NOTE ON A THEOREM OF MAGNUS

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(Received 3rd March 1969)

To Bernhard Hermann Neumann on his 60th birthday

Communicated by G. E. Wall

Magnus [4] proved the following theorem. Suppose that F is a free group and that X is a basis of F. Let R be a normal subgroup of F and write $G = F/R$. Then there is a monomorphism of F/R' in which

$$xR' \to \begin{pmatrix} xR & 0 \\ t_x & 1 \end{pmatrix} \qquad (x \in X);$$

here the t_x are independent parameters permutable with all elements of G. Later investigations [1, 3] have shown what elements can appear in the south-west corner of these 2×2 matrices. In this form the theorem subsequently reappeared in proofs of the cup-product reduction theorem of Eilenberg and MacLane (cf. [7,8]). In this note a direct group-theoretical proof of the theorems will be given.

Let m be a non-negative integer distinct from 1. If T is a group, T^m denotes the group generated by the m-th powers of the elements of T; in particular if $m = 0$, $T^m = 1$. Let $\Lambda = \boldsymbol{Z}/m\boldsymbol{Z}$ and denote by ΛT the group-ring of T with coefficients in Λ. As above let F be a free group with basis X and let R be a normal subgroup of F. Let $G = F/R$ and let μ be the epimorphism of ΛF onto ΛG induced by the natural epimorphism $a \to aR$ of F onto F/R. Let M be a free ΛG-module having a basis in (1,1) correspondence $x \leftrightarrow t_x$ with X. The Abelian group $R/R'R^m$ can be regarded as a ΛG-module by putting

$$(aR'R^m)^{b\mu} = b^{-1}abR'R^m \qquad (a \in R,\ b \in F).$$

It is well-known that the augmentation ideal of ΛF is a free ΛF-module with basis the set of all $x-1$ $(x \in X)$. The differential notation of Fox [1] will be used, and we write

$$(1) \qquad a = a\varepsilon + \sum_{x\in X}(x-1)\frac{\partial a}{\partial x} \qquad (a \in \Lambda F),$$

* This research was supported by NSF Grant GP-6539.

where ε is the augmentation of ΛF onto Λ. From (1) it follows easily that

$$\frac{\partial(ab)}{\partial x} = (a\varepsilon)\frac{\partial b}{\partial x} + \frac{\partial a}{\partial x} b. \tag{2}$$

A mapping α of F into M is defined by

$$a\alpha = \sum_{x \in X} t_x \left(\frac{\partial a}{\partial x}\right)\mu. \tag{3}$$

Using (2) it is seen that for $a \in F$, $b \in F$,

$$(ab)\alpha = (a\alpha)(b\mu) + b\alpha. \tag{4}$$

Hence the restriction of α to R is a homomorphism of the group R into the additive group M. The kernel of the restriction of α to R contains $R'R^m$ since M is an Abelian group of exponent m. Hence α induces a homomorphism $\bar{\alpha}$ of $R/R'R^m$ into M. In fact $\bar{\alpha}$ is a ΛG-homomorphism, for if $a \in R$ and $b \in F$,

$$\begin{aligned}\{(aR'R^m)^{b\mu}\}\bar{\alpha} &= (b^{-1}ab)\alpha \\ &= \sum_{x \in X} t_x \left(-\frac{\partial b}{\partial x} b^{-1}ab + \frac{\partial a}{\partial x} b + \frac{\partial b}{\partial x}\right)\mu \\ &= (a\alpha)(b\mu),\end{aligned}$$

on account of (2). The above theorem of Magnus is a consequence of the following.

THEOREM. *If A is the augmentation ideal of ΛG and $\bar{\mu}$ is the ΛG-homomorphism of M into A for which $t_x\bar{\mu} = x\mu - 1$,*

$$0 \to R/R'R^m \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\mu}} A \to 0$$

is an exact sequence of ΛG-modules.

For each $g \in G$ choose a fixed element $s_g \in F$ such that $s_g\mu = g$ and $s_1 = 1$. If we write

$$s_g s_h = s_{gh} r_{g,h} \qquad (g \in G,\ h \in G), \tag{5}$$

then $r_{g,h} \in R$ and $r_{1,h} = r_{g,1} = 1$. For each $g \in G$ a Λ-homomorphism θ_g of A into $R/R'R^m$ is defined by putting

$$(h-1)\theta_g = r_{h,g} R'R^m \qquad (h \in G). \tag{6}$$

It is readily verified that $\theta_1 = 0$ and

$$u\theta_{gh} = (u\theta_g)^h (ug)\theta_h \qquad (u \in A); \tag{7}$$

indeed it suffices to verify this when $u+1 \in G$ on account of linearity, and

https://doi.org/10.1017/S1446788700007734 Published online by Cambridge University Press

in this case it is an immediate consequence of the fact that $r_{g,h}R'R^m$ is a cocycle.

Now let N be the set of ordered pairs $(aR'R^m, u)$ with $a \in R$, $u \in A$. We give N the additive group structure of the direct sum of $R/R'R^m$ and A. However N will be given a ΛG-structure different from that of the direct sum. Namely, if $g \in G$, we put

$$(aR'R^m, u)g = \left((aR'R^m)^g(u\,\theta_g), ug\right). \tag{8}$$

The relation (7) ensures that N becomes a ΛG-module with this definition.

It will next be shown that for $a \in F$,

$$\sum_{x \in X}(s_{x\mu}^{-1}xR'R^m, (x-1)\mu)\left(\frac{\partial a}{\partial x}\right)\mu = (s_{a\mu}^{-1}aR'R^m, (a-1)\mu). \tag{9}$$

(Note that this makes sense since $s_{a\mu}^{-1}a \in R$ and $(a-1)\mu \in A$). (9) will be proved by induction on the length of a relative to the basis X. If this is 1, we observe that (9) is clear for $a \in X$; if $a^{-1} \in X$, the left-hand side of (9) is

$$\begin{aligned}
&-(s_{a^{-1}\mu}^{-1}a^{-1}R'R^m, (a^{-1}-1)\mu)(a\mu) \\
&\quad = (ar_{a^{-1}\mu, a\mu}s_{a\mu}^{-1}R'R^m, 1-a^{-1}\mu)(a\mu) && \text{by (5)}\\
&\quad = ((r_{a^{-1}\mu, a\mu}s_{a\mu}^{-1}aR'R^m)\{(1-a^{-1}\mu)\theta_{a\mu}\}, a\mu-1) && \text{by (8)}\\
&\quad = (s_{a\mu}^{-1}aR'R^m, a\mu-1), && \text{by (6)}
\end{aligned}$$

which is the right-hand side. To complete the proof of (9) it suffices to deduce its validity for ab from that for a and b. We have

$$\begin{aligned}
&\sum_{x \in X}(s_{x\mu}^{-1}xR'R^m, (x-1)\mu)\left(\frac{\partial(ab)}{\partial x}\right)\mu \\
&\quad = \sum_{x \in X}(s_{x\mu}^{-1}xR'R^m, (x-1)\mu)\left\{\left(\frac{\partial a}{\partial x}\right)\mu(b\mu) + \left(\frac{\partial b}{\partial x}\right)\mu\right\} && \text{by (2)}\\
&\quad = (s_{a\mu}^{-1}aR'R^m, (a-1)\mu)(b\mu) + (s_{b\mu}^{-1}bR'R^m, (b-1)\mu) && \text{by assumption}\\
&\quad = ((b^{-1}s_{a\mu}^{-1}abR'R^m)((a-1)\mu\theta_{b\mu}), (ab-b)\mu)\\
&\qquad + (s_{b\mu}^{-1}bR'R^m, (b-1)\mu) && \text{by (8)}\\
&\quad = (r_{a\mu, b\mu}(s_{b\mu}^{-1}b)(b^{-1}s_{a\mu}^{-1}ab)R'R^m, (ab-1)\mu) && \text{by (6)}\\
&\quad = (s_{(ab)\mu}^{-1}abR'R^m, (ab-1)\mu), && \text{by (5)}
\end{aligned}$$

as required.

A ΛG-homomorphism γ of M into N is defined by putting

$$t_x\gamma = (s_{x\mu}^{-1}xR'R^m, (x-1)\mu).$$

Thus (3) and (9) show that

$$a\alpha\gamma = \left\{\sum_{x\in X} t_x\left(\frac{\partial a}{\partial x}\right)\mu\right\}\gamma = (s_{a\mu}^{-1}aR'R^m,\ (a-1)\mu).$$

Two special cases should be noted. Firstly, if $a = s_g$, then $a\mu = g$ and we obtain

$$(10) \qquad s_g\alpha\gamma = (1, g-1).$$

Secondly, if $a \in R$, $a\mu = 1$ and $s_{a\mu} = 1$; hence

$$a\alpha\gamma = (aR'R^m, 0) \qquad (a \in R).$$

It follows at once that $\bar{\alpha}$ is a monomorphism.

Next we define a Λ-homomorphism φ of A into M for which $(g-1)\varphi = s_g\alpha$. Then (10) gives

$$u\varphi\gamma = (1, u) \qquad (u \in A).$$

Hence for $a \in R$, $u \in A$,

$$(11) \qquad (a\alpha+u\varphi)\gamma = (aR'R^m, u).$$

We now define a mapping β of N into M by putting

$$(aR'R^m, u)\beta = (aR'R^m)\bar{\alpha}+u\varphi.$$

Thus (11) states that $\beta\gamma$ is the identity mapping.

Finally we prove that $\gamma\beta$ is the identity mapping. Note that β is a ΛG-homomorphism, for by (8)

$$\{(aR'R^m, u)g\}\beta - \{(aR'R^m, u)\beta\}g = u\theta_g\bar{\alpha}+(ug)\varphi-(u\,\varphi)g,$$

and the vanishing of the right-hand side is easily verified in the case when $u+1 \in G$ by applying α to (5). Hence it suffices to prove that $t_x\gamma\beta = t_x$, and this readily follows from the definitions of γ, β and α.

β and γ are therefore ΛG-isomorphisms and so it suffices to prove that the sequence

$$0 \to R/R'R^m \xrightarrow{\bar{\alpha}\gamma} N \xrightarrow{\beta\bar{\mu}} A \to 0$$

is exact. But $\beta\bar{\mu}$ is the mapping of N into A which carries $(aR'R^m, u)$ into u; to see this write $(aR'R^m, u)\nu = u$ and observe that $\gamma\nu = \bar{\mu}$ since $t_x\gamma\nu = (x-1)\mu = t_x\bar{\mu}$. On account of (11) $\bar{\alpha}\gamma$ carries $aR'R^m$ into $(aR'R^m, 0)$. Hence the above sequence is exact and the theorem is proved.

The theorem has numerous consequences. For example Theorem 2.5 of [5] can be deduced from it. We prove this in its local form; cf. [2].

Corollary 1. *Suppose that F is a non-cyclic free group and that R is a non-trivial normal subgroup of F. Suppose that there exist a prime p, an integer $n \geqq 1$ and an element s of F such that* $[a, \underbrace{s, \cdots, s}_{n}] \in R'R^p$ *for all* $a \in R$. *Then the order of sR is a power of p.*

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Suppose that this is false. Let $\tilde{F}$ be the group generated by s and R. For each $i \geqq 0$ let S_i be the subgroup generated by $R'R^p$ and all $[a, \underbrace{s, \cdots, s}_{i}]$ $(a \in R)$. Thus

$$R = S_0 \geqq S_1 \geqq \cdots \geqq S_n = R'R^p.$$

Since s centralizes S_{i-1}/S_i, $\tilde{F}/R'R^p$ is nilpotent. We define a subgroup F_1 as follows. If sR is of finite order, let F_1/R be a subgroup of the group generated by sR of prime order not equal to p; thus $F_1/R'R^p$ is Abelian. If sR is of infinite order let F_1 be the centralizeı of $R/R'R^p$ in $\tilde{F}$. It is easy to see that F_1 is of finite index in $\tilde{F}$, so that $F_1 \neq R$. In either case $F_1 \neq R$ and $F_1/R'R^p$ is Abelian. On account of the hypotheses R is non-cyclic. Hence a basis X_1 of F_1 contains more than one element. We apply the theorem to the basis X_1 of F_1. Thus if x and y are distinct elements of X_1, $[x, y]\alpha = 0$. If $z = [x, y]$, then $xy = yxz$, so by (4),

$$(x\alpha)(y\mu)+y\alpha = (y\alpha)\{(xz)\mu\}+(x\alpha)(z\mu)+z\alpha.$$

Since $x\alpha = t_x$, $y\alpha = t_y$ and $z\alpha = 0$, this reduces to

$$t_x(y\mu)+t_y = t_y(x\mu)+t_x,$$

whence $x\mu = 1$ and $x \in R$. Hence $X_1 \subseteqq R$ contrary to $R \neq F_1$.

The following consequence of the theorem was deduced from the exact homology sequence by Roquette [6] in his proof of a theorem of Golod and Šafarevič.

COROLLARY 2. *Suppose that G is a finite p-group and that $(G : G'G^p) = p^d$. Suppose that F is a free group of rank d and that G is isomorphic to F/R. Then the augmentation ideal of $(\boldsymbol{Z}/p\boldsymbol{Z})G$ is isomorphic to M/K, where M is a free $(\boldsymbol{Z}/p\boldsymbol{Z})$ G-module of rank d and K is a submodule generated by r elements, where $p^r = |H_2(G, \boldsymbol{Z}/p\boldsymbol{Z})|$.*

Since $(G : G'G^p) = p^d$ and d is the rank of F, $R \leqq F'F^p$. Hence $H_2(G, \boldsymbol{Z}/p\boldsymbol{Z})$ is isomorphic to $R/[R, F]R^p$. Hence $R/[R, F]R^p$ is generated by r elements, so $R/R'R^p$ is generated as a $(\boldsymbol{Z}/p\boldsymbol{Z})G$-module by r elements. The result then follows from the theorem at once.

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