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CHARACTERISATION OF THE SUB-RIEMANNIAN ISOMETRY GROUPS OF H-TYPE GROUPS

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For a H-type group G, we first give explicit equations for its shortest sub-Riemannian geodesics. We use properties of sub-Riemannian geodesics in G to characterise the isometry group $ISO(G)$ with respect to the Carnot-Carathéodory metric. It turns out that $ISO(G)$ coincides with the isometry group with respect to the standard Riemannian metric of G.

1. Introduction

For a H-type group G, the aim of this paper is to study in detail some properties of its shortest sub-Riemannian geodesics and to give a full characterisation of the isometry group with respect to the Carnot-Carathéodory metric.

The Lie groups of H-type are first introduced by Kaplan in [**11**]. Let G be a *Carnot group* (see [**7**]) of step 2. That is, G is a simply connected Lie group whose Lie algebra $\mathcal{G}$ admits a nilpotent stratification of step 2: $\mathcal{G} = V_1 \oplus V_2$, and $[V_1, V_1] = V_2$, whereas $[V_1, V_2] = 0$. From the definition, the centre of G is $\exp(V_2)$ where exp is the exponential map which is a global diffeomorphism. We assume that a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ is given on G for which V_1, V_2 are mutually orthogonal. We denote by H-type groups the subbundle spanned by the system of left-invariant vector fields $\{X_1, \dots, X_{m_1}\}$ such that $\{X_1, \dots, X_{m_1}\}$ is an orthonormal basis of V_1 where $m_1 = \dim(V_1)$. From the stratification condition and the Chow connectivity theorem ([**6**]), the structure of $\big(H\text{-type groups}, \langle \cdot, \cdot \rangle\big)$ induces the so-called Carnot-Carathéodory metric d_c: for any $p, q \in G$,

$$d_c(p,q) = \inf_{\gamma} \left\{ \int_a^b \mid \dot{\gamma}(s) \mid ds \right\}$$

where the infimum is taken over all *horizontal curves* γ connecting p to q, that is, all absolutely continuous curves joining p and q whose derivatives are in H-type groups almost everywhere. d_c is left-invariant, that is, $d_c(p_0 p, p_0 q) = d_c(p,q)$ for any $p_0, p, q \in G$, and is 1-homogeneous with respect to the natural dilations, that is $d_c(\delta_s p, \delta_s q) = s d_c(p,q)$

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for any $s > 0, p, q \in G$, where $\delta_s p = \exp(s\xi_1 + s^2\xi_2)$ for $p = \exp(\xi_1 + \xi_2), \xi_i \in V_i$. A horizontal curve is called a *sub-Riemannian geodesic* if it locally realises the Carnot-Carathéodory distance. We call G is a *H-type group* if G is a Carnot group of step 2 and moreover its Lie algebra $\mathcal{G}$ satisfies the following statement: for every $\eta \in V_2$, such that $|\eta| = 1$, the map $J(\eta) : V_1 \to V_1$ defined by

$$\langle J(\eta)\xi', \xi''\rangle = \langle [\xi', \xi''], \eta\rangle, \qquad \eta \in V_2, \xi', \xi'' \in V_1 \tag{1.1}$$

is orthogonal. The simplest H-type group is the Heisenberg group $\mathbb{H}^n$ (see [**23**]) which is, by definition, simply $\mathbb{R}^{2n+1}$, with the noncommutative group law

$$pp' = (x, y, t)(x', y', t') = \Big(x + x', y + y', t + t' + \frac{1}{2}\big(\langle x', y\rangle - \langle x, y'\rangle\big)\Big) \tag{1.2}$$

where we have let $x, x', y, y' \in \mathbb{R}^n, t, t' \in \mathbb{R}$. A simple computation shows that the left-invariant vector fields

$$X_j(p) = \frac{\partial}{\partial x_j} + \frac{y_j}{2}\frac{\partial}{\partial t}, X_{n+j}(p) = \frac{\partial}{\partial y_j} - \frac{x_j}{2}\frac{\partial}{\partial t}, \quad j = 1, \ldots, n,$$

and $T = \dfrac{\partial}{\partial t}$ span the Lie algebra $(\mathbb{R}^{2n+1})$ of $\mathbb{H}^n$. Moreover $[X_j, X_{n+k}] = -T\delta_{jk}, j, k = 1, \ldots, n$, and all other commutators are trivial. Note that for the Heisenberg group $\mathbb{H}^n$ which is endowed with a Riemannian metric $\langle\cdot,\cdot\rangle$ such that $\{X_1, \ldots, X_{2n+1}, T\}$ is an orthonormal basis, the map J defined by (1.1) can be explicitly written:

$$\begin{aligned} J(T)X_i &= -X_{n+i}, \\ J(T)X_{n+i} &= X_i \end{aligned} \tag{1.3}$$

for $i = 1, \ldots, n$.

H-type groups appear naturally in the Iwasawa decomposition of semisimple Lie groups of real rank one. Since they were introduced in [**11**] by Kaplan, many authors have contributed to analysis and geometry on these groups, see [**5, 12, 13, 14, 15, 16, 20**]. In [**12**] Kaplan studied the Riemannian geodesics and characterised the isometry group with respect to the Riemannian metric $\langle\cdot,\cdot\rangle$. In [**13**, p. 33–p. 35] Korányi gave an explicit description of sub-Riemannian geodesics. What we are interested in is how to characterise *shortest* sub-Riemannian geodesics. In fact, in analysis on H-type groups the most useful information for sub-Riemannian geodesics is the explicit equations for shortest sub-Riemannian geodesics, see [**2, 3, 20, 24**] for applications in the Heisenberg group. In the case of the Heisenberg group, [**20**] listed without proof the explicit equations for the shortest sub-Riemannian geodesics, and [**2, 3**] independently gave proofs. But the proofs of [**2, 3**] are not trivial. In this paper, we present a direct and simpler proof even for H-type groups and explicitly give equations of shortest sub-Riemannian geodesics (see Theorem 2.3).

Next we shall use properties of sub-Riemannian geodesics to give a full characterisation of the isometry group of sub-Riemannian isometries. A *sub-Riemannian isometry* of G is a map $f : G \to G$ such that $d_c\big(f(p), f(q)\big) = d_c(p,q)$ for any $p, q \in G$. Note that in this case, since d_c is not smooth (see for example [**9, 4, 2**]), we cannot use the method in [**12**]. We shall prove that the sub-Riemannian isometry group coincides with the isometry group for the standard Riemannian metric $\langle\cdot,\cdot\rangle$ (see Theorem 3.4 and Theorem 3.5). Our proof essentially depends on two facts on shortest sub-Riemannian geodesics (see Corollary 2.5). One is that a geodesic is globally shortest if and only if it is a ray. The other is that there are infinitely many shortest geodesics connecting two given points p, q if and only if $p^{-1}\cdot q$ is in the centre of G.

NOTATIONS. The letter G will always represent a H-type group. We use $p, q, p', q', p_0, q_0, \dots$ to denote elements in G; adopt $\xi, \xi', \xi^0, W, W^0, \dots$ to denote elements in $\mathcal{G}$ and ξ_1, ξ_1', ξ_1^0 elements in V_1 while ξ_2, ξ_2', ξ_2^0 in V_2. We shall write $p = \big(\xi_1(p), \xi_2(p)\big)$ or $p = (\xi_1, \xi_2)$ when no confusion will be caused. The unit element of G is denoted by 0. Let $G^* := G\backslash \exp(V_2)$ be the set of all elements of the form $p = (\xi_1(p), 0)$. If $p \in G^*$ we shall sometimes use sp to denote $\delta_s p$.

2. PROPERTIES OF SUB-RIEMANNIAN GEODESICS

This section is devoted to studying some properties of sub-Riemannian geodesics in H-type groups. The equations of sub-Riemannian geodesics can be easily deduced from the Maximum Principle of Optimal Control Theory. That is, every sub-Riemannian geodesic must satisfy a Hamiltonian equation determined by the horizontal bundle H-type groups. It is clear that every sub-Riemannian geodesic is smooth (see for example[**19**]). In [**13**] Korányi also found the equations of sub-Riemannian geodesics by minimising the arc length functional among the curve family of horizontal curves joining two given points. The two methods are equivalent. The existence of shortest sub-Riemannian geodesics can be easily inferred from [**8**, Theorem 1.10]. What we are concerned with is the uniqueness of shortest sub-Riemannian geodesics. For more on the theory of sub-Riemannian geodesics in general sub-Riemannian manifolds we refer to the book [**21**].

Our theorem is based on the following statement developed by Korányi in [**13**].

PROPOSITION 2.1. (Equations of sub-Riemannian geodesics.) *Given a point* $p_0 = (\xi_1^0, \xi_2^0)$ $(p_0 \neq 0)$ *in* G, *the equations of sub-Riemannian geodesics* $\gamma(s) = \big(\xi_1(s), \xi_2(s)\big)$, $s \in [0,1]$ *connecting* 0 *to* p_0 *are:*

(1) *If* $\xi_2^0 = 0$, *then*

$$\xi_1(s) = s\xi_1^0, \quad \xi_2(s) = 0. \tag{2.1}$$

(2) *If* $\xi_2^0 \neq 0$, *then, with the notation* $T_0' = \xi_2^0/|\xi_2^0|$,

$$\xi_1(s) = (\cos(s\tau) - 1)W_0 + \sin(s\tau)(J(T_0')W_0),$$

$$\xi_2(s) = \frac{1}{2}(s\tau - \sin(s\tau))|W_0|^2 T_0' \tag{2.2}$$

where τ is a positive solution of

$$\frac{1-\cos\tau}{\tau - \sin\tau} = \frac{|\xi_1^0|^2}{4|\xi_2^0|}. \tag{2.3}$$

and W_0 is determined in the case $\xi_1^0 \neq 0$ by

$$\xi_1^0 = (\cos\tau - 1)W_0 + \sin\tau(J(T_0')W_0) \tag{2.4}$$

while in the case $\xi_1^0 = 0$, W_0 is subject only to the condition

$$2|\xi_2^0| = (\tau - \sin\tau)|W_0|^2 \tag{2.5}$$

and otherwise arbitrary. The length of the sub-Riemannian geodesics is

$$\tau|W_0|. \tag{2.6}$$

Let $\mu(\tau) = (1-\cos\tau)/(\tau - \sin\tau)$. We consider the distribution of solutions in $[0,\infty)$ of the the equation

$$\mu(\tau) = c \tag{2.7}$$

where $c \in [0,\infty)$. The following lemma is elementary but paramountly important for the proof of Theorem 2.3.

Lemma 2.2. *For $0 \leqslant c < \infty$, we have*

(1) *if $c = 0$, the solutions of (2.7) are $\tau = 2k\pi, k = 1, 2, \ldots$.*

(2) *if $c > 0$, then (2.7) has finitely many solutions and all of them are in $(0,+\infty)$. Moreover, (2.7) has only one solution in $(0, 2\pi)$ if and only if $f\big(2(\pi-\theta)\big) < 0$ where $\theta = \arctan(-(1/c))$ and $f(\tau) = \sin(\tau+\theta) - \tau\cos\theta - \sin\theta$. Finally, if $f\big(2(\pi-\theta)\big) \geqslant 0$, then the least solution must satisfy $\tau_1 \in \big[\pi-\theta, (3/2)\pi - \theta\big]$.*

Proof: (1) and the first part of (2) are trivial. Since

$$\dot\mu(\tau) = \frac{4\sin(\tau/2)\cos(\tau/2)((\tau/2) - \tan(\tau/2))}{(\tau - \sin\tau)^2}$$

$\mu(\tau)$ is decreasing on $(0, 2\pi)$. From $\lim_{\tau\to 0+}\mu(\tau) = +\infty$, $\mu(2\pi) = 0$ we deduce that the equation (2.7) has exactly one solution in $[0, 2\pi]$.

Let $c > 0$ and $\theta = \arctan\bigl(-(1/c)\bigr) \in \bigl(-(\pi/2), 0\bigr)$, then (2.7) can be rewritten as

$$\sin(\theta + \tau) = \tau\cos\theta + \sin\theta$$

Let $f(\tau) = \sin(\tau + \theta) - \tau\cos\theta - \sin\theta$. We note that $f(2\pi) = -2\pi\cos\theta < 0$ and $\dot f(\tau) = \cos(\tau + \theta) - \cos\theta > 0$ whenever τ in $\bigl(2\pi, 2(\pi - \theta)\bigr)$. Thus by Rolle's Theorem (2.7) has exactly one solution in $(0, 2\pi)$ if and only if $f\bigl(2(\pi - \theta)\bigr) < 0$.

If $f\bigl(2(\pi - \theta)\bigr) = -2\bigl((\pi - \theta)\cos\theta + \sin\theta\bigr) \geqslant 0$ then $f(\pi - \theta) = -\bigl((\pi - \theta)\cos\theta + \sin\theta\bigr) \geqslant 0$. Let $g(\theta) = f\bigl((3/2)\pi - \theta\bigr) = -1 - \bigl((3/2)\pi - \theta\bigr)\cos\theta - \sin\theta$. Since $\dot g(\theta) = \bigl((3/2)\pi - \theta\bigr)\sin\theta < 0$ whenever $\theta \in \bigl(-(\pi/2), 0\bigr)$, $g(\theta) < g\bigl(-(\pi/2)\bigr) = -2\pi < 0$. We get $\tau_1 \in \bigl[\pi - \theta, (3/2)\pi - \theta\bigr]$ again by Rolle's Theorem. ∎

Now we can prove one of the main results in this paper.

THEOREM 2.3. (Equations of shortest sub-Riemannian geodesics.) *Let $p_0 = (\xi_1^0, \xi_2^0)(\neq 0)$ be a point in G with the same notation T_0', as in Proposition 2.1 and let $\gamma(s) = \bigl(\xi_1(s), \xi_2(s)\bigr)$, $s \in [0,1]$ be a shortest sub-Riemannian geodesic connecting 0 to p_0, then*

(1) *if $\xi_2^0 = 0$, the shortest sub-Riemannian geodesic is unique and its equation is (2.1). Its length is $\rho = |\xi_1^0|$.*

(2) *if $\xi_2^0 \neq 0$ and $\xi_1^0 \neq 0$, the shortest sub-Riemannian geodesic is also unique and its equation is*

$$\begin{aligned} \xi_1(s) &= (\cos(s\tau_1) - 1)W_0 + \sin(s\tau_1)(J(T_0')W_0), \\ \xi_2(s) &= \frac{1}{2}\bigl(s\tau_1 - \sin(s\tau_1)\bigr)|W_0|^2 T_0' \end{aligned} \tag{2.8}$$

where $\tau_1 \in (0, 2\pi)$ is the least solution in $(0, +\infty)$ of equation (2.3) and W_0 is determined by (2.4) and (2.5) where τ is replaced by τ_1. Its length is

$$\rho = \sqrt{\frac{2|\xi_2^0|\tau_1^2}{\tau_1 - \sin\tau_1}}. \tag{2.9}$$

(3) *if $\xi_2^0 \neq 0$ and $\xi_1^0 = 0$, there are infinitely many shortest sub-Riemannian geodesics and their equations are*

$$\begin{aligned} \xi_1(s) &= (\cos(2\pi s) - 1)W_0 + \sin(2\pi s)(J(T_0')W_0), \\ \xi_2(s) &= \frac{1}{2}\bigl(2\pi s - \sin(2\pi s)\bigr)|W_0|^2 T_0' \end{aligned} \tag{2.10}$$

where W_0 is only subject to

$$|\xi_2^0| = \pi|W_0|^2. \tag{2.11}$$

The length is

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$$\rho = 2\sqrt{2\pi|\xi_2^0|}. \tag{2.12}$$

PROOF: Since every pair of points in G can be joined by a shortest sub-Riemannian geodesic, (2.3) follows from Proposition 2.1.

If $\xi_2^0 \neq 0$, from (2.5) we deduce that the length of a sub-Riemannian geodesic can be rewritten as

$$\rho = \sqrt{\frac{2|\xi_2^0|\tau^2}{\tau - \sin\tau}}. \tag{2.13}$$

Thus if $\xi_1^0 = 0$, it is obvious that τ corresponding to the shortest sub-Riemannian geodesic is $\tau_1 = 2\pi$. Since there are infinitely many solutions to equation (2.11), (2.3) follows.

If $\xi_1^0 \neq 0$ and $\xi_2^0 \neq 0$, we first note that for fixed τ (2.4) has only one solution in V_2. So in order to prove (2.3) it suffices to prove that the length of the sub-Riemannian geodesic corresponding to $\tau \in (2\pi, +\infty)$ (if such τ exists) is strictly larger than the length of the sub-Riemannian geodesic corresponding to $\tau_1 \in (0, 2\pi)$. When (2.3) has only one solution $\tau_1 \in (0, 2\pi)$, it is obvious. If (2.3) has another solution τ_2 except τ_1, then by Lemma 2.2 $\tau_1 \in \left[\pi - \theta, (3/2)\pi - \theta\right] \subset \left((3/2)\pi, 2\pi\right)$ and hence $\sin\tau_1 < 0$. Let ρ_2 be the length of geodesic corresponding to τ_2 determined by (2.13). In the case $\tau_2 \in (2\pi, 2(\pi - \theta)]$, since $\sin\tau_2 > 0$ and $\sin\tau_1 < 0$ we have

$$\begin{aligned}\rho_2^2 - \rho_1^2 &= 2|\xi_2^0|\frac{\tau_2\tau_1(\tau_2 - \tau_1) + \tau_1^2\sin\tau_2 - \tau_2^2\sin\tau_1}{(\tau_2 - \sin\tau_2)(\tau_1 - \sin\tau_1)} \\ &> 0.\end{aligned}$$

In the case $\tau_2 \in (2(\pi - \theta), +\infty)$, since

$$\begin{aligned}\tau_2 - \tau_1 &> 2(\pi - \theta) - \left(\frac{3}{2}\pi - \theta\right) \\ &= \frac{\pi}{2} - \theta > 1,\end{aligned}$$

we have

$$\begin{aligned}\rho_2^2 - \rho_1^2 &= 2|\xi_2^0|\frac{\tau_2\tau_1(\tau_2 - \tau_1) + \tau_1^2\sin\tau_2 - \tau_2^2\sin\tau_1}{(\tau_2 - \sin\tau_2)(\tau_1 - \sin\tau_1)} \\ &\geqslant 2|\xi_2^0|\frac{\tau_2\tau_1(\tau_2 - \tau_1) - \tau_1^2 - \tau_2^2\sin\tau_1}{(\tau_2 - \sin\tau_2)(\tau_1 - \sin\tau_1)} \\ &> 2|\xi_2^0|\frac{\tau_1(\tau_2 - \tau_1) - \tau_2^2\sin\tau_1}{(\tau_2 - \sin\tau_2)(\tau_1 - \sin\tau_1)} \\ &> 0.\end{aligned}$$

Thus we have finished the proof. ∎

For the Heisenberg group $\mathbb{H}^n$, since the map J can be explicitly written as in (1.3), the following corollary follows immediately from Theorem 2.3.

COROLLARY 2.4. *Let $g_0 = (x_0, y_0, t_0) \neq 0$ be a point in $\mathbb{H}^n$. We have*

(1) *if $x_0^2 + y_0^2 \neq 0$, then there exists a unique shortest sub-Riemannian geodesic connecting 0 to g_0.*

(2) *otherwise, there exist infinitely many shortest sub-Riemannian geodesics connecting 0 to g_0.*

Moreover, let $\gamma(s) = (x(s), y(s), t(s))(0 \leqslant s \leqslant 1)$ be any shortest sub-Riemannian geodesic connecting 0 to g_0, we have

$$\begin{cases} x_i(s) &= \dfrac{A_i(\cos(s\phi\rho) - 1) + B_i \sin(s\phi\rho)}{\phi}, \quad i = 1, \dots, n, \\ y_i(s) &= \dfrac{B_i(\cos(s\phi\rho) - 1) - A_i \sin(s\phi\rho)}{\phi}, \quad i = 1, \dots, n, \\ t(s) &= \dfrac{s\phi\rho - \sin(s\phi\rho)}{2\phi^2}, \end{cases}$$

where $\tau = \phi\rho \in [-2\pi, 2\pi]$ is the unique solution in $[-2\pi, 2\pi]$ of the equation

$$\frac{1 - \cos\tau}{\tau - \sin\tau} = \frac{|x_0|^2 + |y_0|^2}{4t_0} \tag{2.14}$$

with

$$\begin{cases} \tau = 0 & \text{if } t_0 = 0, \\ |\tau| = 2\pi & \text{if } |x_0|^2 + |y_0|^2 = 0, \\ \tau \in (0, 2\pi) & \text{if } t_0 > 0, \\ \tau \in (-2\pi, 0) & \text{otherwise;} \end{cases}$$

$\rho = d_c(0, g_0)$ is the arc length of γ determined by

$$\rho = \sqrt{2\frac{\tau^2 t_0}{(\tau - \sin\tau)}}, \quad \text{if } t_0 \neq 0,$$
$$\rho = \sqrt{|x_0|^2 + |y_0|^2}, \quad \text{if } t_0 = 0;$$

if $|x_0|^2 + |y_0|^2 \neq 0$, $\{A_1, \dots, A_n, B_1, \dots, B_n\}$ is subject to

$$\begin{cases} \sum_{i=1}^{n} (A_i^2 + B_i^2) = 1, \\ x_{0i} = \dfrac{A_i(\cos(\phi\rho) - 1) + B_i \sin(\phi\rho)}{\phi}, \quad i = 1, \dots, n, \\ y_{0i} = \dfrac{B_i(\cos(\phi\rho) - 1) - A_i \sin(\phi\rho)}{\phi}, \quad i = 1, \dots, n; \end{cases} \tag{2.15}$$

if $|x_0|^2 + |y_0|^2 = 0$, then $\{A_1, \dots, A_n, B_1, \dots, B_n\}$ is only subject to

$$\sum_{i=1}^{n} (A_i^2 + B_i^2) = 1.$$

The following corollary, which follows immediately from Theorem 2.3 and the left-invariance of the Carnot-Carathéodory metric d_c, will be used in Section 3.

Corollary 2.5. *γ is a sub-Riemannian geodesic connecting p to q if and only if $p^{-1}\gamma$ is a sub-Riemannian geodesic joining 0 and $p^{-1}q$. Moreover*

(1) *Let $\gamma(s)\big(s \in [0,+\infty)\big)$ be a smooth arc-length parameterised curve emitting from 0. Then γ is a globally shortest geodesic (that is, $s_2 - s_1 = d_c\big(\gamma(s_2),\gamma(s_1)\big)$ for any $s_2 > s_1$ in $[0,+\infty)$) if and only if γ ia a ray, that is, there exists an element $p_0 = (\xi_1^0, 0) \in G^*$ such that $|\xi_1^0| = 1$ and $\gamma(s) = sp_0$ $\big(s \in [0,+\infty)\big)$ where we abuse the notation $sg = \delta_s g$ when $p \in G^*$.*

(2) *Given two different points $p_1, p_2 \in G$, then there are infinitely many shortest geodesics connecting them if and only if $p_1^{-1}p_2 \in \exp(V_2)$, that is $\xi_1(p_1^{-1} \cdot p_2) = 0$.*

3. Characterisation of the Sub-Riemannian Isometry Group

In this section we give a full characterisation of the sub-Riemannian isometry group of a H-type group. Note that we shall not impose any smoothness conditions on an isometry.

We shall use $ISO(G)$ to denote the set of all sub-Riemannian isometries. Note that if f is an sub-Riemannian isometry, then $g = f(0)^{-1}f$ is an sub-Riemannian isometry preserving the unit.

Lemma 3.1.

(1) *Let p_1, p_2 be two different points in G and f be a sub-Riemannian isometry. Then γ is a shortest geodesic connecting p_1 to p_2 if and only if $f(\gamma)$ is a shortest geodesic connecting $f(p_1)$ to $f(p_2)$. In particular, if f is an isometry fixing the unit, then γ is a ray emitting from 0 if and only if $f(\gamma)$ is a ray from 0.*

(2) *If $p \in \exp(V_2)$ and f is an isometry preserving the unit, then $f(p)$ is also in $\exp(V_2)$.*

Proof: Without restriction we assume that all shortest sub-Riemannian geodesics are parameterised by arc length. Let $\gamma(s)\Big(s \in \big[0, d_c(p_1,p_2)\big]\Big)$ be a shortest sub-Riemannian geodesic joining p_1 to p_2. By definition

$$s_2 - s_1 = d_c\big(\gamma(s_1),\gamma(s_2)\big) = d_c\Big(f\big(\gamma(s_1)\big), f\big(\gamma(s_2)\big)\Big) \tag{3.1}$$

for any $s_2 > s_1$ in $\big[0, d_c(p_1,p_2)\big]$. So it follows from Pansu's Theorem on differentiability of Lipschitz functions defined on Carnot groups ([**22**]) that $f(\gamma)$ is horizontal. Thus (3.1) means that $f(\gamma)$ is a shortest geodesic connecting $f(p_1)$ to $f(p_2)$. If f is an isometry preserving the unit and γ is a ray starting from 0, then $f(\gamma)$ is a globally shortest geodesic. By Corollary 2.5, $f(\gamma)$ is a ray from 0. Since the inverse of an isometry is also an isometry, we proved (3.1).

Because of $p \in \exp(V_2)$ there are infinitely many geodesics connecting 0 to p by Corollary 2.5. Let γ be any such geodesic. By (3.1), $f(\gamma)$ is a shortest geodesic connecting 0 to $f(p)$. Thus there are infinitely many shortest geodesics joining 0 and $f(p)$. So $f(p)$ is in $\exp(V_2)$ again by Corollary 2.5. ∎

PROPOSITION 3.2. *Any sub-Riemannian isometry with $f(0) = 0$ can be written as*

$$f(p) = \Big(f_1\big(\xi_1(p)\big), f_2\big(\xi_2(p)\big)\Big)$$

for $p = \big(\xi_1(p), \xi_2(p)\big)$, where $f_1 \in \mathcal{O}(V_1)$, $f_2 \in \mathcal{O}(V_2)$ and $\mathcal{O}(V_i)$ is the orthogonal group of $V_i, i = 1, 2$.

PROOF: Let f be a sub-Riemannian isometry with $f(0) = 0$. By Lemma 3.1, $f(p) \in G^*$ for $p \in G^*$ and $f(p') \in \exp(V_2)$ for $p' \in \exp(V_2)$. Now let p be any point in G and p' be any point in G^*. Let $\gamma(s) = p.sp'$ be a ray joining p and p'. Since $f\big(\gamma(s)\big)$ is a ray joining $f(p) \in G$ and $f(p') \in G^*$, there exist $\widetilde{p} \in G$ and $\widetilde{p}'$ such that $f\big(\gamma(s)\big) = \widetilde{p} \cdot s\widetilde{p}'$. We deduce that

$$f(pp') = f(p)f(p') \quad \text{for any } p \in G, p' \in G^*. \tag{3.2}$$

For f we define two functions f_1 and f_2 on V_1 and V_2 respectively:

$$f_1(\xi_1) := \xi_1\Big(f\big((\xi_1, 0)\big)\Big), \quad f_2(\xi_2) := \xi_2\Big(f\big((0, \xi_2)\big)\Big).$$

Then by (3.2), for any $p' = (\xi_1', \xi_2')$ we have

$$\begin{aligned} f\big((\xi_1', \xi_2')\big) &= f\big((0, \xi_2')(\xi_1', 0)\big) = f\big((0, \xi_2')\big)f\big((\xi_1', 0)\big) \\ &= \Big(0, \xi_2\Big(f\big((0, \xi_2')\big)\Big)\Big)\Big(\xi_1\Big(f\big((\xi_1', 0)\big)\Big), 0\Big) \\ &= \Big(\xi_1\Big(f\big((\xi_1', 0)\big)\Big), \xi_2\Big(f\big((0, \xi_2')\big)\Big)\Big) \\ &= \big(f_1(\xi_1'), f_2(\xi_2')\big). \end{aligned}$$

Thus for $p = \big(\xi_1(p), \xi_2(p)\big)$, we can write

$$f(p) = \Big(f_1\big(\xi_1(p)\big), f_2\big(\xi_2(p)\big)\Big). \tag{3.3}$$

Let $p = (\xi_1, 0)$ be a point in G^*. On one hand by (2.3) in Theorem 2.3, we have $|\xi_1| = d_c\big((\xi_1, 0), 0\big) = d_c(p, 0) = d_c\big(f(p), 0\big) = d_c\Big(\big(f_1(\xi_1), 0\big), 0\Big) = \big|f_1(\xi_1)\big|$. One the other hand by (3.2), f_1 is a linear map from V_1 to V_1. Thus f_1 is an orthogonal transformation in V_1.

Let $p_i = (0, \xi_2^i), i = 1, 2$ be two points in $\exp(V_2)$. By (2.3) in Theorem 2.3 we have

$$\begin{aligned} d_c(f(p_1), f(p_2)) &= d_c\Big((0, f_2(\xi_2^1)), (0, f_2(\xi_2^2))\Big) \\ &= d_c\Big(0, \big(0, f_2(\xi_2^1)\big)^{-1} \cdot \big(0, f_2(\xi_2^2)\big)\Big) \\ &= d_c\Big(0, \big(0, f_2(\xi_2^2) - f_2(\xi_2^1)\big)\Big) \\ &= 2\sqrt{2\pi\big|f_2(\xi_2^2) - f_2(\xi_2^1)\big|} \end{aligned}$$

and

$$\begin{aligned} d_c(p_1, p_2) &= d_c(0, p_1^{-1} \cdot p_2) \\ &= d_c\big(0, (0, \xi_2^2 - \xi_2^1)\big) \\ &= 2\sqrt{2\pi|\xi_2^2 - \xi_2^1|}. \end{aligned}$$

Since $d_c\big(f(p_1), f(p_2)\big) = d_c(p_1, p_2)$, we get $\big|f_2(\xi_2^2) - f_2(\xi_2^1)\big| = |\xi_2^2 - \xi_2^1|$. Thus f_2 is an isometry in V_2. By [**10**, Section 2.3], f_2 is an orthogonal transformation. ∎

Proposition 3.3. *Any isometry with $f(0) = 0$ satisfies that*

$$f(\delta_s p) = \delta_s f(p) \text{ and } f(pp') = f(p)f(p')$$

for any $s > 0$ and $p, p' \in G$.

Proof: In fact by (3.2), (3.3) and Proposition 3.2, we have

$$f(\delta_s p) = f\big((s\xi_1, s^2\xi_2)\big) = \big(f_1(s\xi_1), f_2(s^2\xi_2)\big) = \big(sf_1(\xi_1), s^2 f_2(\xi_2)\big) = \delta_s f(p).$$

and

$$\begin{aligned} f(pp') &= f\big((\xi_1, \xi_2)(\xi_1', \xi_2')\big) = f\big((\xi_1, \xi_2)(0, \xi_2')(\xi_1', 0)\big) \\ &= f\big((\xi_1, \xi_2 + \xi_2')\big) f\big((\xi_1', 0)\big) = \big(f_1(\xi_1), f_2(\xi_2 + \xi_2')\big)\big(f_1(\xi_1'), 0\big) \\ &= \big(f_1(\xi_1), f_2(\xi_2) + f_2(\xi_2')\big)\big(f_1(\xi_1'), 0\big) = \big(f_1(\xi_1), f_2(\xi_2)\big)\big(0, f_2(\xi_2')\big)\big(f_1(\xi_1'), 0\big) \\ &= f(p)\big(f_1(\xi_1'), f_2(\xi_2')\big) = f(p)f(p') \end{aligned}$$

for any $p = (\xi_1, \xi_2), p' = (\xi_1', \xi_2')$ in G and any $s > 0$. ∎

Now we can prove another of the main results in this paper.

Theorem 3.4. *Let f be a map from G to G with $f(0) = 0$. Then f is an isometry if and only if*

(3.4) $$f(p) = \big(f_1(\xi_1), f_2(\xi_2)\big), f_1 \in \mathcal{O}(V_1), f_2 \in \mathcal{O}(V_2)$$

and

(3.5) $$J\big(f_2(\xi_2)\big)\big(f_1(\xi_1)\big) = f_1\big(J(\xi_2)(\xi_1)\big)$$

for any $p = (\xi_1, \xi_2) \in G$.

PROOF: Let's first show that for f satisfying (3.4) with $f(0) = 0$, (3.5) is equivalent to the fact that f is a group homomorphism. To this aim, let $p = (\xi_1, \xi_2), p' = (\xi_1', \xi_2')$ and let f be satisfying (3.4) for $f_1 \in \mathcal{O}(V_1)$ and $f_2 \in \mathcal{O}(V_2)$. By the Baker–Hausdorff–Campbell formula

$$pp' = \Big(\xi_1 + \xi_1', \xi_2 + \xi_2' + \frac{1}{2}[\xi_1, \xi_1']\Big),$$

we have

$$f(pp') = \Big(f_1(\xi_1) + f_1(\xi_1'), f_2(\xi_2) + f_2(\xi_2') + \frac{1}{2} f_2([\xi_1, \xi_1'])\Big)$$

and

$$f(p)f(p') = \Big(f_1(\xi_1) + f_1(\xi_1'), f_2(\xi_2) + f_2(\xi_2') + \frac{1}{2}[f_1(\xi_1), f_2(\xi_1')]\Big).$$

Thus f is a group homomorphism if and only if

$$[f_1(\xi_1), f_1(\xi_1')] = f_2([\xi_1, \xi_1']). \tag{3.6}$$

If f satisfies (3.5), recalling (1.1) we obtain

$$\begin{aligned}
\Big\langle \xi_2, f_2^{-1}[f_1(\xi_1), f_1(\xi_1')] \Big\rangle &\overset{(3.4)}{=} \Big\langle f_2(\xi_2), [f_1(\xi_1), f_1(\xi_1')] \Big\rangle \overset{(1.1)}{=} \Big\langle J\Big(f_2(\xi_2)\Big(f_1(\xi_1)\Big)\Big), f_1(\xi_1') \Big\rangle \\
&\overset{(3.5)}{=} \Big\langle f_1\big(J(\xi_2)(\xi_1)\big), f_1(\xi_1') \Big\rangle \overset{(3.4)}{=} \langle J(\xi_2)(\xi_1), \xi_1' \rangle \\
&\overset{(1.1)}{=} \langle \xi_2, [\xi_1, \xi_1'] \rangle.
\end{aligned}$$

So (3.6) holds. The proof of the converse can be done similarly.

If f is a sub-Riemannian isometry, then by Proposition 3.2 and Proposition 3.3 and the last statement, (3.4) and (3.5) hold.

If (3.4) and (3.5) hold, then the fact that f is a group homomorphism implies that f transforms horizontal curves into horizontal curves and (3.4) implies that it preserves their length. Of course, this implies that f is an isometry. ∎

Kaplan in [**12**] proved that a map f fixing the unit is an isometry with respect to the standard Riemannian metric $\langle \cdot, \cdot \rangle$ if and only if (3.4) and (3.5) hold. The set of all maps satisfying (3.4) and (3.5) is denoted by $A(G)$, also called the automorphism group of G.

THEOREM 3.5. *The sub-Riemannian isometry group $ISO(G)$ coincides with the isometry group with respect to the standard Riemannian metric $\langle \cdot, \cdot \rangle$. That is, $ISO(G)$ is the semidirect product $A(G) \times G$ (with G acting by left translation).*

In the Heisenberg group $\mathbb{H}^n$, the set $A(G)$ can be more explicitly described due to the fact that $V_1 \simeq \mathbb{R}^{2n}$ can be endowed with a symplectic structure and V_2 is of one dimension and so the map J can be explicitly written out (see (1.3)).

COROLLARY 3.6.

(1) *In $\mathbb{H}^n$, the unit component $A_0(\mathbb{H}^n)$ of the automorphism group $A(\mathbb{H}^n)$ can be identified with the Unitary group $\mathcal{U}(n)$ in the following sense: let $f \in A_0(\mathbb{H}^n)$, then*

$$f = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} \tag{3.7}$$

where $U \in \mathcal{U}(n)$ and $0 \in \mathbb{R}^{2n}$.

(2) *Another component $A_1(\mathbb{H}^n)$ of $A(\mathbb{H}^n)$ is the product of $A_0(\mathbb{H}^n)$ by the matrix*

$$\begin{bmatrix} E & 0 & 0 \\ 0 & -E & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.8}$$

where E is the unit matrix of $n \times n$.

PROOF: From the group law (1.2) we easily deduce that

$$[z, z'] = \frac{1}{2}\sum_{i=1}^{n}(x_i'y_i - x_iy_i') = \frac{1}{2}\omega(z, z') \tag{3.9}$$

for $z = (x, y), z' = (x', y')$ in $\mathbb{R}^{2n}$. In (3.9), $\omega(z, z')$ denotes the standard symplectic form in $\mathbb{R}^{2n}$.

Let $f \in A(\mathbb{H}^n)$. Then by (3.4), f can be written as $f(p) = \big(f_1(z), f_2(t)\big)$ for $p = (z, t) = (x, y, t), z = (x, y)$ where $f_1 \in \mathcal{O}(\mathbb{R}^{2n})$ and $f_2(t) = t$ or $f_2(t) = -t$ for any $t \in \mathbb{R}$. By (3.5), (3.6) and (3.9) we have if $f_2(t) = t$ for any $t \in \mathbb{R}$, then

$$\omega\big(f_1(z), f_1(z')\big) = \omega(z, z') \tag{3.10}$$

and if $f_2(t) = -t$ for any $t \in \mathbb{R}$, then

$$\omega\big(f_1(z), f_1(z')\big) = -\omega(z, z'). \tag{3.11}$$

Note that (3.10) means that f_1 is a symplectic transformation and (3.11) means that f_1 can be seen as the composition of a symplectic transformation with a map determined by the matrix (3.8). We use $\mathcal{S}p(n, \mathbb{R})$ to denote the symplectic group in $\mathbb{R}^{2n}$.

It is easy to verify that if $f \in A_0(\mathbb{H}^n)$ (or $f \in A_1(\mathbb{H}^n)$), then $f_2(t) = t$ (or $-t$) for any $t \in \mathbb{R}$.

Thus we infer that for $f \in A_0(\mathbb{H}^n)$, $f\big((z, t)\big) = \big(f_1(z), t\big)$ where $f_1 \in O(\mathbb{R}^{2n}) \bigcap \mathcal{S}p(n, \mathbb{R}) = \mathcal{U}(n)$(see for example [**1**]). This completes the proof of (1). (2) follows from (1) and the above argument. ∎

REMARK. The full characterisation of the sub-Riemannian isometry group of the Heisenberg group may be useful in finding out the exact isoperimetric set in the Heisenberg group ([**17**]). In Euclidean case, one can use symmetrisation techniques to prove the isoperimetric set is spherical. This is due to the fact that the isometry group of $\mathbb{R}^n$ is large enough to give information of any direction when one tries to deform a set using an isometry. But in the case of the Heisenberg group it is still an open problem whether there are similar symmetrisation result. For this topic we refer to [**17, 18**].

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