A GENERALIZED TAUBERIAN THEOREM

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Let $\{s(n)\}$ be a real sequence and let x be any number in the interval $0 < x \leqslant 1$. Representing x by a non-terminating binary decimal expansion we shall denote by $\{s(n,x)\}$ the subsequence of $\{s(n)\}$ obtained by omitting $s(k)$ if and only if there is a 0 in the kth decimal place in the expansion of x. With this correspondence it is then possible to speak of "a set of subsequences of the first category," "an everywhere dense set of subsequences," and so on.

Suppose that T is a regular summability transform given by the matrix (a_{mn}) and let $t(m,x) = \sum a_{mn}s(n,x)$. In a previous note (3), extending a theorem of Buck **(2)**, we proved that a real sequence $\{s(n)\}$ is convergent if there exists a T which sums a set of subsequences of the second category. Our object now is to generalize this Tauberian theorem to the following:

THEOREM. *Suppose that $\{s(n)\}$ is a real sequence and there is a T such that*

$$\limsup t(m,x) - \liminf t(m,x) < \epsilon$$

in a set of the second category. Then

$$\limsup s(n) - \liminf s(n) < \epsilon.$$

The possibility of such a generalization of a Tauberian theorem has been pointed out by Bowen and Macintyre **(1)**.

We first show that, under the hypothesis of the theorem, $\{s(n)\}$ is bounded. Suppose, on the contrary, that $\{s(n)\}$ is unbounded. In **(3)** we proved that when $\{s(n)\}$ is unbounded then, on the one hand, if (a_{mn}) has infinitely many rows of finite length, $\limsup t(m,x) - \liminf t(m,x)$ is finite only in a set of the first category and, on the other hand, if (a_{mn}) has only a finite number of rows of finite length, $\{s(n,x)\}$ is in the domain of T only in a set of the first category. In either case we have a contradiction and it follows that $\{s(n)\}$ is bounded. We may now prove the conclusion of the theorem with the added hypothesis that $\{s(n)\}$ is bounded. Under this hypothesis, by the following lemma, we may further assume that (a_{mn}) is row finite.

LEMMA 1. *Given a regular transform T with matrix (a_{mn}) we can find a transform with a row finite matrix (a_{mn}') such that, for every bounded sequence $\{s(n)\}$,*

$$\sum a_{mn}s(n) - \sum a'_{mn}s(n) \to 0.$$

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Suppose that $|s(n)| < K$. For each m choose k_m so that

$$\sum_{n=k_m}^{\infty} |a_{mn}| < \frac{1}{m}$$

and define $a_{mn}' = a_{mn}$ for $n < k_m$, $a_{mn}' = 0$ for $n \geqslant k_m$. Then

$$\left| \sum a_{mn}s(n) - \sum a'_{mn}s(n) \right| = \left| \sum_{n=k_m}^{\infty} a_{mn}s(n) \right| < \frac{K}{m} \to 0.$$

We next prove

LEMMA 2. *Let $\{v(n)\}$ be any sequence of* 0's *and* 1's *containing an infinity of both* 0's *and* 1's. *Then for any integers p, N and any regular row finite matrix (a_{mn}), there is a subsequence $\{v(j_n)\}$ such that*

(i) $$\limsup \sum a_{mn}v(j_n) - \liminf \sum a_{mn}v(j_n) \geqslant 1,$$

(ii) $$j_p > N.$$

The subsequence described in the lemma is obtained in the following way. Writing $a_{mn} = a(m,n)$, let $a(m,N(m))$ be the last non-zero number in the mth row of (a_{mn}). We first choose m_1 so that

$$\left| \sum_{n=1}^{N(m_1)} a(m_1,n) - 1 \right| < 1$$

and start the subsequence with $N(m_1)$ 1's. We then choose m_2, with $N(m_2) > N(m_1)$, so that

$$\left| \sum_{n=1}^{N(m_1)} a(m_2,n) \right| < \tfrac{1}{2},$$

and continue the subsequence with $N(m_2) - N(m_1)$ 0's. At the kth stage, if k is odd, we choose m_k so that $N(m_k) > N(m_{k-1})$ and

$$\left| \left(\sum_{n=1}^{N(m_1)} + \sum_{N(m_2)+1}^{N(m_3)} + \sum_{N(m_4)+1}^{N(m_5)} + \ldots + \sum_{N(m_{k-1})+1}^{N(m_k)} \right) a(m_k,n) - 1 \right| < \frac{1}{k}.$$

We then continue the subsequence with $N(m_k) - N(m_{k-1})$ 1's. If k is even we choose m_k so that $N(m_k) > N(m_{k-1})$ and

$$\left| \left(\sum_{n=1}^{N(m_1)} + \sum_{N(m_2)+1}^{N(m_3)} + \ldots + \sum_{N(m_{k-2})+1}^{N(m_{k-1})} \right) a(m_k,n) \right| < \frac{1}{k}.$$

We then continue the subsequence with $N(m_k) - N(m_{k-1})$ 0's. The possibility of this construction is ensured by the facts that, (a_{mn}) being regular,

$$\lim_{m\to\infty} \sum a_{mn} = 1, \quad \lim_{m\to\infty} a_{mn} = 0.$$

Plainly the subsequence $\{v(j_n)\}$ so constructed satisfies the inequality (i). It is obvious, moreover, since $\{v(n)\}$ contains an infinity of both 0's and 1's, that given any integers p,N, we may choose j_p so that $j_p > N$, the inequality (ii).

We now proceed to the proof of the equivalent of the theorem: if

$$\limsup s(n) - \liminf s(n) \geqslant \epsilon,$$

then $\limsup t(m,x) - \liminf t(m,x) < \epsilon$ only in a set of the first category. We prove first that the set D of x such that $\limsup t(m,x) - \liminf t(m,x) \geqslant \epsilon$ is everywhere dense.

Let $\liminf s(n) = L$, $\limsup s(n) - \liminf s(n) = H$ and define

$$u(n) = \frac{1}{H}(s(n) - L). \tag{1}$$

Then $\limsup u(n) = 1$, $\liminf u(n) = 0$ and we can choose two subsequences $\{u(k_n)\}$, $\{u(p_n)\}$, such that $\lim u(k_n) = 1$, $\lim u(p_n) = 0$, and $k_i \neq p_j$ for all i, j. Let $\{u(i_n)\}$ be the subsequence of $\{u(n)\}$ obtained by combining these two subsequences, arranging them so that the suffixes are in ascending order. Now let $\{v(i_n)\}$ be defined by $v(i_n) = 1$ if $i_n = k_j$ for some j, $v(i_n) = 0$ if $i_n = p_j$ for some j. Then

$$\lim\,(v(i_n) - u(i_n)) = 0. \tag{2}$$

By Lemma 2 (i), since it is a sequence of 0's and 1's and contains an infinity of both 0's and 1's, $\{v(i_n)\}$ has a subsequence $\{v(j_n)\}$, say, such that

$$\limsup \sum a_{mn} v(j_n) - \liminf \sum a_{mn} v(j_n) \geqslant 1. \tag{3}$$

By Lemma 2 (ii), moreover, given p and any subsequence $\{s(q_n)\}$ of $\{s(n)\}$ we may choose j_p so that $j_p > q_{p-1}$, and then $\{s(r_n)\} \equiv s(q_1), s(q_2), \ldots s(q_{p-1}), s(j_p), s(j_{p+1}), \ldots$ is a subsequence of $\{s(n)\}$. By varying $\{s(q_n)\}$ and p, we obtain an everywhere dense set of subsequences, whose representative points will be shown to lie in D. In fact, since

$$\lim_{m\to\infty} a_{mn} = 0,$$

we have by (1) and (2)

$$\limsup \sum a_{mn} s(r_n) = \limsup \sum a_{mn} s(j_n) = \limsup \sum a_{mn}(Hu(j_n) + L)$$
$$= \limsup \sum a_{mn}(Hv(j_n) + L)$$

and similar equalities with lim sup replaced by lim inf. Thus, by (3),

$$\limsup \sum a_{mn} s(r_n) - \liminf \sum a_{mn} s(r_n)$$
$$= H(\limsup \sum a_{mn} v(j_n) - \liminf \sum a_{mn} v(j_n)) \geqslant H \geqslant \epsilon.$$

Finally, let S_n^k, $(k = 1,2, \ldots ; n = 1,2, \ldots)$ denote the set of x such that there exist $\mu,\nu > n$ for which

$$|\, t_\mu(x) - t_\nu(x) \,| > \epsilon - \frac{1}{k}.$$

Since (a_{mn}) is row finite, S_n^k is obviously open and, since it contains D, it is everywhere dense. If

$$x \in \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} S_n^k$$

then

$$\limsup t(m,x) - \liminf t(m,x) \geqslant \epsilon - \frac{1}{k}$$

for all k and so $\limsup t(m,x) - \liminf t(m,x) \geqslant \epsilon$. The set of x for which $\limsup t(m,x) - \liminf t(m,x) < \epsilon$ therefore belongs to

$$\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \mathscr{C} S_n^k$$

and so is of the first category.

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References

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