FACTORIZATION OF HYPONORMAL OPERATORS

MARY R. EMBRY

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In [1] R. G. Douglas proved that if A and B are continuous linear operators on a Hilbert space X, the following three conditions are equivalent:

(i) $A(X) \subset B(X)$;

(ii) $AA^* \ll \lambda BB^*$ for some $\lambda \geqq 0$;

(iii) there exists a continuous linear operator C on X such that $A = BC$.

It was also shown that the operator C is unique within the constraints

a) $||C||^2 = \inf \{u | u \geqq 0 \text{ and } AA^* \ll uBB^*\}$,

b) kernel C = kernel A, and

c) $C(X) \subset \overline{B^*(X)}$.

The purpose of this note is to consider the special case in which $B = A^*$; that is, $A(X) \subset A^*(X)$.

Definition. $\mathscr{D}$ is the set of all continuous linear operators on X such that $A(X) \subset A^*(X)$. If $A \in \mathscr{D}$, C_A is the unique operator such that $A = A^*C_A$,

$$||C_A||^2 = \inf \{u | u \geqq 0 \text{ and } AA^* = uA^*A\},$$

kernel C_A = kernel A, and $C_A(X) \subset \overline{A(X)}$.

It is immediate from Douglas' theorem that every hyponormal operator $A(AA^* \ll A^*A)$ is an element of $\mathscr{D}$. In this note we characterize the hyponormal operators as those elements of $\mathscr{D}$ such that $||C_A|| \leqq 1$. A normal operator A is characterized as an element of $\mathscr{D}$ such that C_A is a partial isometry with $C_A(X) = \overline{A(X)}$ and a self-adjoint operator A is characterized as an element of $\mathscr{D}$ for which C_A is the projection on $\overline{A(X)}$.

In this paper $A(X)$ denotes the range of A and kernel A denotes the null space of A. I denotes the identity operator on the Hilbert space X. We repeat the following definition of and facts concerning partial isometries, found in [2, p. 99].

1) U is a *partial isometry* if and only if U is an isometry on the orthogonal complement of kernel U;

2) the range of a partial isometry is closed;

3) if U is a partial isometry, U^*U is the projection on (kernel $U)^\perp$;

4) the adjoint of a partial isometry U is a partial isometry; and

5) U is a partial isometry if and only if $U = UU^*U$.

THEOREM 1. *Let $A \in \mathscr{D}$. A is hyponormal if and only if $\|C_A\| \leqq 1$.*

PROOF. If $\|C_A\| \leqq 1$, $\|A^*x\| = \|C_A^*Ax\| \leqq \|Ax\|$ for all x in X and A is hyponormal. Assume now that A is hyponormal. Since

$$\|C_A^*Ax\| = \|A^*x\| \leqq \|Ax\| \text{ for all } x \text{ in } X,$$

$\|C_A^*y\| \leqq \|y\|$ for all y in $\overline{A(X)}$. Let P be the projection on $\overline{A(X)}$ and note that $C_A^*(I-P) = 0$ since $C_A(X) \subset \overline{A(X)}$. Thus for each x in X,

$$\|C_A^*x\| = \|C_A^*Px\| \leqq \|Px\| \leqq \|x\|$$

and consequently, $\|C_A\| = \|C_A^*\| \leqq 1$.

THEOREM 2. *Let $A \in \mathscr{D}$. A is normal if and only if C_A is a partial isometry with $C_A(X) = \overline{A(X)}$.*

PROOF. If C_A is a partial isometry and $C_A(X) = \overline{A(X)}$, then C_A^* is an isometry on $\overline{A(X)}$. Therefore,

$$\|A^*x\| = \|C_A^*Ax\| = \|Ax\|$$

for all x in X and A is normal. Assume now that A is normal. Then

$$\|Ax\| = \|A^*x\| = \|C_A^*Ax\|$$

for all x in X and C_A^* is an isometry on $\overline{A(X)}$. Since

$$\overline{A(X)} \supset \overline{C_A(X)} = (\text{kernel } C_A^*)^\perp,$$

C_A^* is a partial isometry. Therefore C_A is a partial isometry and $C_A(X)$ is closed. It is easy to show that $C_A(X) = \overline{A(X)}$, using the facts that $C_A(X)$ is a closed subset of $A(X)$, $A = A^*C_A$ and A is normal.

In view of the fact that C_A is a partial isometry if A is normal it is reasonable to ask when C_A is an isometry and when C_A is unitary. With one restriction ($C_A(X) = \overline{A(X)}$) these are both equivalent to A being normal and one-to-one. This is seen as follows. If A is normal and one-to-one, C_A is (by Theorem 2) a partial isometry with kernel $C_A = \{0\}$ and $C_A(X) = \overline{A(X)} = X$. Thus C_A is an isometry whose range is all of X and hence is unitary. Conversely, if $A \in \mathscr{D}$ and C_A is an isometry with $C_A(X) = \overline{A(X)}$, A is normal (by Theorem 2) and one-to-one (since kernel A = kernel $C_A = \{0\}$). The restriction that $C_A(X) = \overline{A(X)}$ is necessary in this equivalence, for if A is an isometry, then by the criteria for uniqueness it can be shown that $C_A = A^2$. There exist many examples of non-normal isometries and thus many isometries C_A associated with non-normal operators A.

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COROLLARY 1. *If A is normal, then C_A is normal.*

PROOF. By Theorem 2, C_A is a partial isometry with $C_A(X) = \overline{A(X)}$ if A is normal. Therefore $C_A^*C_A$ is the projection on $C_A^*(X)$ and $C_A C_A^*$ is the projection on $C_A(X)$. However, since A is normal,

$$C_A(X) = \overline{A(X)} = \overline{A^*(X)} = (\text{kernel } C_A)^{\perp} = C_A^*(X)$$

and therefore $C_A C_A^* = C_A^* C_A$.

The converse of Corollary 1 is not valid as is seen by the following simple example: if $A = \begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix}$, then $A \in \mathscr{D}$ and $C_A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$, which is normal. More generally, if $A^2 = I$ as in this example, then $C_A = A^*A$.

Characterizations of normal operators different from that found in Theorem 2 can be obtained as corollaries of Theorem 2.

COROLLARY 2. *Let $A \in \mathscr{D}$. The following conditions are equivalent:*

(i) *A is normal;*

(ii) *$AC_A C_A^* = A$;*

(iii) *A commutes with C_A^*.*

PROOF. (i ⇒ ii). If A is normal, then by Theorem 2 and Corollary 1 C_A is a normal partial isometry. Therefore

$$AC_A C_A^* = AC_A^* C_A = (A^*C_A)C_A^* C_A = A^*C_A = A.$$

(ii) ⇒ (iii). If (ii) holds, C_A^* is an isometry on $\overline{A^*(X)}$. Since

$$A \in \mathscr{D},\ \overline{A^*(X)} \supset \overline{A(X)} \supset \overline{C_A(X)} = (\text{kernel } C_A^*)^{\perp}$$

and thus C_A^* and C_A are partial isometries. Moreover,

$$A(X) \subset A^*(X) \subset C_A C_A^* A^*(X) \subset C_A(X).$$

Therefore, since $(C_A C_A^* - I)C_A = 0$, $(C_A C_A^* - I)A = 0$. This last equation yields $A = C_A C_A^* A = C_A A^*$. Since by definition $A = A^*C_A$, we see that $A^*C_A = C_A A^*$ and that A commutes with C_A^*.

(iii) ⇒ (i). Assume that $AC_A^* = C_A^* A$. Then

$$AA^* = (A^*C_A)A^* = A^*(A^*C_A) = A^*A$$

by definition of C_A and we see that A is normal.

COROLLARY 3. *Let $A \in \mathscr{D}$. A is self-adjoint if and only if C_A is the projection on $\overline{A(X)}$.*

PROOF. If A is self-adjoint, then $A^*(C_A - I) = 0$, and by Theorem 2, $C_A(X) = \overline{A(X)}$. It follows that $(C_A^* - I)C_A = 0$ and thus that C_A is the projection on $\overline{A(X)}$.

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Assume now that C_A is the projection on $\overline{A(X)}$: $C_A A = A$. Then $A^* = A^* C_A^* = A^* C_A = A$ and A is self-adjoint.

The last theorem in this note is a characterization of normal partial isometries.

THEOREM 3. *C is a normal partial isometry if and only if there exists a normal operator A such that $C = C_A$.*

PROOF. By Theorem 2 and Corollary 1 we know that if $C = C_A$ (A normal), then C is a normal partial isometry. On the other hand if C is a normal partial isometry and A is any normal square root of C, it can be shown that

$$A = AA^*A = A^*A^2 = A^*C,$$

$A(X) = C(X)$ and kernel A = kernel C. Thus $C = C_A$ if A is any normal square root of C.

References

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University of North Carolina at Charlotte