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A formula for the exact number of primes below a given bound in any arithmetic progression

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The formula of [E.] Meissel [*Math. Ann.* 2 (1870), 636-642] is generalized to arbitrary arithmetic progressions. Meissel's formula is applicable not only to computation of $\pi(x)$ for large x (recently $x = 10^{13}$), but also is a sieve technique (see MR36#2548), useful for studying the subtle effect of primes less then or equal to $x^{1/2}$ on the behavior of primes less than or equal to x. The same is true of the generalized Meissel, with the added advantage that the behavior of primes less than or equal to x can be studied in arbitrary progressions.

1. Introduction and summary

Throughout $bn + c$ will denote an arithmetic progression with $1 \le c < b$ and, unless otherwise specified, with $(b, c) = 1$. Moreover, $\psi_{b,c}(x, k)$ will denote the number of integers greater than 1 and less than or equal to x which are congruent to $c \pmod{b}$ and are relatively prime to the first k primes; $\pi_{b,c}(x)$ will denote the number of positive primes congruent to $c \pmod{b}$ and less than or equal to x, and p_r, $r \ge 1$, will denote the r-th prime. Finally, we let $m = \pi\left(x^{1/3}\right)$, $n = \pi\left(x^{1/2}\right)$, $s = n - m$, and when $x < 8$, since $m = 0$, we define $p_0 = 1$.

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Letting $\phi(x, k)$ denote the number of positive integers less than or equal to x which are relatively prime to the first k primes, that is, $\phi(x, k) = \psi_{2,1}(x, k) + 1$, Meissel [5] obtained a well-known formula which enables one to calculate the exact number of positive primes less than or equal to x when all that is known are the individual primes $q \le x^{1/2}$, namely,

$$(1.1) \qquad \pi(x) = \phi(x, m) + m(s+1) + s(s-1)/2 - \sum_{\sigma=1}^{s} \pi\left(\frac{x}{p_{m+\sigma}}\right) - 1 .$$

Meissel's method is outlined in several books; see, for example, [7]. Proofs of (1.1) have been given by Baranowski [1], Brauer [2], Lugli [4], and Rogel [6].

In Section 2 of this paper we derive a formula which enables one to find the exact number of primes less than or equal to x in any arithmetic progression $bn + c$ when all that is known are the individual primes $q \le x^{1/2}$. In particular, letting c' be the unique positive integer less than b such that $c'\left(p_{m+\sigma}\right) \equiv c \pmod b$ for each σ between 1 and s for which $\left(p_{m+\sigma}, b\right) = 1$, we show that for every real $x > 0$,

$$(1.2) \quad \pi_{b,c}(x) = \psi_{b,c}(x, m) - \sum_{\substack{1\le\sigma\le s \\ (p_{m+\sigma},b)=1}} \pi_{b,c'}\left(\frac{x}{p_{m+\sigma}}\right) + \sum_{\substack{1\le\sigma\le s \\ (p_{m+\sigma},b)=1}} \pi_{b,c'}\left(p_{m+\sigma}\right) + \pi_{b,c}\left(p_m\right) - \sum_{\substack{1\le\sigma\le s \\ p_{m+\sigma}\equiv c' \pmod b}} 1$$

Note that setting $b = 2$ and $c = 1$ in (1.2) gives (1.1), excluding the prime 2, since the third and fourth terms on the right hand side of (1.2) sum to $m(s+1) + s(s-1)/2 - 1$ while the first, second, and fifth terms sum to $\phi(x, m) - \sum_{\sigma=1}^{s} \pi\left(\frac{x}{p_{m+\sigma}}\right) - 1$. Setting $b = 4$, $c = 1$ or 3 and setting $b = 6$, $c = 1$ or 5, we obtain from (1.2) the generalization of Meissel's result given by the author and Brauer [3].

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2. The proof of (1.2)

By definition, there are $\psi_{b,c}(x, k-1)$ integers $t \equiv c \pmod{b}$ with $1 < t \le x$ and $(t, p_1 p_2 \cdots p_{k-1}) = 1$.

If $x < (p_{k+1})^3$ and $(p_k, b) = 1$, exactly $\psi_{b,c'}\left(\frac{x}{p_k}, k-1\right)$ of these integers are divisible by p_k where c' is the unique positive integer less than b with $c'p_k \equiv c \pmod{b}$, and it follows that for each such $k > 1$,

$$\psi_{b,c}(x, k) = \psi_{b,c}(x, k-1) - \psi_{b,c'}\left(\frac{x}{p_k}, k-1\right), \tag{2.1}$$

since the integers included in the calculation of $\psi_{b,c}(x, k-1)$ and excluded in the calculation of $\psi_{b,c}(x, k)$ are precisely the integers of the form $\gamma p_k \le x$ for which γ is congruent to $c' \pmod{b}$ and has no prime factor less than p_k. Each such integer arises exactly once in the calculation of $\psi_{b,c'}\left(\frac{x}{p_k}, k-1\right)$.

Noting, however, that $x < (p_{m+1})^3$ since $m = \pi\left(x^{1/3}\right)$, and that $\psi_{b,c}(x, k) = \psi_{b,c}(x, k-1)$ if $(p_k, b) > 1$, we can sum from $m+1$ to n to obtain

$$\psi_{b,c}(x, n) = \psi_{b,c}(x, m) - \sum_{\substack{1 \le \sigma \le s \\ (p_{m+\sigma}, b) = 1}} \psi_{b,c'}\left(\frac{x}{p_{m+\sigma}}, m+\sigma-1\right). \tag{2.2}$$

Moreover, it is clear that

$$x^{1/3} < p_{m+\sigma} \le x^{1/2} \le \frac{x}{p_{m+\sigma}} < x^{2/3}, \quad \sigma = 1, 2, \ldots, s, \tag{2.3}$$

so that if d and f are positive integers with $d \le f \le d^2$, the integers greater than 1 and less than or equal to f which are congruent to $c \pmod{b}$ and are relatively prime to each prime less than or equal to d are the $\pi_{b,c}(f) - \pi_{b,c}(d)$ primes q for which $d \le q \le f$.

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Consequently, for each b and c with $1 \le c < b$, we have

$$\psi_{b,c}(x,\ n) = \pi_{b,c}(x) - \pi_{b,c}\left(x^{1/2}\right) . \tag{2.4}$$

Let $f = \dfrac{x}{p_{m+\sigma}}$ and $d = p_{m+\sigma} - \varepsilon$ where $0 < \varepsilon < \min\left(1,\ p_{m+\sigma}-x^{1/3}\right)$.

Then we have

$$\psi_{b,c}\left(\frac{x}{p_{m+\sigma}},\ m+\sigma-1\right) = \pi_{b,c}\left(\frac{x}{p_{m+\sigma}}\right) - \pi_{b,c}\left(p_{m+\sigma}-\varepsilon\right) . \tag{2.5}$$

Since $\pi_{b,c}(x) = \psi_{b,c}(x,\ n) + \pi_{b,c}\left(x^{1/2}\right)$, by (2.4), we obtain from (2.2),

$$\pi_{b,c}(x) = \psi_{b,c}(x,\ m) - \sum_{\substack{1\le\sigma\le s \\ (p_{m+\sigma},b)=1}} \psi_{b,c'}\left(\frac{x}{p_{m+\sigma}},\ m+\sigma-1\right) + \pi_{b,c}\left(x^{1/2}\right) . \tag{2.6}$$

Now $\pi_{b,c'}\left(p_{m+\sigma}-\varepsilon\right) = \pi_{b,c'}\left(p_{m+\sigma}\right) - 1$ if we have $p_{m+\sigma} \equiv c' \pmod b$ and is equal to 0 otherwise. Remembering to replace the primes less than or equal to m and congruent to $c \pmod b$, we have from (2.5),

$$\pi_{b,c}(x) = \psi_{b,c}(x,\ m) - \sum_{\substack{1\le\sigma\le s \\ (p_{m+\sigma},b)=1}} \pi_{b,c'}\left(\frac{x}{p_{m+\sigma}}\right)$$
$$+ \sum_{\substack{1\le\sigma\le s \\ (p_{m+\sigma},b)=1}} \pi_{b,c'}\left(p_{m+\sigma}\right) + \pi_{b,c}\left(p_m\right) - \sum_{\substack{1\le\sigma\le s \\ p_{m+\sigma}\equiv c' (\mathrm{mod}\, b)}} 1 . \tag{2.7}$$

REMARKS. In general x is large enough that $p_{m+1} > b$ so that the restriction $\left(b,\ p_{m+\sigma}\right) = 1$ is unnecessary. Parenthetically, we note that if $p_m > b$, even the restriction $(b,\ c) = 1$ may be lifted since the right hand side of (1.2) is easily seen to reduce to $\pi_{b,c}\left(p_m\right)$ which is 1 or 0 according as c is or is not prime.

3. A brief illustration of the numerical use of (1.2)

Professor Carter Bays has already demonstrated to me the feasibility of using the generalization in [3] to calculate $\pi_{4,1}(x)$, $\pi_{4,3}(x)$, $\pi_{6,1}(x)$,

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and $\pi_{6,5}(x)$ for quite large values of x (say $x > 10^{12}$) as well as its application to related problems (products of two primes) where conventional sieve techniques have failed. I therefore give an abbreviated numerical example for the benefit of the numerically inclined reader.

Consider the calculation of the exact number of primes less than 1000 in the arithmetic progressions $5n + 1$, $5n + 2$, $5n + 3$, and $5n + 4$ (presuming explicit knowledge only of the primes less than or equal to 31).

Note that $m = 4$, $n = 11$, $s = 7$, and since $p_{m+1} > b$, the restriction $(p_{m+\sigma}, b) = 1$ may be lifted. Then, for $c = 1, 2, 3$, and 4 we have

$$(3.1)\quad \pi_{5,c}(1000) = \psi_{5,c}(1000, 4) - \sum_{\sigma=5}^{11} \pi_{5,c'}\left(\frac{x}{p_\sigma}\right) + \sum_{\sigma=5}^{11} \pi_{5,c'}(p_\sigma) + \pi_{5,c}(7) - \sum_{\substack{\sigma=5\\ p_\sigma \equiv c' (\mathrm{mod}\, 5)}}^{11} 1 ,$$

where $c'p_\sigma \equiv c \pmod 5$ for each $\sigma = 5, 6, 7, \ldots, 11$.

Calculation of $\psi_{5,c}(1000, 4)$ for each c by repeated application of (2.1) yields

$$\psi_{5,1}(1000, 4) = \psi_{5,2}(1000, 4) = \psi_{5,3}(1000, 4) = 57 , \quad \psi_{5,4}(1000, 4) = 56 .$$

The fourth term in (3.1) is zero if $c = 1$ or 4, is 1 if $c = 3$, and is 2 if $c = 2$. The fifth term sums to 4 in the calculation of $\pi_{5,1}(1000)$ and to 3 in the calculation of $\pi_{5,4}(1000)$ whereas it is zero otherwise - see remark below.

To complete the calculation of $\pi_{5,1}(1000)$ note that

$$(3.2)\quad \sum_{\sigma=5}^{11} \pi_{5,c'}(p_\sigma) = \pi_{5,1}(11) + \pi_{5,2}(13) + \pi_{5,3}(17) + \ldots + \pi_{5,1}(31) = 13 ,$$

and that

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$$(3.3)\quad \sum_{\sigma=5}^{11} \pi_{5,c'}\left(\frac{x}{p_\sigma}\right) = \pi_{5,1}\left(\frac{1000}{11}\right) + \pi_{5,2}\left(\frac{1000}{13}\right) + \dots + \pi_{5,1}\left(\frac{1000}{31}\right) = 26 \ .$$

Hence $\pi_{5,1}(1000) = 57 - 26 + 13 - 4 = 40$. Similarly, $\pi_{5,4}(1000) = 56 - 27 + 12 - 3 = 38$, $\pi_{5,3}(1000) = 57 - 28 + 12 + 1 = 42$, $\pi_{5,2}(1000) = 57 - 24 + 12 + 2 = 47$.

REMARK. Note that the fifth term in (3.1) has the surprising property that it sums to zero if c is a quadratic non-residue of b whereas it sums to (approximately) $s/\phi(b) = \left(\pi\left(x^{1/2}\right)-\pi\left(x^{1/3}\right)\right)/\phi(b)$ otherwise. The investigation of the precise nature of this term is of considerable theoretical interest in the light of the recent work of Knapowski and Turán on "average preponderance problems". However, this takes us beyond the scope of this elementary exposition.

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