij} Z_{\boldsymbol{\xi}}(D_i^{(\boldsymbol{\mu})}\phi_i^{\nu_i})$$

where $\mu_{ij} = 0$ if $(i, j) \in \mathscr{S}$.
Accordingly,

$$[\boldsymbol{\zeta}^{\mathbf{v}}]\boldsymbol{\xi}^{\mathbf{r}} = \sum_{\boldsymbol{\mu}} (\boldsymbol{\mu}!)^{-1} \|\delta_{ij} - \nu_i^{-1}\mu_{ij}\| \prod_{i=1}^{\kappa} \{Z_{\boldsymbol{\xi}}(D_i^{(\boldsymbol{\mu})}\phi_i^{\nu_i})\}$$

and the result follows.

3. Plane rooted trees with given colour and edge partitions.

Let $\theta_{c,r}^{(K)}(\mathbf{L}, \mathbf{D})$ be the number of plane rooted K-coloured trees with root colour c and degree r, colour partition $\mathbf{L}$ and edge partition $\mathbf{D}$.

LEMMA 3.1. *Let $\mathscr{C}_i^{(r)}$ be the set of plane rooted K-coloured trees with root colour i and root degree r, and let $\mathscr{C}_i^{(1)}$ be denoted by $\mathscr{C}_i$. Then*

$$\text{i)} \quad \dot{\bigcup_{k=0}^{\infty}} \dot{\bigcup_{j=1}^{K}} \mathscr{C}_j^{\,k} \cong \mathscr{C}_i \qquad \text{ii)} \quad \mathscr{C}_i^{\,r}/C_r \cong \mathscr{C}_i^{(r)}$$

where C_r denotes the cyclic group of order r.

Proof. (i) We note that the vertex adjacent to the root has degree $k + 1$, and colour j for some $k \geqq 0$ and $j = 1, \ldots, K$.
(ii) Select r plane planted trees and identify roots.

THEOREM 3.2.

$$\theta_{c,r}(\mathbf{L}, \mathbf{D}) = r^{-1} \sum_{d \mid r, \mathbf{L}, \mathbf{D}} \phi(d) \gamma_{c, rd^{-1}}(d^{-1}\mathbf{L}, d^{-1}\mathbf{D}),$$

where

$$\gamma_{c,r}^{(K)}(\mathbf{L}, \mathbf{D}) = \frac{r!(\mathbf{n} - \mathbf{1})!\mathbf{q}!}{\mathbf{L}!\mathbf{D}!} \binom{q_c + r - 1}{r - 1} n_c \operatorname{cof}_{cc}[\delta_{ij} n_i - d_{ij}].$$

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Proof. Let $[\mathbf{X}]_{ij} = x_{ij}$, $[\mathbf{W}]_{ij} = w_{ij}$, $\mathbf{a} = (a_1, \ldots, a_K)$ and

$$h_i(\mathbf{a}) = \sum_{\mathbf{L}} \sum_{\mathbf{D}} \theta_{i,1}{}^{(K)}(\mathbf{L}, \mathbf{D})\mathbf{X}^{\mathbf{L}}\mathbf{W}^{\mathbf{D}}\mathbf{a}^{\mathbf{n}}.$$

Let $h_{i,r}(\mathbf{a}) = h_i(\mathbf{a})^r$ and $\gamma_{i,r}{}^{(K)}(\mathbf{L}, \mathbf{D}) = [\mathbf{X}^{\mathbf{L}}\mathbf{W}^{\mathbf{D}}\mathbf{a}^{\mathbf{n}}]h_{i,r}(\mathbf{a})$. Thus from Lemma 3.1 (i) we have

$$h_i(\mathbf{a}) = \sum_{j=1}^{K} w_{ij}a_j g_j(h_j(\mathbf{a}))$$

where

$$g_j(\lambda) = \sum_{k \geqq 1} x_{jk}\lambda^{k-1}.$$

Let $\alpha_j = a_j g_j(h_j(\mathbf{a}))$ so

$$h_i(\mathbf{a}) = \sum_{j=1}^{K} w_{ij}\alpha_j.$$

Thus

$$\gamma_{c,r}{}^{(K)}(\mathbf{L}, \mathbf{D}) = [\mathbf{X}^{\mathbf{L}}\mathbf{W}^{\mathbf{D}}\mathbf{a}^{\mathbf{n}}]\left(\sum_{j=1}^{K} w_{cj}\alpha_j\right)^r$$

where

$$\alpha_j = a_j g_j\left(\sum_{l=1}^{K} w_{jl}\alpha_l\right).$$

From Corollary 2.2 we obtain

$$[\mathbf{X}^{\mathbf{L}}\mathbf{W}^{\mathbf{D}}\mathbf{a}^{\mathbf{n}}]\left(\sum_{j=1}^{K} w_{cj}\alpha_j\right)^r$$
$$= (n_1 \ldots n_K)^{-1} \sum_{r_1+\ldots+r_K=r} \begin{bmatrix} r \\ \mathbf{r} \end{bmatrix} \sum_{\mathbf{M}} \|\delta_{ij}n_i - m_{ij}\|$$
$$\times \left\{\prod_{i=1}^{K} [x_{i1}{}^{l_{i1}}x_{i2}{}^{l_{i2}} \ldots] [w_{i1}{}^{d_{i1}-r_1\delta_{ic}} \ldots w_{ik}{}^{d_{iK}-r_K\delta_{ic}}]\right.$$
$$\left. \times [\alpha_1{}^{m_{i1}} \ldots \alpha_K{}^{m_{iK}}]\, g_i{}^{n_i}\left(\sum_{j=1}^{K} w_{ij}\alpha_j\right)\right\}$$

where the summation is over all $\mathbf{M}$ such that $\sum_{i=1}^{K} m_{ij} = n_j - r_j$. Accordingly

$$\gamma_{c,r}{}^{(K)}(\mathbf{L}, \mathbf{D}) = r!(\mathbf{n} - \mathbf{1})!\mathbf{q}!(\mathbf{D}!\mathbf{L}!)^{-1} \sum_{r_1+\ldots+r_K=r} \left\{\prod_{j=1}^{K} \binom{d_{cj}}{r_j}\right\}$$
$$\times \|\delta_{ij}n_i - (d_{ij} - r_j\delta_{ic})\|.$$

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But

$$\sum_{r_1+\dots+r_K=r} \|\delta_{ij}n_i - (d_{ij} - r_j\delta_{ic})\| \prod_{j=1}^{K} \binom{d_{cj}}{r_j}$$
$$= [x^r] \sum_{m=1}^{K} \operatorname{cof}_{cm}[\delta_{ij}n_i - d_{ij}] \sum_{\mathbf{r}\geqq\mathbf{0}} r_m \prod_{j=1}^{K} \binom{d_{cj}}{r_j} x^{r_j}$$
$$= [x^r](1+x)^{d_{c1}+\dots+d_{cK}-1} \sum_{m=1}^{K} \operatorname{cof}_{cm}[\delta_{ij}n_i - d_{ij}]d_{cm}$$
$$= n_c \binom{q_c + r - 1}{r-1} \operatorname{cof}_{cc}[\delta_{ij}n_i - d_{ij}]$$

since $\|\delta_{ij}n_i - d_{ij}\| = 0$. The result follows from Lemma 3.1 (ii) using the cycle index polynomial for the cyclic group.

An immediate specialisation of Theorem 3.2 to the case $K = 1$, $r = 1$ gives us Tutte's result [**5**], that the number of plane planted trees with i_j non-root vertices of degree j, for $j \geqq 1$, is $(n-1)!(\mathbf{i}!)^{-1}$, where $\mathbf{i} = (i_1, i_2, \dots)$ and $n = i_1 + i_2 + \dots$.

By a straightforward argument for embedding rooted trees in the plane it follows that the number of labelled rooted K-coloured trees with root colour c, root degree r, colour partition $\mathbf{L}$ and edge partition $\mathbf{D}$ may be obtained from $\gamma_{c,r}^{(K)}(\mathbf{L}, \mathbf{D})$. The required number is given by

$$(N+2)!\left\{r! \prod_{\substack{1\leqq i\leqq K\\ j\geqq 1}} (j-1)!^{l_{ij}}\right\}^{-1} \gamma_{c,r}^{(K)}(\mathbf{L}, \mathbf{D}).$$

4. Plane planted trees with given chromatic partition. The number of K-chromatic plane planted trees with root colour c and chromatic partition $\mathbf{L}$ is denoted by $\chi_c^{(K)}(\mathbf{L})$.

THEOREM 4.1.

$$\chi_c^{(K)}(\mathbf{L}) = (\mathbf{L}!)^{-1} \sum_{i=0}^{N} (N-i)![x^i]P(-x)$$

where

$$P(x) = \prod_{i=1}^{K} \left(\sum_{j\geqq 0} \binom{q_i}{j} \binom{n_i + \delta_{ic} - 1}{j} j! x^j \right).$$

Proof. Clearly

$$\chi_c^{(K)}(\mathbf{L}) = \sum_{\mathbf{D}} \theta_{c,1}^{(K)}(\mathbf{L}, \mathbf{D})$$

where the summation is over all $\mathbf{D}$ for which

$$\sum_{i=1}^{K} d_{ij} = n_j, \ \sum_{i=1}^{K} d_{ji} = q_j + \delta_{jc} \quad \text{and}$$
$$d_{jj} = 0 \quad \text{for} \quad j = 1, \dots, K.$$

Let $\omega = (\mathbf{n} - \mathbf{1})!\mathbf{q}!n_c(\mathbf{L}!)^{-1}$ so that, by Theorem 3.2,

$$\chi_c^{(K)}(\mathbf{L}) = \omega[\mathbf{v}^{\mathbf{n}}\mathbf{u}^{\mathbf{q}}u_c] \sum_{\substack{d_{ij} \geqq 0 \\ \text{for } i \neq j}} \operatorname{cof}_{cc}\left[\delta_{ij}\left(\sum_{l=1}^{K} d_{lj}\right) - d_{ij}\right] \prod_{i,j} \frac{(u_i v_j)^{d_{ij}}}{d_{ij}!}$$

$$= \omega[\mathbf{v}^{\mathbf{n}}\mathbf{u}^{\mathbf{q}}u_c] \operatorname{cof}_{cc}\left[\delta_{ij}\left(\sum_{l=1}^{K} u_l v_j\right) - u_i v_j\right] \exp\left\{\sum_{j=1}^{K} v_j \sum_{i \neq j} u_i\right\},$$

since

$$\sum_{\mathbf{i} \geqq \mathbf{0}} g(\mathbf{i}) \frac{\mathbf{x}^{\mathbf{i}}}{\mathbf{i}!} = g(\mathbf{x}) \exp\left(\sum_{i=1}^{n} x_i\right),$$

where $g(\mathbf{x})$ is multilinear. Let $u = u_1 + \ldots + u_K$. Noting that $\det(\mathbf{I} + \mathbf{A}) = 1 + \operatorname{trace} \mathbf{A}$ for $\mathbf{A}$ with rank one, we have

$$\chi_c^{(K)}(\mathbf{L}) = \omega[\mathbf{v}^{\mathbf{n}-\mathbf{1}} v_c \mathbf{u}^{\mathbf{q}} u_c] \exp\left\{\sum_{j=1}^{K} v_j(u - u_j)\right\} \operatorname{cof}_{cc}[\delta_{ij} u - u_i]$$

$$= \omega[\mathbf{v}^{\mathbf{n}-\mathbf{1}} v_c \mathbf{u}^{\mathbf{q}}] u^{K-2} \exp\left\{\sum_{j=1}^{K} v_j(u - u_j)\right\}$$

$$= (\mathbf{L}!)^{-1} \prod_{i=1}^{K} q_i! [\mathbf{u}^{\mathbf{q}}] u^{K-2} \prod_{i=1}^{K} (u - u_i)^{n_i - 1 + \delta_i}.$$

Let $\mathbf{J}_{i,j}$ be the $i \times j$ matrix of all 1's. Then, considering the exclusion of objects from positions, we obtain

$$\chi_c^{(K)}(\mathbf{L}) = (\mathbf{L}!)^{-1} \operatorname{per}(\mathbf{J}_{N,N} - \mathbf{Q})$$

where $\mathbf{Q}$ is formed by appending $K - 2$ rows of zeros to the matrix $\bigoplus_{i=1}^{K} \mathbf{J}_{n_i - 1 + \delta_{ic}, q_i}$. Expanding the permanent of a sum,

$$\operatorname{per}(\mathbf{J}_{N,N} - \mathbf{Q}) = \sum_{k=0}^{N} (-1)^k (N - k)! s_k,$$

since $\operatorname{per} \mathbf{J}_{i,i} = i!$, where

$$s_k = \sum_{\left\{\substack{\alpha, \beta \subset \{1, \ldots, N\} \\ |\alpha| = |\beta| = k}\right.} \operatorname{per} \mathbf{Q}[\alpha|\beta]$$

$$= \sum_{\left\{\substack{m_1 + \ldots + m_K = k \\ m_i \geqq 0,\, i = 1, \ldots, K}\right.} \prod_{i=1}^{K} \left\{\binom{q_i}{m_i} \binom{n_i - 1 + \delta_{ic}}{m_i} m_i!\right\}$$

since, for a non-zero contribution to the permanent, we must select m_i rows and m_i columns from the i-th block of the direct sum, a selection which may be carried out in $\binom{q_i}{m_i}\binom{n_i - 1 + \delta_{ic}}{m_i}$ ways. The remaining factor, $m_i!$, comes from the permanent of the submatrix constructed in this way. Thus $s_k = [x^k]P(x)$ and the result follows.

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It follows immediately from Theorem 4.2 that

$$\chi_1^{(2)}(\mathbf{L}) = n_1!(n_2 - 1)!(\mathbf{L}!)^{-1},$$

the 2-chromatic result of Tutte [**5**].

Theorem 4.2 allows us to conclude that $\chi_c^{(K)}(\mathbf{L})$ may be computed in time $O(M(\log M)^2)$ where M is the number of non-root vertices.

5. A generalisation of the BEST theorem. Let $\tau_c^{(K)}(\mathbf{D})$ be the number of out-directed spanning arborescences, rooted at c, of a graph with adjacency matrix $\mathbf{D}$. Then by the matrix tree theorem ([**2**], [**6**]) we have

$$\tau_c^{(K)}(\mathbf{D}) = \operatorname{cof}_{cc}[\delta_{ij}n_i - d_{ij}].$$

This result allows us to identify the determinant which is involved in Theorem 3.2.

THEOREM 5.1.

$$\theta_{c,1}^{(K)}(\mathbf{L}, \mathbf{D}) = (\mathbf{L}!\mathbf{D}!)^{-1}\left\{\prod_{i=1}^{K} q_i!(n_i - 1 + \delta_{ic})!\right\}\tau_c^{(K)}(\mathbf{D}).$$

Proof. This follows directly from Theorem 3.2.

We now derive the BEST theorem [**1**], a correspondence between Eulerian dicircuits and spanning arborescences, from Theorem 5.1.

THEOREM 5.2. *Let $\mathscr{G}$ be a digraph on the vertex set $\{1, \ldots, K\}$ with in-degree* (i) = *out-degree* (i) = k_i *for* $i = 1, \ldots, K$, *e Eulerian dicircuits and t_c out-directed spanning arborescences rooted at c. Then*

$$e = (\mathbf{k} - \mathbf{1})!t_c \quad \text{for any} \quad c = 1, \ldots, K.$$

Proof. Let $\mathbf{D}$ be the adjacency matrix of $\mathscr{G}$. Let n_{cc} be the number of sequences over $\{1, \ldots, K\}$ which begin and end with c, with d_{ij} occurrences of the substring ij for $i, j = 1, \ldots, K$ and thus

$$\sum_{j=1}^{K} d_{ij} = \sum_{j=1}^{K} d_{ji} = k_i$$

occurrences of i for $i \neq c$ and

$$\sum_{j=1}^{K} d_{cj} - 1 = \sum_{j=1}^{K} d_{jc} - 1 = k_c - 1$$

non-terminal occurrences of c.

It is immediate that $e = \mathbf{D}!k_c^{-1}n_{cc}$ since edges are distinct. A sequence in the set counted by n_{cc} is a plane planted tree rooted at a vertex of colour c such that, in the notation of Theorem 5.1 $l_{i2} = k_i$, $l_{ij} = 0, j \neq 2,$

$i \neq c$; $l_{c2} = k_c - 1$, $l_{c1} = 1$ and $l_{cj} = 0$, $j \geqq 3$. Thus, we have $r = 1$, $q_i = k_i$, $i \neq c$; $q_c = k_c - 1$ and $n_i = k_i$, $i = 1, \ldots, K$. The result follows directly from Theorem 5.1.

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