COMPOSITIO MATHEMATICA

Chow motives without projectivity

Jörg Wildeshaus

FOUNDATION
COMPOSITIO
MATHEMATICA

The London
Mathematical
Society

Chow motives without projectivity

Jörg Wildeshaus

Abstract

In a recent paper, Bondarko [*Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, Preprint (2007), 0704.4003] defined the notion of weight structure, and proved that the category $DM_{gm}(k)$ of geometrical motives over a perfect field k, as defined and studied by Voevodsky, Suslin and Friedlander [*Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol. 143 (Princeton University Press, Princeton, NJ, 2000)], is canonically equipped with such a structure. Building on this result, and under a condition on the weights avoided by the boundary motive [J. Wildeshaus, *The boundary motive: definition and basic properties*, Compositio Math. **142** (2006), 631–656], we describe a method to construct intrinsically in $DM_{gm}(k)$ a motivic version of interior cohomology of smooth, but possibly non-projective schemes. In a sequel to this work [J. Wildeshaus, *On the interior motive of certain Shimura varieties: the case of Hilbert–Blumenthal varieties*, Preprint (2009), 0906.4239], this method will be applied to Shimura varieties.

Contents

Introduction

The full title of this work would be 'Approximation of motives of varieties which are smooth, but not necessarily projective, by Chow motives, using homological rather than purely geometrical methods'. This might give a better idea of our program, but as a title, it appeared too long. So there.

One way to place the problem historically is to start with the question asked by Serre [Ser91, p. 341], whether the 'virtual motive' $\chi_c(X)$ of an arbitrary variety X over a fixed base field k

can be (well) defined in the Grothendieck group of the category $CHM^{\mathrm{eff}}(k)$ of effective *Chow motives*. When k admits resolution of singularities, Gillet and Soulé [GS96] (see also [GN02] when $\mathrm{char}(k) = 0$) provided an affirmative answer. In fact, their solution yields much more information: they define the *weight complex* $W(X)$ ($h_c(X)$ in [GN02]) in the category of complexes over $CHM^{\mathrm{eff}}(k)$, well defined up to canonical homotopy equivalence. By definition, $\chi_c(X)$ equals the class of $W(X)$ in $K_0(CHM^{\mathrm{eff}}(k))$. Thus, given any representative

$$M_\bullet: \qquad \cdots \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots$$

of $W(X)$, we have the formula $\chi_c(X) = \sum_n (-1)^n [M_n]$.

Consider the fully faithful embedding ι of $CHM^{\mathrm{eff}}(k)$ into $DM_{gm}^{\mathrm{eff}}(k)$, the category of effective *geometrical motives*, as defined and studied by Voevodsky *et al.* [VSF00]. As observed in [GS96], localization for both $\chi_c(X)$ and the *motive with compact support* $M_{gm}^c(X)$ of X shows that the element $\chi_c(X)$ is mapped to the class of $M_{gm}^c(X)$ under $K_0(\iota)$. Gillet and Soulé went on and asked [GS96, p. 153] whether

$$K_0(\iota): K_0(CHM^{\mathrm{eff}}(k)) \longrightarrow K_0(DM_{gm}^{\mathrm{eff}}(k))$$

is an isomorphism.

Similarly to Serre, Gillet and Soulé provoked much more than just an affirmative answer to their question. Bondarko defined in [Bon07] the notion of *weight structure* on a triangulated category $\mathcal{C}$. He proved that the inclusion of the *heart* of the weight structure into $\mathcal{C}$ induces an isomorphism on the level of Grothendieck groups, whenever the weight structure is bounded, and the heart is pseudo-Abelian. The definitions will be recalled in our §1; let us just note that shift by $[m]$, for $m \in \mathbb{Z}$, adds m to the weight of an object of $\mathcal{C}$. According to one of the main results of [Bon07] (recalled in Theorem 1.13), $DM_{gm}^{\mathrm{eff}}(k)$ carries a canonical weight structure, which is indeed bounded, and admits $CHM^{\mathrm{eff}}(k)$ as its heart. By definition, this means that among all geometrical motives, Chow motives distinguish themselves as being the motives which are pure of weight zero.

In particular, this gives an intrinsic characterization of objects of the category $DM_{gm}^{\mathrm{eff}}(k)$ belonging to $CHM^{\mathrm{eff}}(k)$. We think of this insight as nothing less than revolutionary.

To come back to the beginning, the component M_n of the weight complex $W(X)$ can be considered as a '$\mathrm{Gr}_n W(X)$' with respect to the weight structure. In the context of Bondarko's theory, the formula for the 'virtual motive' of X thus reads

$$\chi_c(X) = \sum_n (-1)^n \, \mathrm{Gr}_n W(X).$$

Basically, our approach to construct a Chow motive out of X, when X is smooth, is very simple: according to Bondarko (see Corollary 1.14), the *motive* $M_{gm}(X)$ of X is of weights at most zero. We would like to consider $\mathrm{Gr}_0 M_{gm}(X)$, the 'quotient' of $M_{gm}(X)$ of maximal weight zero.

However, one of the main subtleties of the notion of weight structure is that 'the' *weight filtration* of an object is almost never unique. (In the example of the weight complex, this corresponds to the fact that $W(X)$ is only well-defined up to homotopy.) Hence, '$\mathrm{Gr}_n M_{gm}(X)$' is not well defined, and the above approach cannot work as stated. In fact, for any smooth compactification $\widetilde{X}$ of X, the Chow motive $M_{gm}(\widetilde{X})$ occurs as '$\mathrm{Gr}_0 M_{gm}(X)$' for a suitable weight filtration.

Our main contribution is to identify a criterion assuring existence and unicity of a 'best choice' of $\mathrm{Gr}_0 M_{gm}(X)$. It is best understood in the abstract setting created by Bondarko, that is, in

the context of a weight structure on a triangulated category $\mathcal{C}$. Let M be an object of $\mathcal{C}$ of non-positive weights. Then Bondarko's axioms imply that if M admits a weight filtration in which the adjacent weight -1 does not occur, then the associated $Gr_0 \, M$ is unique up to unique isomorphism. In the motivic context, this observation leads to our criterion on weights avoided by the *boundary motive* $\partial M_{gm}(X)$ of X, introduced and studied in [Wil06]. The behavior of the realizations of the Chow motive $Gr_0 \, M_{gm}(X)$ motivates its name: we chose to call it the *interior motive* of X.

Let us now give a more detailed description of the content of this article. Section 1 claims no originality whatsoever. We give Bondarko's definition of weight structures (Definition 1.1) and review the results from [Bon07] needed in the sequel. We treat particularly carefully the phenomenon which will turn out to be the main theme of this article, namely *the absence of certain weights*. Thus, we introduce the central notion of *weight filtration avoiding weights* $m, m+1, \ldots, n-1, n$, for fixed integers $m \leqslant n$. Bondarko's axioms imply that whenever such a filtration exists, it behaves functorially (Proposition 1.7). In particular, for any fixed object, it is unique up to unique isomorphism (Corollary 1.9). We conclude §1 with Bondarko's application of his theory to geometrical motives, which we already mentioned before (Theorem 1.13 and Corollary 1.14).

Section 2 is the technical center of this article. It is devoted to a further study of functoriality of weight filtrations avoiding certain weights. We work in the context of an abstract weight structure $w = (\mathcal{C}_{w \leqslant 0}, \mathcal{C}_{w \geqslant 0})$ on a triangulated category $\mathcal{C}$. We first show (Proposition 2.2) that the inclusion ι_- of the heart $\mathcal{C}_{w=0}$ into the full sub-category $\mathcal{C}_{w \leqslant 0, \neq -1}$ of objects of non-positive weights other than minus one admits a left adjoint

$$Gr_0 : \mathcal{C}_{w \leqslant 0, \neq -1} \longrightarrow \mathcal{C}_{w=0}.$$

There is a dual version of this statement for non-negative weights other than one. We then consider the situation which will be of interest in our application to motives. We thus fix a morphism $u : M_- \to M_+$ between objects $M_- \in \mathcal{C}_{w \leqslant 0}$ and $M_+ \in \mathcal{C}_{w \geqslant 0}$, and a cone $C[1]$ of u. While the axioms characterizing weight structures easily show that u can be factored through *some* object of $\mathcal{C}_{w=0}$, our aim is to do so *in a canonical way*. We therefore formulate Assumption 2.3: the object C is without weights minus one and zero. Theorem 2.4 states that Assumption 2.3 not only allows to factor u as desired; in addition this factorization is through an object which is simultaneously identified with $Gr_0 \, M_-$ and with $Gr_0 \, M_+$. As a formal consequence of this, and of the functoriality properties of Gr_0 from Proposition 2.2, we obtain a statement on abstract factorization of u (Corollary 2.5), whose rigidity may appear surprising at first sight, given that we work in a triangulated category: whenever $u : M_- \to N \to M_+$ factors through an object N of $\mathcal{C}_{w=0}$, then $Gr_0 \, M_- = Gr_0 \, M_+$ is canonically identified with a direct factor of N, admitting in addition a canonical direct complement.

The reader willing to turn directly to the application of these results to geometrical motives may choose to skip §3, in which Scholl's construction of motives for modular forms [Sch90] is discussed at length. As in [Sch90], we consider a self-product X_n^r of the universal elliptic curve X_n over a modular curve. We show (Theorem 3.3, Corollaries 3.4 and 3.6) that certain direct factors $M_{gm}(X_n^r)^e$ of the motive $M_{gm}(X_n^r)$ and $M_{gm}^c(X_n^r)^e$ of the motive with compact support $M_{gm}^c(X_n^r)$, together with the canonical morphism $u : M_{gm}(X_n^r)^e \to M_{gm}^c(X_n^r)^e$, satisfy the conclusions of Theorem 2.4. In particular, the Chow motives $Gr_0 \, M_{gm}(X_n^r)^e$ and $Gr_0 \, M_{gm}^c(X_n^r)^e$ are defined, and canonically isomorphic. In fact, they are both canonically isomorphic to the motive denoted by ${}_n^r \mathcal{W}$ in [Sch90]. We insist on giving a proof of the above statements which

is independent of the theory developed in §2, that is, we prove the conclusions of Theorem 2.4 without first checking Assumption 2.3. Instead, we use the detailed analysis from [Sch90] of the geometry of the boundary of a 'good choice' of smooth compactification of X_n^r. If one forgets about the language of weight structures, whose use could not be completely avoided, §3 is thus technically independent of the material preceding it. The reader may find an interest in the re-interpretation of Scholl's construction in the context of Voevodsky's geometrical motives, which were not yet defined at the time when [Sch90] was written. This concerns in particular the exact triangle

$$M_{gm}(S_n^\infty)(r+1)[r+1] \longrightarrow M_{gm}(X_n^r)^e \longrightarrow {}_n^r\mathcal{W} \longrightarrow M_{gm}(S_n^\infty)(r+1)[r+2]$$

from Corollary 3.4 ($S_n^\infty :=$ the cuspidal locus of the modular curve). As a by-product of Scholl's analysis, we prove that the triangle is defined using $\mathbb{Z}[1/(2n \cdot r!)]$-coefficients. An interpretation of Beilinson's *Eisenstein symbol* [Bei86] in this context (namely, as a splitting of this triangle) is clearly desirable.

Section 4 is devoted to the application of the results from §2 to geometrical motives. As $u : M_- \to M_+$, we take the canonical morphism $M_{gm}(X)^e \to M_{gm}^c(X)^e$, for a fixed smooth variety X over k, and a fixed idempotent e. Then the role of the object C from §2 is canonically played by $\partial M_{gm}(X)^e$, the e-part of the boundary motive. In this context, Assumption 2.3 reads as follows: the object $\partial M_{gm}(X)^e$ is without weights minus one and zero (Assumption 4.2). Our main result Theorem 4.3 is then simply the translation of the sum of the results from §2 into this particular motivic context. Thus, the Chow motive $\mathrm{Gr}_0 M_{gm}(X)^e = \mathrm{Gr}_0 M_{gm}^c(X)^e$ is defined. Furthermore (Corollary 4.6), we obtain the motivic version of abstract factorization: whenever $\widetilde{X}$ is a smooth compactification of X, then $\mathrm{Gr}_0 M_{gm}(X)^e$ is canonically a direct factor of the Chow motive $M_{gm}(\widetilde{X})$, with a canonical direct complement. We then study the implications of these results for the Hodge theoretic and ℓ-adic realizations. Theorems 4.7 and 4.8 state that they are equal to the respective e-parts of the interior cohomology of X. Abstract factorization allows us to say more about the quality of the Galois representation on the ℓ-adic realization of $\mathrm{Gr}_0 M_{gm}(X)^e$ (Theorem 4.14). For example, simple semi-stable reduction of *some* smooth compactification of X implies that the representation is semi-stable.

To conclude, we get back to Scholl's construction. We show (Remark 4.17) that essentially all results of §3 can be deduced (just) from Assumption 4.2, and the theory developed in §4. We consider that in spite of the technical independence of §3, there is a good reason to include that material in this article: for higher-dimensional Shimura varieties, 'good choices' of smooth compactifications as that used in [Sch90] may simply not be available. Therefore, the purely geometrical strategy of proof of the results of §3 can safely be expected not to be generalizable. We think that a promising way to generalize is via a verification of Assumption 4.2 by other than purely geometrical means. We refer to [Wil09] for the development of such an alternative in a context including that of (powers of universal elliptic curves over) modular curves.

Notation and conventions

In this article k denotes a fixed perfect base field, Sch/k the category of separated schemes of finite type over k, and $Sm/k \subset Sch/k$ the full sub-category of objects which are smooth over k. When we assume that k admits resolution of singularities, then in the sense of [FV00, Definition 3.4]: (i) for any $X \in Sch/k$, there exists an abstract blow-up $Y \to X$ (see [FV00, Definition 3.1]) whose source Y is in Sm/k, (ii) for any $X, Y \in Sm/k$, and any abstract blow-up $q : Y \to X$, there exists a sequence of blow-ups $p : X_n \to \cdots \to X_1 = X$ with

smooth centers, such that p factors through q. Let us note that the main reason for us to assume that k admits resolution of singularities is to have the motive with compact support satisfy localization [Voe00, Proposition 4.1.5].

As far as motives are concerned, the notation of this paper follows that of [Voe00]. We refer to [Wil06, § 1] for a review of this notation and, in particular, of the definition of the categories $DM_{gm}^{\mathrm{eff}}(k)$ and $DM_{gm}(k)$ of (effective) geometrical motives over k, and of the motive $M_{gm}(X)$ and the motive with compact support $M_{gm}^c(X)$ of $X \in Sch/k$. Let F be a commutative flat $\mathbb{Z}$-algebra, i.e. a commutative unitary ring whose additive group is without torsion. The notation $DM_{gm}^{\mathrm{eff}}(k)_F$ and $DM_{gm}(k)_F$ stands for the F-linear analogues of $DM_{gm}^{\mathrm{eff}}(k)$ and $DM_{gm}(k)$ defined in [And04, §§ 16.2.4 and 17.1.3]. Similarly, let us denote by $CHM^{\mathrm{eff}}(k)$ and $CHM(k)$ the categories opposite to the categories of (effective) Chow motives, and by $CHM^{\mathrm{eff}}(k)_F$ and $CHM(k)_F$ the pseudo-Abelian completion of the category $CHM^{\mathrm{eff}}(k) \otimes_{\mathbb{Z}} F$ and $CHM(k) \otimes_{\mathbb{Z}} F$, respectively. Using [Voe02, Corollary 2] ([Voe00, Corollary 4.2.6] if k admits resolution of singularities), we canonically identify $CHM^{\mathrm{eff}}(k)_F$ and $CHM(k)_F$ with a full additive sub-category of $DM_{gm}^{\mathrm{eff}}(k)_F$ and $DM_{gm}(k)_F$, respectively.

1. Weight structures

In this section, we review the main definitions and results of Bondarko's recent paper [Bon07].

DEFINITION 1.1. Let $\mathcal{C}$ be a triangulated category. A *weight structure on $\mathcal{C}$* is a pair $w = (\mathcal{C}_{w \leqslant 0}, \mathcal{C}_{w \geqslant 0})$ of full sub-categories of $\mathcal{C}$, such that, putting

$$\mathcal{C}_{w \leqslant n} := \mathcal{C}_{w \leqslant 0}[n], \quad \mathcal{C}_{w \geqslant n} := \mathcal{C}_{w \geqslant 0}[n] \quad \forall n \in \mathbb{Z},$$

the following conditions are satisfied.

(1) The categories $\mathcal{C}_{w \leqslant 0}$ and $\mathcal{C}_{w \geqslant 0}$ are Karoubi-closed: for any object M of $\mathcal{C}_{w \leqslant 0}$ or $\mathcal{C}_{w \geqslant 0}$, any direct summand of M formed in $\mathcal{C}$ is an object of $\mathcal{C}_{w \leqslant 0}$ or $\mathcal{C}_{w \geqslant 0}$, respectively.

(2) (Semi-invariance with respect to shifts.) We have the inclusions

$$\mathcal{C}_{w \leqslant 0} \subset \mathcal{C}_{w \leqslant 1}, \quad \mathcal{C}_{w \geqslant 0} \supset \mathcal{C}_{w \geqslant 1}$$

of full sub-categories of $\mathcal{C}$.

(3) (Orthogonality.) For any pair of objects $M \in \mathcal{C}_{w \leqslant 0}$ and $N \in \mathcal{C}_{w \geqslant 1}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(M, N) = 0.$$

(4) (Weight filtration.) For any object $M \in \mathcal{C}$, there exists an exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in $\mathcal{C}$, such that $A \in \mathcal{C}_{w \leqslant 0}$ and $B \in \mathcal{C}_{w \geqslant 1}$.

By condition (2) in Definition 1.1,

$$\mathcal{C}_{w \leqslant n} \subset \mathcal{C}_{w \leqslant 0}$$

for negative n, and

$$\mathcal{C}_{w \geqslant n} \subset \mathcal{C}_{w \geqslant 0}$$

for positive n. There are obvious analogues of the other conditions for all of the categories $\mathcal{C}_{w \leqslant n}$ and $\mathcal{C}_{w \geqslant n}$. In particular, they are all Karoubi-closed, and any object $M \in \mathcal{C}$ is part of an exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in $\mathcal{C}$, such that $A \in \mathcal{C}_{w \leqslant n}$ and $B \in \mathcal{C}_{w \geqslant n+1}$. By a slight generalization of the terminology introduced in condition (4) in Definition 1.1, we shall refer to any such exact triangle as a *weight filtration* of M.

Remark 1.2. We make the following remarks.

(a) Our convention concerning the sign of the weight is actually opposite to that from [Bon07, Definition 1.1.1], that is, we exchanged the roles of $\mathcal{C}_{w \leqslant 0}$ and $\mathcal{C}_{w \geqslant 0}$.

(b) Note that in condition (4) in Definition 1.1, 'the' weight filtration is not assumed to be unique.

(c) Recall the notion of *t-structure* on a triangulated category $\mathcal{C}$ (see [BBD82, Définition 1.3.1]). It consists of a pair $t = (\mathcal{C}^{t \leqslant 0}, \mathcal{C}^{t \geqslant 0})$ of full sub-categories satisfying formal analogues of conditions (2)–(4) in Definition 1.1, but putting

$$\mathcal{C}^{t \leqslant n} := \mathcal{C}^{t \leqslant 0}[-n], \quad \mathcal{C}^{t \geqslant n} := \mathcal{C}^{t \geqslant 0}[-n] \quad \text{for all } n \in \mathbb{Z}.$$

Note that in the context of t-structures, the analogues of the exact triangles in condition (4) in Definition 1.1 are then unique up to unique isomorphism, and that the analogue of condition (1) in Definition 1.1 is formally implied by the others.

The following is contained in [Bon07, Definition 1.2.1].

DEFINITION 1.3. Let $w = (\mathcal{C}_{w \leqslant 0}, \mathcal{C}_{w \geqslant 0})$ be a weight structure on $\mathcal{C}$. The *heart of* w is the full additive sub-category $\mathcal{C}_{w=0}$ of $\mathcal{C}$ whose objects lie both in $\mathcal{C}_{w \leqslant 0}$ and in $\mathcal{C}_{w \geqslant 0}$.

Among the basic properties developed in [Bon07], let us note the following.

PROPOSITION 1.4. *Let* $w = (\mathcal{C}_{w \leqslant 0}, \mathcal{C}_{w \geqslant 0})$ *be a weight structure on* $\mathcal{C}$,

$$L \longrightarrow M \longrightarrow N \longrightarrow L[1]$$

an exact triangle in $\mathcal{C}$.

(a) *If both* L *and* N *belong to* $\mathcal{C}_{w \leqslant 0}$, *then so does* M.

(b) *If both* L *and* N *belong to* $\mathcal{C}_{w \geqslant 0}$, *then so does* M.

Proof. This is the content of [Bon07, Proposition 1.3.3 3]. □

The reader may wonder whether there is an easy criterion on a given sub-category of a triangulated category to be the heart of a suitable weight structure. Bondarko has results [Bon07, Theorem 4.3.2] answering this question. For our purposes, the result with the most restrictive finiteness condition will be sufficient.

PROPOSITION 1.5. *Let* $\mathcal{H}$ *be a full additive sub-category of a triangulated category* $\mathcal{C}$. *Suppose that* $\mathcal{H}$ *generates* $\mathcal{C}$, *that is,* $\mathcal{C}$ *is the smallest full triangulated sub-category containing* $\mathcal{H}$.

(a) *If there is a weight structure on* $\mathcal{C}$ *whose heart contains* $\mathcal{H}$, *then it is unique. In this case, the heart is equal to the* Karoubi envelope *of* $\mathcal{H}$, *that is, the category of retracts of* $\mathcal{H}$ *in* $\mathcal{C}$.

(b) *The following conditions are equivalent.*

    (i) *There is a weight structure on* $\mathcal{C}$ *whose heart contains* $\mathcal{H}$.

    (ii) *The sub-category* $\mathcal{H}$ *is* negative, *that is,*

$$\mathrm{Hom}_{\mathcal{C}}(A, B[i]) = 0$$

    for any two objects A, B *of* $\mathcal{H}$, *and any integer* $i > 0$.

Proof. Condition (ii) on $\mathcal{H}$ is clearly necessary for $\mathcal{H}$ to belong to the heart, given orthogonality (3) in Definition 1.1. As for (a), note that by Proposition 1.4, there is only one possible definition of the category $\mathcal{C}_{w\leqslant 0}$ (resp. $\mathcal{C}_{w\geqslant 0}$): it is necessarily the full sub-category of successive extensions of objects of the form $A[n]$, for $A \in \mathcal{H}$ and $n \leqslant 0$ (respectively $n \geqslant 0$).

The main point is to show that under condition (ii), the above construction indeed yields a weight structure on $\mathcal{C}$. We refer to [Bon07, Theorem 4.3.2 II] for details. □

For the rest of this section, we consider a fixed weight structure w on a triangulated category $\mathcal{C}$.

DEFINITION 1.6. Let $M \in \mathcal{C}$, and $m \leqslant n$ two integers (which may be identical). A *weight filtration of M avoiding weights* $m, m+1, \ldots, n-1, n$ is an exact triangle

$$M_{\leqslant m-1} \longrightarrow M \longrightarrow M_{\geqslant n+1} \longrightarrow M_{\leqslant m-1}[1]$$

in $\mathcal{C}$, with $M_{\leqslant m-1} \in \mathcal{C}_{w\leqslant m-1}$ and $M_{\geqslant n+1} \in \mathcal{C}_{w\geqslant n+1}$.

The following observation is vital.

PROPOSITION 1.7. *Assume that $m \leqslant n$, and that $M, N \in \mathcal{C}$ admit weight filtrations*

$$M_{\leqslant m-1} \xrightarrow{x_-} M \xrightarrow{x_+} M_{\geqslant n+1} \longrightarrow M_{\leqslant m-1}[1]$$

and

$$N_{\leqslant m-1} \xrightarrow{y_-} N \xrightarrow{y_+} N_{\geqslant n+1} \longrightarrow N_{\leqslant m-1}[1]$$

avoiding weights $m, \ldots, n$. Then any morphism $M \to N$ in $\mathcal{C}$ extends uniquely to a morphism of exact triangles.

$$
\begin{array}{ccccccc}
M_{\leqslant m-1} & \longrightarrow & M & \longrightarrow & M_{\geqslant n+1} & \longrightarrow & M_{\leqslant m-1}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
N_{\leqslant m-1} & \longrightarrow & N & \longrightarrow & N_{\geqslant n+1} & \longrightarrow & N_{\leqslant m-1}[1]
\end{array}
$$

Proof. This follows from [Bon07, Lemma 1.5.1 2]. For the convenience of the reader, let us recall the proof. Let $\alpha \in \mathrm{Hom}_{\mathcal{C}}(M, N)$. The composition $y_+ \circ \alpha \circ x_- : M_{\leqslant m-1} \to N_{\geqslant n+1}$ is zero by orthogonality (3) in Definition 1.1: $m-1$ is strictly smaller than $n+1$. Hence, $\alpha \circ x_-$ factors through $N_{\leqslant m-1}$. We claim that this factorization is unique. Indeed, the error term comes from $\mathrm{Hom}_{\mathcal{C}}(M_{\leqslant m-1}, N_{\geqslant n+1}[-1])$. However, this group is trivial, thanks to orthogonality, and our assumption on the weights: the object $N_{\geqslant n+1}[-1]$ lies in

$$\mathcal{C}_{w\geqslant n+1}[-1] = \mathcal{C}_{w\geqslant n},$$

and $m-1$ is still strictly smaller than n. Similarly, the composition $y_+ \circ \alpha$ factors uniquely through $M_{\geqslant n+1}$. □

Remark 1.8. Note that the hypothesis of Proposition 1.7 does not imply unicity of weight filtrations in the (more general) sense of Definition 1.1(4). For example, assume that $m = n = -1$, and let

$$(*) \qquad M_{\leqslant -2} \longrightarrow M \longrightarrow M_{\geqslant 0} \longrightarrow M_{\leqslant -2}[1]$$

be a weight filtration avoiding weight minus one. Choose any object M_0 in $\mathcal{C}_{w=0}$ and replace $M_{\geqslant 0}$ by $M_0 \oplus M_{\geqslant 0}$, and $M_{\leqslant -2}$ by $M_0[-1] \oplus M_{\leqslant -2}$. Arguing as in the proof of Proposition 1.7, one shows that any weight filtration of M is isomorphic to one obtained in this way. Thus, the exact triangle $(*)$ satisfies a minimality property among all weight filtrations of M.

COROLLARY 1.9. *Assume that $m \leqslant n$. Then if $M \in \mathcal{C}$ admits a weight filtration avoiding weights $m, \ldots, n$, it is unique up to unique isomorphism.*

DEFINITION 1.10. Assume that $m \leqslant n$. We say that $M \in \mathcal{C}$ *does not have weights* $m, \ldots, n$, *or that M is without weights* $m, \ldots, n$, if it admits a weight filtration avoiding weights $m, \ldots, n$.

Let us now state what we consider as one of the main results of [Bon07].

THEOREM 1.11. *Assume that the triangulated category $\mathcal{C}$ is generated by its heart $\mathcal{C}_{w=0}$.*

(a) *The pseudo-Abelian completion $\mathcal{C}'_{w=0}$ of $\mathcal{C}_{w=0}$ generates the pseudo-Abelian completion $\mathcal{C}'$ of $\mathcal{C}$.*

(b) *There is a weight structure w' on $\mathcal{C}'$, uniquely characterized by any of the following conditions.*

    (i) *The weight structure w' extends w.*
    (ii) *The heart of w' equals $\mathcal{C}'_{w=0}$.*
    (iii) *The heart of w' contains $\mathcal{C}'_{w=0}$.*

Proof. This is [Bon07, Proposition 5.2.2]. Let us describe the main steps of the proof. Recall that by [BS01, Theorem 1.5], the category $\mathcal{C}'$ is indeed triangulated. The criterion from Proposition 1.5 implies the existence of a weight structure w' on the full triangulated sub-category $\mathcal{D}$ of $\mathcal{C}'$ generated by $\mathcal{C}'_{w=0}$ (hence, containing $\mathcal{C}$), and uniquely characterized by condition (iii), hence also by condition (i) or (ii). The claim then follows from [Bon07, Lemma 5.2.1], which states that $\mathcal{D}$ is pseudo-Abelian, and hence equal to $\mathcal{C}'$. $\qquad\square$

Remark 1.12. Note that given Proposition 1.5, part (b) of Theorem 1.11 follows formally from its part (a). One may see Theorem 1.11(a) as a generalization of [BS01, Corollary 2.12], which states that the pseudo-Abelian completion of the bounded derived category $D^b(\mathcal{A})$ of an exact category $\mathcal{A}$ equals the bounded derived category $D^b(\mathcal{A}')$ of the pseudo-Abelian completion $\mathcal{A}'$ of $\mathcal{A}$.

For our purposes, the main application of the preceding is the following (cf. [Bon07, § 6]).

THEOREM 1.13. *Let F be a commutative flat $\mathbb{Z}$-algebra, and assume that k admits resolution of singularities.*

(a) *There is a canonical weight structure on the category $DM_{gm}^{\mathrm{eff}}(k)_F$. It is uniquely characterized by the requirement that its heart equals $CHM^{\mathrm{eff}}(k)_F$.*

(b) *There is a canonical weight structure on the category $DM_{gm}(k)_F$, extending the weight structure from statement (a). It is uniquely characterized by the requirement that its heart equals $CHM(k)_F$.*

(c) *Statements (a) and (b) hold without assuming resolution of singularities provided that F is a $\mathbb{Q}$-algebra.*

Proof. For $F = \mathbb{Z}$ and k of characteristic zero, this is the content of [Bon07, §§ 6.5 and 6.6].

(1) As in [Bon07], denote by DM^s the full triangulated sub-category of $DM_{gm}^{\mathrm{eff}}(k)$ generated by the motives $M_{gm}(X)$ (see [Voe00, Definition 2.1.1]) of objects X of Sm/k, by J_0 the full additive sub-category of DM^s generated by $M_{gm}(X)$ for X smooth and projective, and by J'_0 the Karoubi envelope of J_0. Thus, $DM_{gm}^{\mathrm{eff}}(k)$ is the pseudo-Abelian completion of DM^s, and $CHM^{\mathrm{eff}}(k)$ is the pseudo-Abelian completion of both J_0 and J'_0.

(2) We need two of the main results from [Voe00]. First, by [Voe00, Corollary 3.5.5], the additive category J_0 generates the triangulated category DM^s. Next, by [Voe00, Corollary 4.2.6], the category J_0 is negative:

$$\mathrm{Hom}_{DM^s}(A, B[i]) = 0$$

for any two objects A, B of J_0, and any integer $i > 0$.

(3) By Proposition 1.5, there is a weight structure on DM^s, uniquely characterized by the fact that J_0 is contained in the heart. Furthermore, the heart equals the Karoubi envelope J_0'.

(4) By Theorem 1.11, the pseudo-Abelian completion $CHM^{\mathrm{eff}}(k)$ of J_0 generates the pseudo-Abelian completion $DM_{gm}^{\mathrm{eff}}(k)$ of DM^s (let us remark that this is stated, but not proved, in [Voe00, Corollary 3.5.5]). Thus, part (a) of our claim holds for $F = \mathbb{Z}$.

(5) Recall that $CHM(k)$ and $DM_{gm}(k)$ are obtained from $CHM^{\mathrm{eff}}(k)$ and $DM_{gm}^{\mathrm{eff}}(k)$ by inverting an object, namely the Tate object T, with respect to the tensor structures. Hence, $CHM(k)$ generates the triangulated category $DM_{gm}(k)$. Its negativity follows formally from that of $CHM^{\mathrm{eff}}(k)$: indeed, for two objects A and B of $CHM(k)$, and any integer $i > 0$, the group $\mathrm{Hom}_{DM_{gm}(k)}(A, B[i])$ is, by definition, the direct limit over large integers r of the groups

$$\mathrm{Hom}_{DM_{gm}^{\mathrm{eff}}(k)}(A \otimes T^{\otimes r}, B \otimes T^{\otimes r}[i]),$$

which are all zero by part (a). Thus, we may again apply Proposition 1.5. The resulting weight structure extends that on $DM_{gm}^{\mathrm{eff}}(k)$: in fact, its restriction to $DM_{gm}^{\mathrm{eff}}(k)$ is a weight structure, whose heart equals $CHM^{\mathrm{eff}}(k)$. This proves part (b) of our claim for $F = \mathbb{Z}$.

If F is flat over $\mathbb{Z}$, then the same proof works. $(1')$ Replace DM^s by the full F-linear triangulated sub-category DM_F^s of $DM_{gm}^{\mathrm{eff}}(k)_F$ generated by the motives $M_{gm}(X)$ of objects X of Sm/k, and J_0 by $J_0 \otimes_{\mathbb{Z}} F$. $(2')$ The two results cited in step (2) formally imply that $J_0 \otimes_{\mathbb{Z}} F$ generates DM_F^s, and that $J_0 \otimes_{\mathbb{Z}} F$ is negative. Steps $(3')$ and $(4')$ are formally identical to steps (3) and (4), proving part (a) of the claim. Step $(5')$ shows part (b), once we observe that $CHM(k)_F$ and $DM_{gm}(k)_F$ are obtained from $CHM^{\mathrm{eff}}(k)_F$ and $DM_{gm}^{\mathrm{eff}}(k)_F$ by inverting the Tate object.

As for part (c) of our claim, everything reduces to showing analogues of the two statements made in step (2). By [And04, Corollary 18.1.1.2], the additive category $J_0 \otimes_{\mathbb{Z}} F$ generates the triangulated category $DM^s \otimes_{\mathbb{Z}} F$. The argument uses alterations à la de Jong; since this involves finite extensions of fields, whose degrees need to be inverted, one requires F to be a $\mathbb{Q}$-algebra. The generalization of [Voe00, Corollary 4.2.6] to arbitrary fields [Voe02, Corollary 2] shows that the category J_0 is negative. Hence, so is $J_0 \otimes_{\mathbb{Z}} F$. $\square$

The following is the content of [Bon09, Theorems 6.2.1(1) and (2)], when k is of characteristic zero.

COROLLARY 1.14. *Assume that k admits resolution of singularities. Let X in Sch/k be of (Krull) dimension d.*

(a) *The motive with compact support $M_{gm}^c(X)$ lies in*

$$DM_{gm}^{\mathrm{eff}}(k)_{w \geqslant 0} \cap DM_{gm}^{\mathrm{eff}}(k)_{w \leqslant d}.$$

(b) *If $X \in Sm/k$, then the motive $M_{gm}(X)$ lies in*

$$DM_{gm}^{\mathrm{eff}}(k)_{w \geqslant -d} \cap DM_{gm}^{\mathrm{eff}}(k)_{w \leqslant 0}.$$

Proof. (a) We proceed by induction on d. If $d = 0$, then $M_{gm}^c(X)$ is an effective Chow motive, hence of weight zero by Theorem 1.13(a).

For $d \geqslant 1$, Nagata's theorem on the existence of a compactification of X, and resolution of singularities imply that there is an open dense sub-scheme U of X admitting a smooth compactification $\widetilde{X}$. Denote by Z the complement of U in X, and by Y the complement of U in $\widetilde{X}$ (both with the reduced scheme structure). By localization for the motive with compact support [Voe00, Proposition 4.1.5], there are exact triangles

$$M_{gm}^c(Z) \longrightarrow M_{gm}^c(X) \longrightarrow M_{gm}^c(U) \longrightarrow M_{gm}^c(Z)[1]$$

and

$$M_{gm}^c(Y) \longrightarrow M_{gm}^c(\widetilde{X}) \longrightarrow M_{gm}^c(U) \longrightarrow M_{gm}^c(Y)[1].$$

By induction,

$$M_{gm}^c(Y), M_{gm}^c(Z) \in DM_{gm}^{\mathrm{eff}}(k)_{w \geqslant 0} \cap DM_{gm}^{\mathrm{eff}}(k)_{w \leqslant d-1}.$$

Therefore,

$$M_{gm}^c(Y)[1], M_{gm}^c(Z) \in DM_{gm}^{\mathrm{eff}}(k)_{w \geqslant 0} \cap DM_{gm}^{\mathrm{eff}}(k)_{w \leqslant d}.$$

Given that $M_{gm}^c(\widetilde{X})$ is of weight zero, Proposition 1.4 shows first that

$$M_{gm}^c(U) \in DM_{gm}^{\mathrm{eff}}(k)_{w \geqslant 0} \cap DM_{gm}^{\mathrm{eff}}(k)_{w \leqslant d},$$

and then that

$$M_{gm}^c(X) \in DM_{gm}^{\mathrm{eff}}(k)_{w \geqslant 0} \cap DM_{gm}^{\mathrm{eff}}(k)_{w \leqslant d}.$$

(b) By [Voe00, Theorems 4.3.7 1 and 2], the category $DM_{gm}(k)$ is rigid tensor triangulated. The claim thus follows formally from part (a), and from the following observations: (i) assuming (as we may) X to be of pure dimension d, the motive $M_{gm}(X)$ is dual to $M_{gm}^c(X)(-d)[-2d]$ (see [Voe00, Theorem 4.3.7 3]); (ii) the object $\mathbb{Z}(-d)[-2d]$ is a Chow motive; (iii) the heart of the weight structure on $DM_{gm}(k)$ is stable under duality, hence for any natural number n, induction on n shows that the dual of an object of the intersection $DM_{gm}(k)_{w \geqslant 0} \cap DM_{gm}(k)_{w \leqslant n}$ belongs to $DM_{gm}(k)_{w \geqslant -n} \cap DM_{gm}(k)_{w \leqslant 0}$; (iv) the weight structure on $DM_{gm}^{\mathrm{eff}}(k)$ is induced from the weight structure on $DM_{gm}(k)$ (Theorem 1.13(b)). $\qquad\square$

Remark 1.15. (a) Corollary 1.14(a) and its proof should be compared to the construction of the weight complex $W(X)$ from [GS96, §2.1]. Let $\widetilde{j} : \widetilde{Y}_\bullet \to \widetilde{X}_\bullet$ be a smooth hyper-envelope (in the sense of [GS96]) of a closed immersion $Y \hookrightarrow \overline{X}$ of proper schemes whose complement equals X. Both $\widetilde{Y}_\bullet$ and $\widetilde{X}_\bullet$ give rise to complexes $\mathbb{Z}\widetilde{Y}_\bullet$ and $\mathbb{Z}\widetilde{X}_\bullet$ in the category denoted $\mathbb{Z}\mathbf{V}$ in [GS96], that is, the $\mathbb{Z}$-linearized category associated with the category $\mathbf{V}$ of smooth proper schemes over k. Hence, we may form the complex $\mathrm{Cone}(\widetilde{j})$ in $\mathbb{Z}\mathbf{V}$. Applying the functor $\underline{C} \circ L$ (see [Voe00, pp. 207, 223–224]), we obtain a complex of Nisnevich sheaves with transfers whose cohomology sheaves are homotopy invariant. On the one hand, it should be possible to employ [Voe00, Theorem 4.1.2] in order to show that the complex $\underline{C}(L(\mathrm{Cone}(\widetilde{j})))$ represents the motive with compact support $M_{gm}^c(X)$ in $DM_{gm}^{\mathrm{eff}}(k)$. On the other hand, by definition [GS96, §2.1], the (opposite of the) complex $M_{gm}(\mathrm{Cone}(\widetilde{j}))$ represents $W(X)$ in the homotopy category over $CHM^{\mathrm{eff}}(k)$. The statement

$$M_{gm}^c(X) \in DM_{gm}^{\mathrm{eff}}(k)_{w \geqslant 0} \cap DM_{gm}^{\mathrm{eff}}(k)_{w \leqslant d}$$

from Corollary 1.14(a) should be compared to [GS96, Theorem 2(i)].

1205

(b) Similarly, the construction from [GN02, Theorem (5.10)(3)] of the object $h(X)$ should be compared to Corollary 1.14(b).

COROLLARY 1.16. *Assume that k admits resolution of singularities. Suppose that we are given a direct factor M of $M_{gm}(X)$, for $X \in Sm/k$, which is abstractly isomorphic to a direct factor of $M_{gm}^c(Y)$, for some $Y \in Sch/k$. Then M is an effective Chow motive.*

2. Weight zero

Throughout this section, we fix a weight structure $w = (\mathcal{C}_{w \leqslant 0}, \mathcal{C}_{w \geqslant 0})$ on a triangulated category $\mathcal{C}$. Anticipating the situation which will be of interest in our applications, we formulate Assumption 2.3 on the cone of a morphism u in $\mathcal{C}$. As we shall see (Theorem 2.4), this hypothesis ensures in particular unique factorization of u through an object of the heart $\mathcal{C}_{w=0}$.

DEFINITION 2.1. Denote by $\mathcal{C}_{w \leqslant 0, \neq -1}$ the full sub-category of $\mathcal{C}_{w \leqslant 0}$ of objects without weight minus one, and by $\mathcal{C}_{w \geqslant 0, \neq 1}$ the full sub-category of $\mathcal{C}_{w \geqslant 0}$ of objects without weight one.

PROPOSITION 2.2. *We have the following results.*

(a) *The inclusion of the heart $\iota_- : \mathcal{C}_{w=0} \hookrightarrow \mathcal{C}_{w \leqslant 0, \neq -1}$ admits a left adjoint*

$$\mathrm{Gr}_0 : \mathcal{C}_{w \leqslant 0, \neq -1} \longrightarrow \mathcal{C}_{w=0}.$$

On objects, it is given by sending M to the term $M_{\geqslant 0}$ of a weight filtration

$$M_{\leqslant -2} \longrightarrow M \longrightarrow M_{\geqslant 0} \longrightarrow M_{\leqslant -2}[1]$$

avoiding weight minus one. The composition $\mathrm{Gr}_0 \circ \iota_-$ equals the identity on $\mathcal{C}_{w=0}$.

(b) *The inclusion of the heart $\iota_+ : \mathcal{C}_{w=0} \hookrightarrow \mathcal{C}_{w \geqslant 0, \neq 1}$ admits a right adjoint*

$$\mathrm{Gr}_0 : \mathcal{C}_{w \geqslant 0, \neq 1} \longrightarrow \mathcal{C}_{w=0}.$$

On objects, it is given by sending M to the term $M_{\leqslant 0}$ of a weight filtration

$$M_{\leqslant 0} \longrightarrow M \longrightarrow M_{\geqslant 2} \longrightarrow M_{\leqslant 0}[1]$$

avoiding weight one. The composition $\mathrm{Gr}_0 \circ \iota_+$ equals the identity on $\mathcal{C}_{w=0}$.

Proof. Given Proposition 1.7 and Corollary 1.9, all that remains to be proved is that the objects $M_{\geqslant 0}$ (in (a)) respectively $M_{\leqslant 0}$ (in (b)) actually do lie in $\mathcal{C}_{w=0}$. However, this follows from Proposition 1.4. $\square$

Let us now fix the following data.

(1) A morphism $u : M_- \to M_+$ in $\mathcal{C}$ between $M_- \in \mathcal{C}_{w \leqslant 0}$ and $M_+ \in \mathcal{C}_{w \geqslant 0}$.
(2) An exact triangle

$$C \xrightarrow{v_-} M_- \xrightarrow{u} M_+ \xrightarrow{v_+} C[1]$$

in $\mathcal{C}$. Thus, the object $C[1]$ is a fixed choice of cone of u.

We make the following rather restrictive hypothesis.

ASSUMPTION 2.3. The object C is without weights minus one and zero, that is, it admits a weight filtration

$$C_{\leqslant -2} \xrightarrow{c_-} C \xrightarrow{c_+} C_{\geqslant 1} \xrightarrow{\delta_C} C_{\leqslant -2}[1]$$

avoiding weights minus one and zero.

1206

The validity of this assumption is independent of the choice of C. Here is our main technical tool.

Theorem 2.4. *Fix the data (1), (2), and assume that Assumption 2.3 holds.*

(a) *The object M_- is without weight minus one, and M_+ is without weight one.*

(b) *The morphisms $v_- \circ c_- : C_{\leqslant -2} \to M_-$ and $\pi_0 : M_- \to \mathrm{Gr}_0 M_-$ (respectively $i_0 : \mathrm{Gr}_0 M_+ \to M_+$ and $(c_+[1]) \circ v_+ : M_+ \to C_{\geqslant 1}[1]$) can be canonically extended to exact triangles*

$$(3) \qquad C_{\leqslant -2} \xrightarrow{v_- c_-} M_- \xrightarrow{\pi_0} \mathrm{Gr}_0 M_- \xrightarrow{\delta_-} C_{\leqslant -2}[1]$$

and

$$(4) \qquad C_{\geqslant 1} \xrightarrow{\delta_+} \mathrm{Gr}_0 M_+ \xrightarrow{i_0} M_+ \xrightarrow{(c_+[1])v_+} C_{\geqslant 1}[1].$$

Thus, (3) is a weight filtration of M_- avoiding weight minus one, and (4) is a weight filtration of M_+ avoiding weight one.

(c) *There is a canonical isomorphism $\mathrm{Gr}_0 M_- \xrightarrow{\sim} \mathrm{Gr}_0 M_+$. As a morphism, it is uniquely determined by the property of making the diagram*

$$
\begin{array}{ccc}
M_- & \xrightarrow{\;\;u\;\;} & M_+ \\
\pi_0 \downarrow & & \uparrow i_0 \\
\mathrm{Gr}_0 M_- & \longrightarrow & \mathrm{Gr}_0 M_+
\end{array}
$$

commute. Its inverse makes the diagram

$$
\begin{array}{ccc}
C_{\geqslant 1} & \xrightarrow{\;\;\delta_C\;\;} & C_{\leqslant -2}[1] \\
\delta_+ \downarrow & & \uparrow \delta_- \\
\mathrm{Gr}_0 M_+ & \longrightarrow & \mathrm{Gr}_0 M_-
\end{array}
$$

commute.

Proof. We start by choosing and fixing exact triangles

$$(3') \qquad C_{\leqslant -2} \xrightarrow{v_- c_-} M_- \xrightarrow{\pi_0'} G_- \xrightarrow{\delta_-} C_{\leqslant -2}[1]$$

and

$$(4') \qquad C_{\geqslant 1} \xrightarrow{\delta_-} G_+ \xrightarrow{i_0'} M_+ \xrightarrow{(c_+[1])v_+} C_{\geqslant 1}[1].$$

Thus, G_- is a cone of $v_- \circ c_-$, and G_+ a cone of $c_+ \circ v_+[-1]$. Observe first that by Proposition 1.4,

$$G_- \in \mathcal{C}_{w \leqslant 0} \quad \text{and} \quad G_+ \in \mathcal{C}_{w \geqslant 0}.$$

Given this, the existence of *some* isomorphism $G_- \cong G_+$ clearly implies parts (a) and (b) of our statement.

Let us now show that there is an isomorphism $\alpha : G_- \xrightarrow{\sim} G_+$ making the diagrams

$$
\begin{array}{ccc}
M_- & \xrightarrow{\;\;u\;\;} & M_+ \\
\pi_0' \downarrow & & \uparrow i_0' \\
G_- & \xrightarrow{\;\;\alpha\;\;} & G_+
\end{array}
$$

1207

and

$$C_{\geqslant 1} \xrightarrow{\ \delta_C\ } C_{\leqslant -2}[1]$$
$$\delta_+ \downarrow \qquad\qquad \uparrow \delta_-$$
$$G_+ \xrightarrow{\ \alpha^{-1}\ } G_-$$

commute. To see this, consider the following diagram.

The three arrows marked [1] link the source to the shift by [1] of the target, the upper and lower triangles are commutative, and the left and right triangles are exact. In the terminology of [BBD82, §1], this is a *calotte inférieure*, which thanks to the axiom TR4' of triangulated categories in the formulation of [BBD82, §1.1.6] can be completed to an octahedron. In particular, its contour

is part of a *calotte supérieure*:

Here, the upper and lower triangles are exact, and the left and right triangles are commutative. Hence, the same object $G[1]$ can be chosen as the cone of $v_- \circ c_-$ and of $c_+ \circ v_+[-1]$ and, in addition, such that the morphisms u and δ_C factor through $G[1]$. For our fixed choices of cones, this means precisely that there is an isomorphism $G_- \cong G_+$ factorizing both u and δ_C.

As observed before, this implies parts (a) and (b). It remains to show the unicity statement from part (c). For this, use the exact triangle (3) and apply Proposition 2.2(b) to see that

$$\operatorname{Hom}_{\mathcal{C}}(M_-, M_+) = \operatorname{Hom}_{\mathcal{C}}\big(\operatorname{Gr}_0 M_-, M_+\big) = \operatorname{Hom}_{\mathcal{C}_{w=0}}(\operatorname{Gr}_0 M_-, \operatorname{Gr}_0 M_+).$$

Under this identification, a morphism $M_- \to M_+$ is sent to its unique factorization $\operatorname{Gr}_0 M_- \to \operatorname{Gr}_0 M_+$. $\qquad\qquad\Box$

Given Theorem 2.4(c), we may and do identify $\operatorname{Gr}_0 M_-$ and $\operatorname{Gr}_0 M_+$.

1208

COROLLARY 2.5. *Fix the data (1), (2), and assume that Assumption 2.3 holds. Let $M_- \to N \to M_+$ be a factorization of u through an object N of $\mathcal{C}_{w=0}$. Then $\mathrm{Gr}_0 M_- = \mathrm{Gr}_0 M_+$ is canonically identified with a direct factor of N, admitting a canonical direct complement.*

Proof. By Proposition 2.2, the morphism $M_- \to N$ factors uniquely through $\mathrm{Gr}_0 M_-$, and $N \to M_+$ factors uniquely through $\mathrm{Gr}_0 M_+$. The composition $\mathrm{Gr}_0 M_- \to N \to \mathrm{Gr}_0 M_+$ is therefore a factorization of u. By Theorem 2.4(c), it thus equals the canonical identification $\mathrm{Gr}_0 M_- = \mathrm{Gr}_0 M_+$. Hence $\mathrm{Gr}_0 M_- = \mathrm{Gr}_0 M_+$ is a retract of N. Consider a cone of $\mathrm{Gr}_0 M_- \to N$ in $\mathcal{C}$:

$$\mathrm{Gr}_0 M_- \longrightarrow N \longrightarrow P \longrightarrow \mathrm{Gr}_0 M_-[1]$$

The exact triangle is split in the sense that $P \to \mathrm{Gr}_0 M_-[1]$ is zero. Hence, the morphism $N \to P$ admits a right inverse $P \to N$, unique up to morphisms $P \to \mathrm{Gr}_0 M_-$. There is therefore a unique right inverse $i: P \to N$ such that its composition with the projection $p: N \to \mathrm{Gr}_0 M_+ = \mathrm{Gr}_0 M_-$ is zero. The image of i is then a kernel of p, whose existence is thus established. This is the canonical complement of $\mathrm{Gr}_0 M_- = \mathrm{Gr}_0 M_+$ in N.

As a retract of N, the object P belongs to $\mathcal{C}_{w=0}$ (condition (1) in Definition 1.1). □

Remark 2.6. Our proof of Corollary 2.5 uses the triangulated structure of the category $\mathcal{C}$. This can be avoided when the heart $\mathcal{C}_{w=0}$ is pseudo-Abelian. Namely, consider the composition

$$p: N \longrightarrow Gr_0 M_+ = \mathrm{Gr}_0 M_- \longrightarrow N.$$

It is an idempotent whose image is identified with $\mathrm{Gr}_0 M_- = \mathrm{Gr}_0 M_+$. Since $\mathcal{C}_{w=0}$ is pseudo-Abelian, the morphism p also admits a kernel.

For future use, we check the compatibility of Assumption 2.3 with tensor products. Assume therefore that our category $\mathcal{C}$ is tensor triangulated. Thus, a bilinear bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

is given, and it is assumed to be triangulated in both arguments. Assume also that the weight structure w is compatible with $\otimes$, that is, that

$$\mathcal{C}_{w \leqslant 0} \otimes \mathcal{C}_{w \leqslant 0} \subset \mathcal{C}_{w \leqslant 0} \quad \text{and} \quad \mathcal{C}_{w \geqslant 0} \otimes \mathcal{C}_{w \geqslant 0} \subset \mathcal{C}_{w \geqslant 0}.$$

It follows that the heart $\mathcal{C}_{w=0}$ is a tensor category. Now fix a second set of data as above.

(1′) A morphism $u': M'_- \to M'_+$ in $\mathcal{C}$ between $M'_- \in \mathcal{C}_{w \leqslant 0}$ and $M'_+ \in \mathcal{C}_{w \geqslant 0}$.

(2′) An exact triangle

$$C' \xrightarrow{v'_-} M'_- \xrightarrow{u'} M'_+ \xrightarrow{v'_+} C'[1].$$

Fix an exact triangle

$$D \longrightarrow M_- \otimes M'_- \xrightarrow{u \otimes u'} M_+ \otimes M'_+ \longrightarrow D[1].$$

PROPOSITION 2.7. *If C and C' are without weights minus one and zero, then so is D. In other words, the validity of Assumption 2.3 for u and u' implies the validity of Assumption 2.3 for $u \otimes u'$.*

Proof. We leave it to the reader to first construct an exact triangle

$$D \longrightarrow M_+ \otimes C' \xrightarrow{v_+ \otimes v'_-} C \otimes M'_-[1] \longrightarrow D[1],$$

that is, to show that $D[1]$ is isomorphic to the cone of the morphism $v_+ \otimes v'_-$. Then, consider the morphisms

$$\delta_- \otimes v'_- c'_- : \mathrm{Gr}_0\, M_- \otimes C'_{\leqslant -2} \longrightarrow C_{\leqslant -2} \otimes M'_-[1]$$

and

$$(c_+[1])v_+ \otimes \delta'_+ : M_+ \otimes C'_{\geqslant 1} \longrightarrow C_{\geqslant 1} \otimes \mathrm{Gr}_0\, M'_+[1]$$

(notation as in Theorem 2.4). They are completed to give exact triangles

$$D_{\leqslant -2} \longrightarrow \mathrm{Gr}_0\, M_- \otimes C'_{\leqslant -2} \longrightarrow C_{\leqslant -2} \otimes M'_-[1] \longrightarrow D_{\leqslant -2}[1]$$

and

$$D_{\geqslant 1} \longrightarrow M_+ \otimes C'_{\geqslant 1} \longrightarrow C_{\geqslant 1} \otimes \mathrm{Gr}_0\, M'_+[1] \longrightarrow D_{\leqslant 1}[1].$$

By compatibility of w and $\otimes$, and by Proposition 1.4, the object $D_{\leqslant -2}$ is of weights at most minus two, and $D_{\geqslant 1}$ is of weights at least one. Finally, it remains to construct an exact triangle

$$D_{\leqslant -2} \longrightarrow D \longrightarrow D_{\geqslant 1} \longrightarrow D_{\leqslant -2}[1].$$

We leave this to the reader (hint: use Theorem 2.4(c)). $\qquad\square$

COROLLARY 2.8. *Under the hypotheses of Proposition 2.7, the canonical morphisms*

$$\mathrm{Gr}_0(M_- \otimes M'_-) \longrightarrow \mathrm{Gr}_0\, M_- \otimes \mathrm{Gr}_0\, M'_-$$

and

$$\mathrm{Gr}_0\, M_+ \otimes \mathrm{Gr}_0\, M'_+ \longrightarrow \mathrm{Gr}_0(M_+ \otimes M'_+)$$

are isomorphisms.

3. Example: motives for modular forms

In his article [Sch90], Scholl constructs the Grothendieck motive $M(f)$ for elliptic normalized newforms f of fixed level n and weight $w = r + 2$, for positive integers $n \geqslant 3$ and $r \geqslant 1$. It is a direct factor of a Grothendieck motive, which underlies a Chow motive denoted ${}^r_n\mathcal{W}$ in [Sch90] (this Chow motive depends only on n and r).

In order to establish the relation of ${}^r_n\mathcal{W}$ to the theory of weights, let us begin by setting up the notation. It is identical to that introduced in [Sch90], up to one exception: the letter M used in [Sch90] to denote certain sub-schemes of the modular curve will change to S in order to avoid confusion with the motivic notation used earlier in the present paper. Thus, for our fixed $n \geqslant 3$ and $r \geqslant 1$, let $S_n \in Sm/\mathbb{Q}$ denote the modular curve parametrizing elliptic curves with level n structure, $j : S_n \hookrightarrow \overline{S}_n$ its smooth compactification, and S_n^∞ the complement of S_n in $\overline{S}_n$. Thus, S_n^∞ is of dimension zero. Write $X_n \to S_n$ for the universal elliptic curve, and $\overline{X}_n \to \overline{S}_n$ for the universal generalized elliptic curve. Thus, $\overline{X}_n$ is smooth and proper over $\mathbb{Q}$. The r-fold fibre product $\overline{X}_n^r := \overline{X}_n \times_{\overline{S}_n} \cdots \times_{\overline{S}_n} \overline{X}_n$ of $\overline{X}_n$ over $\overline{S}_n$ is singular for $r \geqslant 2$, and can be desingularized canonically [Del69, Lemmas 5.4 and 5.5] (see also [Sch90, §3]). Denote by $\overline{\overline{X}}_n^r$ this desingularization ($\overline{\overline{X}}_n^r = \overline{X}_n^r$ for $r = 1$). Write X_n^r for the r-fold fibre product X_n over S_n. The symmetric group $\mathfrak{S}_r$ acts on $\overline{X}_n^r$ by permutations, the rth power of the group $\mathbb{Z}/n\mathbb{Z}$ by translations, and the rth power of the group μ_2 by inversion in the fibres. Altogether [Sch90, §1.1.1], this gives a canonical action of the semi-direct product

$$\Gamma_r := ((\mathbb{Z}/n\mathbb{Z})^2 \rtimes \mu_2)^r \rtimes \mathfrak{S}_r$$

by automorphisms on $\overline{X}_n^r$. By the canonical nature of the desingularization, this extends to an action of Γ_r by automorphisms on $\overline{\overline{X}}_n^r$. Of course, this action respects the open sub-scheme X_n^r of $\overline{\overline{X}}_n^r$.

As in [Sch90, § 1.1.2], let $\varepsilon : \Gamma_r \to \{\pm 1\}$ be the morphism which is trivial on $(\mathbb{Z}/n\mathbb{Z})^{2r}$, is the product map on μ_2^r, and is the sign character on $\mathfrak{S}_r$.

DEFINITION 3.1. We make the following definitions.

(a) Let F denote the $\mathbb{Z}$-algebra $\mathbb{Z}[1/(2n \cdot r!)]$.
(b) Let e denote the idempotent in the group ring $F[\Gamma_r]$ associated with ε:

$$ e := \frac{1}{(2n^2)^r \cdot r!} \sum_{\gamma \in \Gamma_r} \varepsilon(\gamma)^{-1} \cdot \gamma = \frac{1}{(2n^2)^r \cdot r!} \sum_{\gamma \in \Gamma_r} \varepsilon(\gamma) \cdot \gamma $$

(observe that $\varepsilon^{-1} = \varepsilon$).

Let M be an object of an F-linear pseudo-Abelian category. If M comes equipped with an action of the group Γ_r, let us agree to denote by M^e the direct factor of M on which Γ_r acts via ε (in other words, the image of e). Let us define the object ${}_n^r\mathcal{W}$ as in [Sch90, § 1.2.2].

DEFINITION 3.2. Denote by ${}_n^r\mathcal{W} := M_{gm}(\overline{\overline{X}}_n^r)^e \in DM_{gm}^{\mathrm{eff}}(\mathbb{Q})_F$ the image of the idempotent e on $M_{gm}(\overline{\overline{X}}_n^r)$.

Given that $\overline{\overline{X}}_n^r$ is smooth and proper, we see that ${}_n^r\mathcal{W}$ is an effective Chow motive over $\mathbb{Q}$. As above, denote by $M_{gm}(X_n^r)^e$ and $M_{gm}^c(X_n^r)^e$ the images of e on $M_{gm}(X_n^r)$ and $M_{gm}^c(X_n^r)$, respectively. The following can be seen as a translation into the language of geometrical motives of the detailed analysis from [Sch90, §§ 2 and 3] of the geometry of the boundary of $\overline{\overline{X}}_n^r$.

THEOREM 3.3. We have the following results.

(a) The motive $M_{gm}^c(X_n^r)^e$ is without weight one. In particular, the object $\mathrm{Gr}_0 M_{gm}^c(X_n^r)^e$ is defined.
(b) The restriction

$$ j_n^{r,*} : {}_n^r\mathcal{W} = M_{gm}(\overline{\overline{X}}_n^r)^e \longrightarrow M_{gm}^c(X_n^r)^e $$

induced by the open immersion j_n^r of X_n^r into $\overline{\overline{X}}_n^r$ factors canonically through an isomorphism

$$ \mathrm{Gr}_0 \, j_n^{r,*} : {}_n^r\mathcal{W} \xrightarrow{\;\sim\;} \mathrm{Gr}_0 \, M_{gm}^c(X_n^r)^e. $$

(c) There is an exact triangle in $DM_{gm}^{\mathrm{eff}}(\mathbb{Q})_F$

$$ C_r \xrightarrow{\delta_+} \mathrm{Gr}_0 \, M_{gm}^c(X_n^r)^e \xrightarrow{i_0} M_{gm}^c(X_n^r)^e \xrightarrow{p_+} C_r[1], $$

where

$$ C_r = M_{gm}(S_n^\infty)[r] $$

is pure of weight r. The exact triangle is canonical up to a replacement of the triple of morphisms (δ_+, i_0, p_+) by $((-1)^r \delta_+, i_0, (-1)^r p_+)$.

Before giving the proof of Theorem 3.3, let us list some of its consequences. First, duality for smooth schemes [Voe00, Theorem 4.3.7(3)] implies the following.

J. WILDESHAUS

COROLLARY 3.4. *We have the following results.*

(a) *The motive $M_{gm}(X_n^r)^e$ is without weight minus one. In particular, the object $\mathrm{Gr}_0\, M_{gm}(X_n^r)^e$ is defined.*

(b) *The morphism*

$$j_n^r : M_{gm}(X_n^r)^e \longrightarrow M_{gm}(\overline{\overline{X}}_n^r)^e = {}^r_n\mathcal{W}$$

factors canonically through an isomorphism

$$\mathrm{Gr}_0\, j_n^r : \mathrm{Gr}_0\, M_{gm}(X_n^r)^e \overset{\sim}{\longrightarrow} {}^r_n\mathcal{W}.$$

(c) *There is an exact triangle in $DM_{gm}^{\mathrm{eff}}(\mathbb{Q})_F$*

$$C_{-(r+1)} \overset{\iota_-}{\longrightarrow} M_{gm}(X_n^r)^e \overset{\pi_0}{\longrightarrow} \mathrm{Gr}_0\, M_{gm}(X_n^r)^e \overset{\delta_-}{\longrightarrow} C_{-(r+1)}[1],$$

where

$$C_{-(r+1)} = M_{gm}(S_n^\infty)(r+1)[r+1]$$

is pure of weight $-(r+1)$. The exact triangle is canonical up to a replacement of the triple of morphisms $(\iota_-, \pi_0, \delta_-)$ by $((-1)^r\iota_-, \pi_0, (-1)^r\delta_-)$.

Remark 3.5. We have the following remarks.

(a) It is well known that the motive $M_{gm}(S_n^\infty)$ is isomorphic to a finite sum of copies of $M_{gm}(\mathbf{Spec}\,\mathbb{Q}(\mu_n))$.

(b) As the proof will show, Theorem 3.3(c) and Corollary 3.4(c) remain true for $r=0$ if one replaces $\mathrm{Gr}_0\, M_{gm}^c(X_n^r)^e$ and $\mathrm{Gr}_0\, M_{gm}(X_n^r)^e$ by $M_{gm}(\tilde{S}_n)$.

Next, note that Theorem 3.3(b) and Corollary 3.4(b) together imply the following.

COROLLARY 3.6. *The canonical morphism $M_{gm}(X_n^r)^e \to M_{gm}^c(X_n^r)^e$ factors canonically through an isomorphism*

$$\mathrm{Gr}_0\, M_{gm}(X_n^r)^e \overset{\sim}{\longrightarrow} \mathrm{Gr}_0\, M_{gm}^c(X_n^r)^e.$$

For any object M of $DM_{gm}^{\mathrm{eff}}(\mathbb{Q})_F$, define motivic cohomology

$$H_{\mathcal{M}}^p(M, F(q)) := \mathrm{Hom}_{DM_{gm}^{\mathrm{eff}}(\mathbb{Q})_F}(M, \mathbb{Z}(q)[p]).$$

When $M = M_{gm}(Y)$ for a scheme $Y \in Sm/\mathbb{Q}$, this gives motivic cohomology $H_{\mathcal{M}}^p(Y, \mathbb{Z}(q))$ of Y, tensored with $F = \mathbb{Z}[1/(2n\cdot r!)]$. Thus, for example,

$$H_{\mathcal{M}}^{r+1}(C_{-(r+1)}, F(r+1)) = H_{\mathcal{M}}^0(S_n^\infty, \mathbb{Z}(0)) \otimes_{\mathbb{Z}} F.$$

Similarly,

$$H_{\mathcal{M}}^{r+2}(C_{-(r+1)}, F(r+\ell+2)) = H_{\mathcal{M}}^1(S_n^\infty, \mathbb{Z}(\ell+1)) \otimes_{\mathbb{Z}} F$$

for any integer ℓ. We get the following refinement of [Sch90, Corollary 1.4.1].

COROLLARY 3.7. *Let $\ell \geqslant 0$ be a second integer. Then the kernel of the morphism*

$$H_{\mathcal{M}}(\iota_-) : (H_{\mathcal{M}}^{r+2}(X_n^r, \mathbb{Z}(r+\ell+2)) \otimes_{\mathbb{Z}} F)^e \longrightarrow H_{\mathcal{M}}^1(S_n^\infty, \mathbb{Z}(\ell+1)) \otimes_{\mathbb{Z}} F$$

equals $H_{\mathcal{M}}^{r+2}(\mathrm{Gr}_0\, M_{gm}(X_n^r)^e, F(r+\ell+2))$.

Proof. This follows from the exact triangle of Corollary 3.4(c), and from the vanishing of $H_{\mathcal{M}}^0(S_n^\infty, \mathbb{Z}(\ell+1))$ (since $\ell+1 \geqslant 1$). $\square$

1212

Recall that following ideas of Beilinson [Bei86], this result can be employed as follows: using the *Eisenstein symbol* defined in [Bei86] one constructs elements in $(H_{\mathcal{M}}^{r+2}(X_n^r, \mathbb{Z}(r + \ell + 2)) \otimes_{\mathbb{Z}} \mathbb{Q})^e$. By Corollary 3.7, linear combinations of such elements vanishing under $H_{\mathcal{M}}(\iota_-)$ lie in the sub-$\mathbb{Q}$-vector space $H_{\mathcal{M}}^{r+2}(\mathrm{Gr}_0\, M_{gm}(X_n^r)^e, \mathbb{Q}(r + \ell + 2))$. It can then be shown that there are sufficiently many such linear combinations, in the sense that their images under the regulator generate *Deligne cohomology* [Sch88, §§ 2 and 4] $H_{\mathcal{D}}^{r+2}(\mathrm{Gr}_0\, M_{gm}(X_n^r)_{/\mathbb{R}}^e, \mathbb{R}(r + \ell + 2))$. Furthermore, the $\mathbb{Q}$-span of these images has the relation to the leading coefficient at $s = -\ell$ of the L-function of $\mathrm{Gr}_0\, M_{gm}(X_n^r)^e$, predicted by Beilinson's conjecture. For details, see forthcoming work of Scholl [Sch97].

Proof of Theorem 3.3. Denote by $\overline{\overline{X}}_n^{r,\infty}$ the complement of the smooth scheme X_n^r in the smooth and proper scheme $\overline{\overline{X}}_n^r$. Localization for the motive with compact support [Voe00, Proposition 4.1.5] shows that there is a canonical exact triangle

$$M_{gm}(\overline{\overline{X}}_n^r)^e \xrightarrow{j_n^{r,*}} M_{gm}^c(X_n^r)^e \longrightarrow M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e[1] \longrightarrow M_{gm}(\overline{\overline{X}}_n^r)^e[1].$$

Following the strategy from [Sch90], we show the following claim:

$$(\mathrm{C}) \qquad M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e = M_{gm}^c(\overline{\overline{X}}_n^{r,\infty})^e \xrightarrow{\sim} M_{gm}^c(S_n^\infty)[r] = M_{gm}(S_n^\infty)[r]$$

canonically up to a sign $(-1)^r$. In particular, the motive $M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e$ is pure of weight $r \geqslant 1$. Claim (C) implies that the above exact triangle is a weight filtration of $M_{gm}^c(X_n^r)^e$ avoiding weight one. Furthermore, the restriction $j_n^{r,*}$ identifies $M_{gm}(\overline{\overline{X}}_n^r)^e$ with $\mathrm{Gr}_0\, M_{gm}^c(X_n^r)^e$.

To show claim (C), observe first that the motivic version of [Sch90, Statement 1.3.0] remains valid: for any $S \in Sm/\mathbb{Q}$, there is a decomposition in $DM_{gm}^{\mathrm{eff}}(\mathbb{Q})_F$ (in fact, already in $DM_{gm}^{\mathrm{eff}}(\mathbb{Q})_{\mathbb{Z}[1/2]}$)

$$M_{gm}^c(\mathbb{G}_m \times_{\mathbb{Q}} S) \cong M_{gm}^c(S)(1)[2] \oplus M_{gm}^c(S)[1],$$

such that inversion $x \mapsto x^{-1}$ on $\mathbb{G}_m$ acts on the first factor by plus one, and on the second by minus one. The projection onto the first factor is canonical, and the projection onto the second factor is canonical up to a sign. Proof: localization [Voe00, Proposition 4.1.5] for the inclusion of $\mathbb{G}_m$ into the projective line; the choice of the second projection is equivalent to the choice of one of the residue morphisms to zero or infinity. Fix one of the two choices of projection

$$\pi_- : M_{gm}^c(\mathbb{G}_m) \longrightarrow M_{gm}^c(\mathbf{Spec}\,\mathbb{Q})[1]$$

(and use the same notation for $M_{gm}^c(\mathbb{G}_m \times_{\mathbb{Q}} S) \longrightarrow M_{gm}^c(S)[1]$ obtained by base change via S). Then to define the morphism

$$j^{0,*} : M_{gm}^c(\overline{\overline{X}}_n^{r,\infty})^e \longrightarrow M_{gm}^c(S_n^\infty)[r],$$

consider the following.

(i) The intersection $\overline{\overline{X}}_n^{r,\infty,\mathrm{reg}}$ of $\overline{\overline{X}}_n^{r,\infty}$ with the non-singular part $\overline{X}_n^{r,\mathrm{reg}}$ of $\overline{X}_n^r$. Explanation: by [Sch90, Theorem 3.1.0 (ii)], the desingularization $\overline{\overline{X}}_n^r \longrightarrow \overline{X}_n^r$ is an isomorphism over $\overline{X}_n^{r,\mathrm{reg}}$. Thus, $\overline{\overline{X}}_n^{r,\infty,\mathrm{reg}}$ is an open sub-scheme of $\overline{\overline{X}}_n^{r,\infty}$.

(ii) The neutral component $\overline{\overline{X}}_n^{r,\infty,0}$ of $\overline{\overline{X}}_n^{r,\infty,\mathrm{reg}}$, that is, its intersection with the Néron model of X_n^r. Thus, $\overline{\overline{X}}_n^{r,\infty,0}$ is an open sub-scheme of $\overline{\overline{X}}_n^{r,\infty,\mathrm{reg}}$.

(iii) The identification of $\overline{\overline{X}}_n^{r,\infty,0}$ with the rth power (over the base S_n^∞) of the neutral component $\overline{\overline{X}}_n^{1,\infty,0}$ for $r = 1$. The latter can be identified with $\mathbb{G}_m \times_{\mathbb{Q}} S_n^\infty$, canonically up to an automorphism $x \mapsto x^{-1}$. Hence, $\overline{\overline{X}}_n^{r,\infty,0} \cong \mathbb{G}_m^r \times_{\mathbb{Q}} S_n^\infty$, canonically up to an automorphism

$$(x_1, \ldots, x_r) \longmapsto (x_1, \ldots, x_r)^{-1}.$$

The three steps (i)–(iii) give an open immersion

$$j^0 : \mathbb{G}_m^r \times_{\mathbb{Q}} S_n^\infty \hookrightarrow \overline{\overline{X}}_n^{r,\infty},$$

which by contravariance of M_{gm}^c induces a morphism

$$j^{0,*} : M_{gm}^c(\overline{\overline{X}}_n^{r,\infty})^e \longrightarrow M_{gm}^c(\mathbb{G}_m^r \times_{\mathbb{Q}} S_n^\infty).$$

Its composition with the rth power of π_- gives the desired morphism

$$M_{gm}^c(\overline{\overline{X}}_n^{r,\infty})^e \longrightarrow M_{gm}^c(S_n^\infty)[r],$$

equally denoted $j^{0,*}$, and canonical up to a sign $(-1)^r$. In the context of twisted Poincaré duality theories (H^*, H_*), Scholl's main technical result [Sch90, Theorem 1.3.3] is equivalent to stating that the morphism induced by $j^{0,*}$ on the level of the theory H_* is an isomorphism.

Our observation is simply that the same proof as that given in [Sch90] runs through, with (H^*, H_*) replaced by (M_{gm}, M_{gm}^c).

More precisely, [Sch90, Lemma 1.3.1] holds for M_{gm}^c, and hence the proof of [Sch90, Proposition 2.4.1] runs through for M_{gm}^c. The latter result implies that the schemes occurring in a suitable stratification of the complement of $\overline{\overline{X}}_n^{r,\infty,\mathrm{reg}}$ in $\overline{\overline{X}}_n^{r,\infty}$ all have trivial $M_{gm}^{c,e}$ (cf. [Sch90, Proof of Theorem 3.1.0(ii)]). This shows that step (i) induces an isomorphism

$$M_{gm}^c(\overline{\overline{X}}_n^{r,\infty})^e \xrightarrow{\sim} M_{gm}^c(\overline{\overline{X}}_n^{r,\infty,\mathrm{reg}})^e.$$

To deal with step (ii), one shows that the group Γ_r acts transitively on the set of components of the singular part of $\overline{\overline{X}}_n^{r,\infty,\mathrm{reg}}$, and that the stabilizer of each component admits a subgroup of order two acting trivially on the component, but having trivial intersection with the kernel of ε (cf. [Sch90, Proof of Theorem 3.1.0(iii)]). This shows first that the singular part of $\overline{\overline{X}}_n^{r,\infty,\mathrm{reg}}$ does not contribute to $M_{gm}^{c,e}$, and then that

$$M_{gm}^c(\overline{\overline{X}}_n^{r,\infty,\mathrm{reg}})^e \longrightarrow M_{gm}^c(\overline{\overline{X}}_n^{r,\infty,0})^{e'}$$

is an isomorphism, where e' denotes the projection onto the eigenspace for the restriction of ε to the subgroup $\mu_2^r \rtimes \mathfrak{S}_r$ of Γ_r. To conclude, we apply the motivic version of [Sch90, Lemma 1.3.1] to see that $\pi_-^{\otimes r}$ induces an isomorphism $M_{gm}^c(\mathbb{G}_m^r \times_{\mathbb{Q}} S_n^\infty)^{e'} \xrightarrow{\sim} M_{gm}^c(S_n^\infty)[r]$. $\qquad\square$

Remark 3.8. We make the following remarks.

(a) The proof of Theorem 3.3 shows that the open immersion of the non-singular part $\overline{X}_n^{r,\mathrm{reg}}$ of $\overline{X}_n^r$ into $\overline{\overline{X}}_n^r$ induces an isomorphism

$$M_{gm}(\overline{\overline{X}}_n^r)^e \xrightarrow{\sim} M_{gm}^c(\overline{X}_n^{r,\mathrm{reg}})^e.$$

In particular, $M_{gm}^c(\overline{X}_n^{r,\mathrm{reg}})^e$ is a Chow motive. By duality for smooth schemes [Voe00, Theorem 4.3.7 3],

$$M_{gm}(\overline{X}_n^{r,\mathrm{reg}})^e \longrightarrow M_{gm}(\overline{\overline{X}}_n^r)^e$$

1214

is also an isomorphism, and hence so is the canonical morphism

$$M_{gm}(\overline{X}_n^{r,\mathrm{reg}})^e \longrightarrow M_{gm}^c(\overline{X}_n^{r,\mathrm{reg}})^e.$$

The construction of the motives $M(f)$ can therefore also be done using the smooth non-proper scheme $\overline{X}_n^{r,\mathrm{reg}}$ instead of $\overline{\overline{X}}_n^r$.

(b) A slightly closer look at the proof of [Sch90, Theorem 3.1.0(ii)] reveals that the open immersion of $\overline{X}_n^{r,\mathrm{reg}}$ into $\overline{X}_n^r$ also induces an isomorphism

$$M_{gm}(\overline{X}_n^r)^e \xrightarrow{\ \sim\ } M_{gm}^c(\overline{X}_n^{r,\mathrm{reg}})^e.$$

By remark (a), $M_{gm}(\overline{X}_n^r)^e = M_{gm}^c(\overline{X}_n^r)^e$ is a Chow motive. The construction of the motives $M(f)$ can therefore also be done using the non-smooth (for $r \geqslant 2$) proper scheme $\overline{X}_n^r$ instead of $\overline{\overline{X}}_n^r$.

4. Weights, boundary motive and interior motive

This section contains our main result (Theorem 4.3). We list its main consequences, and define in particular the motivic analogue of (certain direct factors of) interior cohomology (Definition 4.9). Throughout, we assume k to admit resolution of singularities.

Let us fix $X \in Sm/k$. The *boundary motive* $\partial M_{gm}(X)$ of X (see [Wil06, Definition 2.1]) fits into a canonical exact triangle

$$(*) \qquad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1]$$

in $DM_{gm}^{\mathrm{eff}}(k)$. The algebra of *finite correspondences* $c(X, X)$ acts on $M_{gm}(X)$ (see [Voe00, p. 190]). Denote by ${}^t c(X, X)$ the transposed algebra: a cycle $\mathfrak{Z}$ on $X \times_k X$ lies in ${}^t c(X, X)$ if and only if ${}^t \mathfrak{Z} \in c(X, X)$. The definition of composition of correspondences [Voe00] shows that the intersection $c(X, X) \cap {}^t c(X, X)$ acts on $M_{gm}^c(X)$.

DEFINITION 4.1. We make the following definitions.

(a) Define the algebra $c_{1,2}(X, X)$ as the intersection of the algebras $c(X, X)$ and ${}^t c(X, X)$. As an Abelian group, $c_{1,2}(X, X)$ is thus free on the symbols (Z), where Z runs through the integral closed sub-schemes of $X \times_k X$, such that both projections to the components X are finite on Z, and map Z surjectively to a connected component of X. Multiplication in $c_{1,2}(X, X)$ is defined by composition of correspondences as in [Voe00, p. 190].

(b) Denote by ^{t} the canonical anti-involution on $c_{1,2}(X, X)$ mapping a cycle $\mathfrak{Z}$ to ${}^t \mathfrak{Z}$.

This results directly from the definitions that the algebra $c_{1,2}(X, X)$ acts on the triangle $(*)$ in the sense that it acts on the three objects, and the morphisms are $c_{1,2}(X, X)$-equivariant. Denote by $\bar{c}_{1,2}(X, X)$ the quotient of $c_{1,2}(X, X)$ by the kernel of this action. Fix a commutative flat $\mathbb{Z}$-algebra F, and an idempotent e in $\bar{c}_{1,2}(X, X) \otimes_{\mathbb{Z}} F$. Denote by $M_{gm}(X)^e$, $M_{gm}^c(X)^e$ and $\partial M_{gm}(X)^e$ the images of e on $M_{gm}(X)$, $M_{gm}^c(X)$ and $\partial M_{gm}(X)$, respectively, considered as objects of the category $DM_{gm}^{\mathrm{eff}}(k)_F$. We are ready to set up the data (1), (2) considered in §2, for $\mathcal{C} := DM_{gm}^{\mathrm{eff}}(k)_F$.

(1) The morphism u is the morphism $M_{gm}(X)^e \to M_{gm}^c(X)^e$. By Corollary 1.14 and Definition 1.1(1), the object $M_{gm}(X)^e$ belongs indeed to $DM_{gm}^{\mathrm{eff}}(k)_{F,w\leqslant 0}$, and $M_{gm}^c(X)^e$ to $DM_{gm}^{\mathrm{eff}}(k)_{F,w\geqslant 0}$.

1215

(2) Our choice of cone of u is $\partial M_{gm}(X)^e[1]$, together with the exact triangle

$$\partial M_{gm}(X)^e \xrightarrow{v_-} M_{gm}(X)^e \xrightarrow{u} M_{gm}^c(X)^e \xrightarrow{v_+} \partial M_{gm}(X)^e[1]$$

in $DM_{gm}^{\mathrm{eff}}(k)_F$ induced by $(*)$.

Observe that our data (1), (2) are stable under the natural action of

$$GCen_{\bar{c}_{1,2}(X,X)}(e) := \{z \in \bar{c}_{1,2}(X,X) \otimes_{\mathbb{Z}} F, \ ze = eze\}.$$

In particular, they are stable under the action of the centralizer $Cen_{\bar{c}_{1,2}(X,X)}(e)$ of e in $\bar{c}_{1,2}(X,X) \otimes_{\mathbb{Z}} F$. In this context, Assumption 2.3 reads as follows.

ASSUMPTION 4.2. *The direct factor $\partial M_{gm}(X)^e$ of the boundary motive of X is without weights minus one and zero.*

Thus, we may and do fix a weight filtration

$$C_{\leqslant -2} \xrightarrow{c_-} \partial M_{gm}(X)^e \xrightarrow{c_+} C_{\geqslant 1} \xrightarrow{\delta_C} C_{\leqslant -2}[1]$$

avoiding weights minus one and zero. Theorem 2.4, Corollary 2.5 and the adjunction property from Proposition 2.2 then give the following.

THEOREM 4.3. *Fix the data (1), (2), and assume that Assumption 4.2 holds.*

(a) *The motive $M_{gm}(X)^e$ is without weight minus one, and the motive $M_{gm}^c(X)^e$ is without weight one. In particular, the effective Chow motives $\mathrm{Gr}_0 M_{gm}(X)^e$ and $\mathrm{Gr}_0 M_{gm}^c(X)^e$ are defined, and they carry a natural action of $GCen_{\bar{c}_{1,2}(X,X)}(e)$.*

(b) *There are canonical exact triangles*

$$(3) \qquad C_{\leqslant -2} \xrightarrow{v_- c_-} M_{gm}(X)^e \xrightarrow{\pi_0} \mathrm{Gr}_0 M_{gm}(X)^e \xrightarrow{\delta_-} C_{\leqslant -2}[1]$$

and

$$(4) \qquad C_{\geqslant 1} \xrightarrow{\delta_+} \mathrm{Gr}_0 M_{gm}^c(X)^e \xrightarrow{i_0} M_{gm}^c(X)^e \xrightarrow{(c_+[1])v_+} C_{\geqslant 1}[1],$$

which are stable under the natural action of $GCen_{\bar{c}_{1,2}(X,X)}(e)$.

(c) *There is a canonical isomorphism $\mathrm{Gr}_0 M_{gm}(X)^e \xrightarrow{\sim} \mathrm{Gr}_0 M_{gm}^c(X)^e$ in $CHM^{\mathrm{eff}}(k)_F$. As a morphism, it is uniquely determined by the property of making the diagram*

$$
\begin{array}{ccc}
M_{gm}(X)^e & \xrightarrow{u} & M_{gm}^c(X)^e \\
\downarrow{\scriptstyle \pi_0} & & \uparrow{\scriptstyle i_0} \\
\mathrm{Gr}_0 M_{gm}(X)^e & \longrightarrow & \mathrm{Gr}_0 M_{gm}^c(X)^e
\end{array}
$$

commute; in particular, it is $GCen_{\bar{c}_{1,2}(X,X)}(e)$-equivariant. Its inverse makes the diagram

$$
\begin{array}{ccc}
C_{\geqslant 1} & \xrightarrow{\delta_C} & C_{\leqslant -2}[1] \\
\downarrow{\scriptstyle \delta_+} & & \uparrow{\scriptstyle \delta_-} \\
\mathrm{Gr}_0 M_{gm}^c(X)^e & \longrightarrow & \mathrm{Gr}_0 M_{gm}(X)^e
\end{array}
$$

commute.

(d) *Let $N \in CHM(k)_F$ be a Chow motive. Then π_0 and i_0 induce isomorphisms*

$$\mathrm{Hom}_{CHM(k)_F}(\mathrm{Gr}_0 M_{gm}(X)^e, N) \xrightarrow{\sim} \mathrm{Hom}_{DM_{gm}(k)_F}(M_{gm}(X)^e, N)$$

1216

and

$$\operatorname{Hom}_{CHM(k)_F}(N, \operatorname{Gr}_0 M_{gm}^c(X)^e) \overset{\sim}{\longrightarrow} \operatorname{Hom}_{DM_{gm}(k)_F}(N, M_{gm}^c(X)^e).$$

(e) *Let* $M_{gm}(X)^e \to N \to M_{gm}^c(X)^e$ *be a factorization of* u *through a Chow motive* $N \in CHM(k)_F$. *Then* $\operatorname{Gr}_0 M_{gm}(X)^e = \operatorname{Gr}_0 M_{gm}^c(X)^e$ *is canonically a direct factor of* N, *with a canonical direct complement.*

We explicitly mention the following immediate consequence of Theorem 4.3.

COROLLARY 4.4. *Fix* X *and* e, *and suppose that* $\partial M_{gm}(X)^e = 0$, *that is, that*

$$u : M_{gm}(X)^e \overset{\sim}{\longrightarrow} M_{gm}^c(X)^e.$$

Then $M_{gm}(X)^e \cong M_{gm}^c(X)^e$ *are effective Chow motives.*

Of course, this also follows from Corollary 1.16. The author knows of no proof of Corollary 4.4 'avoiding weights' when $e \neq 1$. (For $e = 1$, we leave it to the reader to show (using for example [Voe00, Corollary 4.2.5]) that the assumption $\partial M_{gm}(X) = 0$ is equivalent to X being proper.)

Remark 4.5. It is not difficult to see that Assumption 4.2 is actually implied by parts (a) and (c) of Theorem 4.3.

Henceforth, we identify $\operatorname{Gr}_0 M_{gm}(X)^e$ and $\operatorname{Gr}_0 M_{gm}^c(X)^e$ via the canonical isomorphism of Theorem 4.3(c).

COROLLARY 4.6. *In the situation considered in Theorem 4.3, let* $\widetilde{X}$ *be any smooth compactification of* X. *Then* $\operatorname{Gr}_0 M_{gm}(X)^e$ *is canonically a direct factor of the Chow motive* $M_{gm}(\widetilde{X})$, *with a canonical direct complement.*

Proof. Indeed, the morphism u factors canonically through $M_{gm}(\widetilde{X})$:

$$M_{gm}(X)^e \hookrightarrow M_{gm}(X) \longrightarrow M_{gm}(\widetilde{X}) \longrightarrow M_{gm}^c(X) \longrightarrow M_{gm}^c(X)^e.$$

Hence, we may apply Theorem 4.3(e). □

Recall that the category $CHM(k)_F$ is pseudo-Abelian. Thus, the construction of a submotive of $M_{gm}(\widetilde{X})$ does not *a priori* necessitate the *identification*, but only the *existence* of a complement. In our situation, Corollary 4.6 states that the complement of $\operatorname{Gr}_0 M_{gm}(X)^e$ *is* canonical. This shows that Assumption 4.2 is indeed rather restrictive, an observation confirmed by part (c) of the following results on the Hodge theoretic and ℓ-adic realizations ([Hub04, §2 and Corrigendum]; see [DG05, §1.5] for a simplification of this approach). They can be seen as applications of the *cohomological weight spectral sequence* [Bon07, Theorem 2.4.1 and Remark 2.4.2] in a very special case.

THEOREM 4.7. *Keep the situation considered in Theorem 4.3. Assume that* k *can be embedded into the field* $\mathbb{C}$ *of complex numbers. Fix one such embedding. Let* H^* *be the Hodge theoretic realization [Hub04, Corollary 2.3.5 and Corrigendum], followed by the canonical cohomology functor, that is, the functor on* $DM_{gm}^{\mathrm{eff}}(k)_F$ *given by Betti cohomology of the topological space of* $\mathbb{C}$*-valued points, tensored with* $\mathbb{Q} \otimes_{\mathbb{Z}} F$, *and with its natural mixed Hodge structure. Let* $n \in \mathbb{N}$.

(a) *The morphisms* π_0 *and* i_0 *induce isomorphisms*

$$H^n(\operatorname{Gr}_0 M_{gm}(X)^e) \overset{\sim}{\longrightarrow} W_n H^n(M_{gm}(X)^e) = (W_n H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e$$

and

$$\frac{(H^n_c(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e}{(W_{n-1} H^n_c(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e} = \frac{H^n(M^c_{gm}(X)^e)}{W_{n-1} H^n(M^c_{gm}(X)^e)} \xrightarrow{\sim} H^n(\mathrm{Gr}_0 M_{gm}(X)^e).$$

Here, W_r denotes the rth filtration step of the weight filtration of a mixed Hodge structure (thus, the weights of $H^n(X(\mathbb{C}), \mathbb{Q})$ are at least n, and those of $H^n_c(X(\mathbb{C}), \mathbb{Q})$ are at most n).

(b) The isomorphisms of part (a) identify $H^n(\mathrm{Gr}_0 M_{gm}(X)^e)$ with the image of the natural morphism

$$(H^n_c(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e \longrightarrow (H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e.$$

(c) The image of $(H^n_c(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e$ in $(H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e$ equals the lowest weight filtration step W_n of $(H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e$.

The reader should be aware that the algebra $\bar{c}_{1,2}(X, X)$ acts contravariantly on Betti cohomology $H^n(X(\mathbb{C}), \mathbb{Q})$. The same remark applies of course to the ℓ-adic realization, which we consider now.

THEOREM 4.8. *Keep the situation considered in Theorem 4.3, and fix a prime $\ell > 0$. Assume that k is finitely generated over its prime field, and of characteristic zero. Let H^* be the ℓ-adic realization [Hub04, Corollary 2.3.4 and Corrigendum], followed by the canonical cohomology functor, that is, the functor on $DM^{\mathrm{eff}}_{gm}(k)_F$ given by ℓ-adic cohomology of the base change to a fixed algebraic closure $\bar{k}$ of k, tensored with $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} F$, and with its natural action of the absolute Galois group G_k of k. Let $n \in \mathbb{N}$.*

(a) *The morphisms π_0 and i_0 induce isomorphisms*

$$H^n(\mathrm{Gr}_0 M_{gm}(X)^e) \xrightarrow{\sim} W_n H^n(M_{gm}(X)^e) = (W_n H^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e$$

and

$$\frac{(H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e}{(W_{n-1} H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e} = \frac{H^n(M^c_{gm}(X)^e)}{W_{n-1} H^n(M^c_{gm}(X)^e)} \xrightarrow{\sim} H^n(\mathrm{Gr}_0 M_{gm}(X)^e).$$

Here, W_r denotes the rth filtration step of the weight filtration of a G_k-module, and $X_{\bar{k}}$ denotes the base change $X \otimes_k \bar{k}$ of X to $\bar{k}$ (thus, the weights of $H^n(X_{\bar{k}}, \mathbb{Q}_\ell)$ are at least n, and those of $H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell)$ are at most n).

(b) *The isomorphisms of part (a) identify $H^n(\mathrm{Gr}_0 M_{gm}(X)^e)$ with the image of the natural morphism*

$$(H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e \longrightarrow (H^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e.$$

(c) *The image of $(H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e$ in $(H^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e$ equals the lowest weight filtration step W_n of $(H^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e$.*

DEFINITION 4.9. Fix the data (1), (2), and assume that Assumption 4.2 holds. We call $\mathrm{Gr}_0 M_{gm}(X)^e$ the *e-part of the interior motive of X*.

This terminology is motivated by Theorems 4.7(b) and 4.8(b), which show that after passage to rational coefficients, the realizations of $\mathrm{Gr}_0 M_{gm}(X)^e$ are classes of complexes, whose cohomology equals the part of the interior cohomology of X fixed by e.

Proof of Theorems 4.7 and 4.8. Consider the exact triangle

$$(*) \qquad C_{\leqslant -2} \longrightarrow M_{gm}(X)^e \xrightarrow{\pi_0} \mathrm{Gr}_0 M_{gm}(X)^e \longrightarrow C_{\leqslant -2}[1]$$

1218

from Theorem 4.3(b). Recall that as suggested by the notation, the motive $C_{\leqslant -2}$ is of weights at most minus two. The cohomological functor H^* transforms it into a long exact sequence

$$H^{n-1}(C_{\leqslant -2}) \longrightarrow H^n(\mathrm{Gr}_0\, M_{gm}(X)^e) \stackrel{H^n(\pi_0)}{\longrightarrow} H^n(M_{gm}(X)^e) \longrightarrow H^n(C_{\leqslant -2}).$$

The essential information we need to use is that H^n transforms Chow motives into objects which are pure of weight n, for any $n \in \mathbb{N}$. Since $C_{\leqslant -2}$ admits a filtration whose cones are Chow motives sitting in degrees at least two, the object $H^n(C_{\leqslant -2})$ admits a filtration whose graded pieces are of weights at least $n + 2$. Since our coefficients are $\mathbb{Q}$-vector spaces, there are no non-trivial morphisms between objects of disjoint weights. The above long exact sequence then shows that

$$H^n(\pi_0) : H^n(\mathrm{Gr}_0\, M_{gm}(X)^e) \longrightarrow H^n(M_{gm}(X)^e)$$

is injective, and its image is identical to the part of weight n of $H^n(M_{gm}(X)^e)$. This shows the part of claim (a) concerning π_0.

Now recall that the realization functor is compatible with the tensor structures [Hub04, Corollaries 2.3.5 and 2.3.4], and sends the Tate motive $\mathbb{Z}(1)$ to the dual of the Tate object [Hub04, Theorem 2.3.3]. It follows that it is compatible with duality. Since, on the one hand, $M_{gm}(X)$ and $M_{gm}^c(X)$ are in duality [Voe00, Theorem 4.3.7 3] and, on the other hand, the same is true for Betti, respectively ℓ-adic cohomology and Betti, respectively ℓ-adic cohomology with compact support, we see that H^n sends the motive with compact support $M_{gm}^c(X)$ to cohomology with compact support H_c^n of X.

Now repeat the above argument for the exact triangle

$$(4) \qquad C_{\geqslant 1} \longrightarrow \mathrm{Gr}_0\, M_{gm}^c(X)^e \stackrel{i_0}{\longrightarrow} M_{gm}^c(X)^e \longrightarrow C_{\geqslant 1}[1]$$

from Theorem 4.3(b). This shows the remaining part of claim (a).

Claims (b) and (c) follow, once we observe that the composition of $H^n(i_0)$ and $H^n(\pi_0)$ equals the canonical morphism from cohomology with compact support to cohomology without support. $\qquad\square$

Remark 4.10. We make the following remarks.

(a) The proof of Theorems 4.7 and 4.8 uses the fact that in the respective target categories (mixed Hodge structures in Theorem 4.7, Galois representations in Theorems 4.8), there are no non-trivial morphisms between objects of disjoint weights. This is true as long as we work with coefficients which are $\mathbb{Q}$-vector spaces. Note that recent work of Lecomte [Lec08] establishes the existence of a Betti realization which does not require the passage to $\mathbb{Q}$-coefficients. In particular [Lec08, Theorem 1.1], for a smooth quasi-projective variety X, it yields the classical singular cohomology of the topological space $X(\mathbb{C})$. We have no statement (and not even a guess) to offer on the image of $\mathrm{Gr}_0\, M_{gm}(X)^e$ under the realization of [Lec08].

(b) The author does not know whether for general $Y \in Sm/k$ it is possible (or even reasonable to expect) to find a complex computing interior (Betti or ℓ-adic) cohomology of Y, and through which the natural morphism $R\Gamma_c(Y) \to R\Gamma(Y)$ factors.

Remark 4.11. When k is a number field, Theorems 4.7(b) and 4.8(b) tell us in particular that the L-function of the Chow motive $\mathrm{Gr}_0\, M_{gm}(X)^e$ is computed via (the e-part of) interior cohomology of X.

1219

J. WILDESHAUS

Example 4.12. Let C be a smooth projective curve, and $P \in C$ a k-rational point. Put $X := C - P$. Localization [Voe00, Proposition 4.1.5] shows that there is an exact triangle

$$M_{gm}(P) \longrightarrow M_{gm}(C) \longrightarrow M_{gm}^c(X) \longrightarrow M_{gm}(P)[1].$$

The morphism $M_{gm}(P) \to M_{gm}(C)$ is split; hence $M_{gm}^c(X)$ is a direct factor of $M_{gm}(C)$. It is therefore a Chow motive. By duality [Voe00, Theorem 4.3.7(3)], the same is then true for $M_{gm}(X)$. In particular, both $M_{gm}(X)$ and $M_{gm}^c(X)$ are pure of weight zero. However, the morphism $u : M_{gm}(X) \to M_{gm}^c(X)$ is not an isomorphism (look at degree zero or minus two, or check that the conclusions of Theorems 4.7(c) and 4.8(c) do not hold). Therefore, Assumption 4.2 is not fulfilled for $e = 1$. Of course, this can be seen directly: the boundary motive $\partial M_{gm}(X)$ has a weight filtration

$$\mathbb{Z}(1)[1] \longrightarrow \partial M_{gm}(X) \longrightarrow \mathbb{Z}(0) \longrightarrow \mathbb{Z}(1)[2]$$

(which is necessarily split since there are no non-trivial morphisms from $\mathbb{Z}(0)$ to $\mathbb{Z}(1)[2]$). Orthogonality (3) in Definition 1.1 then shows that there are no non-trivial morphisms from an object of weights at most minus two to $\partial M_{gm}(X)$, and no non-trivial morphisms from $\partial M_{gm}(X)$ to an object of weights at least one. The object $\partial M_{gm}(X)$ being non-trivial, we conclude that it does not admit a weight filtration avoiding weights minus one and zero.

More generally, we have the following, which again illustrates just how restrictive Assumption 4.2 is.

PROPOSITION 4.13. *Fix data (1) and (2) as before. Assume that X admits a smooth compactification $\widetilde{X}$ such that the complement $Y = \widetilde{X} - X$ is smooth. Then the following statements are equivalent.*

(i) *Assumption 4.2 is valid, that is, the object $\partial M_{gm}(X)^e$ is without weights minus one and zero.*

(ii) *The object $\partial M_{gm}(X)^e$ is trivial (hence, the conclusion of Corollary 4.4 holds).*

Proof. Statement (ii) clearly implies statement (i). In order to show that it is implied by statement (i), let us show that the hypothesis on $\widetilde{X}$ and Y forces the boundary motive $\partial M_{gm}(X)$ to lie in the intersection

$$DM_{gm}^{\mathrm{eff}}(k)_{w \geqslant -1} \cap DM_{gm}^{\mathrm{eff}}(k)_{w \leqslant 0}.$$

By orthogonality (3) in Definition 1.1, the same is then true for its direct factor $\partial M_{gm}(X)^e$. Thus, the only way for $\partial M_{gm}(X)^e$ to avoid weights minus one and zero is to be trivial (again by orthogonality).

In order to show our claim, apply [Wil06, Proposition 2.4] to see that $\partial M_{gm}(X)$ is isomorphic to the shift by $[-1]$ of a choice of cone of the canonical morphism

$$M_{gm}(Y) \oplus M_{gm}(X) \longrightarrow M_{gm}(\widetilde{X}).$$

In particular, there is a morphism $c_+ : \partial M_{gm}(X) \to M_{gm}(Y)$, and an exact triangle

$$(W) \qquad C_- \longrightarrow \partial M_{gm}(X) \xrightarrow{c_+} M_{gm}(Y) \longrightarrow C_-[1],$$

where C_- equals the shift by $[-1]$ of a cone of

$$M_{gm}(X) \longrightarrow M_{gm}(\widetilde{X}).$$

By assumption, Y is smooth and proper, hence $M_{gm}(Y)$, as a Chow motive, is pure of weight zero. Duality for smooth schemes [Voe00, Theorem 4.3.7 3] shows that C_- is pure of weight minus one.

1220

Hence, (W) is a weight filtration of $\partial M_{gm}(X)$ by objects of weights minus one and zero. Proposition 1.4 then shows that $\partial M_{gm}(X)$ belongs indeed to

$$DM_{gm}^{\text{eff}}(k)_{w \geqslant -1} \cap DM_{gm}^{\text{eff}}(k)_{w \leqslant 0}. \qquad \qquad \square$$

Corollary 4.6 allows us to say more about the étale realizations.

THEOREM 4.14. *Keep the situation considered in Theorem 4.3, and fix a prime $\ell > 0$. Assume that k is the quotient field of a Dedekind domain A, and that k is of characteristic zero. Fix a non-zero prime ideal $\mathfrak{p}$ of A, and let p denote its residue characteristic.*

(a) *Assume that $p \neq \ell$. Then a sufficient condition for the ℓ-adic realization $H^*(\text{Gr}_0\, M_{gm}(X)^e)$ to be unramified at $\mathfrak{p}$ is the existence of some smooth compactification of X having good reduction at $\mathfrak{p}$. A sufficient condition for $H^*(\text{Gr}_0\, M_{gm}(X)^e)$ to be semi-stable at $\mathfrak{p}$ is the existence of some smooth compactification of X having simple semi-stable reduction at $\mathfrak{p}$.*

(b) *Assume that $p = \ell$, and that the residue field $A/\mathfrak{p}$ is perfect. Then a sufficient condition for the p-adic realization $H^*(\text{Gr}_0\, M_{gm}(X)^e)$ to be crystalline at $\mathfrak{p}$ is the existence of some smooth compactification of X having good reduction at $\mathfrak{p}$. A sufficient condition for $H^*(\text{Gr}_0\, M_{gm}(X)^e)$ to be semi-stable at $\mathfrak{p}$ is the existence of some smooth compactification of X having simple semi-stable reduction at $\mathfrak{p}$.*

Proof. By Corollary 4.6, $\text{Gr}_0\, M_{gm}(X)^e$ is a direct factor of the motive $M_{gm}(\widetilde{X})$ of any smooth compactification $\widetilde{X}$ of X. Hence, $H^*(\text{Gr}_0\, M_{gm}(X)^e)$ is a direct factor of the cohomology of any such $\widetilde{X}$.

Part (a) uses the spectral sequence [Del72, Scholie 2.5] relating cohomology with coefficients in the vanishing cycle sheaves ψ^q to cohomology of the generic fibre $\widetilde{X}$. By [RZ82, Korollarie 2.25] ([Del72, Theorem 3.3] when $p = 0$), our assumption on the reduction of $\widetilde{X}$ implies that the inertia group acts trivially on the ψ^q. It therefore acts unipotently on the cohomology of $\widetilde{X}$. Part (b) follows from the C_{st}-conjecture, proved by Tsuji [Tsu99, Theorem 0.2] (see also [Niz08] for a proof via K-theory). $\qquad \square$

By [Voe00, Theorem 4.3.7], the category $DM_{gm}(k)_F$ is a rigid tensor triangulated category. Furthermore, if X is smooth of pure dimension n, then the objects $M_{gm}(X)$ and $M_{gm}^c(X)(-n)[-2n]$ are canonically dual to each other. By [Wil06, Theorem 6.1], the boundary motive $\partial M_{gm}(X)$ is canonically dual to $\partial M_{gm}(X)(-n)[-(2n-1)]$. Furthermore, these dualities fit together to give an identification of the dual of the exact triangle

$$(*) \qquad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1]$$

and the exact triangle $(*)(-n)[-2n]$. The construction of the duality isomorphism [Wil06] shows that the dual of the action of the algebra $\bar{c}_{1,2}(X, X)$ on $(*)$ equals the natural (anti-)action given by the composition of the canonical action on $(*)(-n)[-2n]$, preceded by the anti-involution ^{t}. Consider the idempotent $^t e$ (i.e. the transposition of e).

PROPOSITION 4.15. *We have the following results.*

(a) *Assumption 4.2 is equivalent to any of the following statements.*
   (i) *Both $\partial M_{gm}(X)^e$ and $\partial M_{gm}(X)^{t e}$ are without weight minus one.*
   (ii) *Both $\partial M_{gm}(X)^e$ and $\partial M_{gm}(X)^{t e}$ are without weight zero.*

   In particular, Assumption 4.2 is satisfied for e if and only if it is satisfied for the transposition $^t e$.

1221

(b) *Assume that e is symmetric, that is, that $e = {}^t e$. Then Assumption 4.2 is equivalent to any of the following statements:*
   (i) $\partial M_{gm}(X)^e$ *is without weight minus one;*
   (ii) $\partial M_{gm}(X)^e$ *is without weight zero.*

Proof. Indeed, $\partial M_{gm}(X)^e$ is dual to $\partial M_{gm}(X)^{{}^t e}(-n)[-(2n-1)]$. Now observe that $\mathbb{Z}(-n)[-(2n-1)]$ is pure of weight one. Then use unicity of weight filtrations avoiding weight minus one respectively weight zero (Corollary 1.9). $\qquad\square$

Example 4.16. Assume that an abstract group G acts by automorphisms on X.

(a) The action of G translates into a morphism of algebras $\mathbb{Z}[G] \to c_{1,2}(X, X)$. This morphism transforms the natural anti-involution $*$ of $\mathbb{Z}[G]$ induced by $g \mapsto g^{-1}$ into the anti-involution ^{t} of $c_{1,2}(X, X)$.

(b) Let F be a flat $\mathbb{Z}$-algebra, and e an idempotent in $F[G]$. The morphism of part (a), then allows us to consider the image of e (equally denoted by e) in $c_{1,2}(X, X)$, then in $\bar{c}_{1,2}(X, X)$, and to ask whether Assumption 4.2 is valid for e.

(c) As a special case of part (b), consider the case when G is finite, its order r is invertible in F, and e is the idempotent in $F[G]$ associated with a character ε on G with values in the multiplicative group F^*:

$$e = \frac{1}{r} \sum_{g \in G} \varepsilon(g)^{-1} \cdot g.$$

Observe that the idempotent $e \in \bar{c}_{1,2}(X, X)$ is symmetric if $\varepsilon^{-1} = \varepsilon$.

(d) Let us consider the situation from §3, and show that Assumption 4.2 is satisfied for $X = X_n^r \in Sm/\mathbb{Q}$ and

$$e = \frac{1}{(2n^2)^r \cdot r!} \sum_{\gamma \in \Gamma_r} \varepsilon(\gamma)^{-1} \cdot \gamma.$$

As in §3, denote by $\overline{\overline{X}}_n^{r,\infty}$ the complement of X_n^r in $\overline{\overline{X}}_n^r$. By [Wil06, Proposition 2.4], the object $\partial M_{gm}(X_n^r)^e$ is canonically isomorphic to the shift by $[-1]$ of a canonical choice of cone of the canonical morphism

$$M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e \oplus M_{gm}(X_n^r)^e \longrightarrow M_{gm}(\overline{\overline{X}}_n^r)^e.$$

In particular, there is a canonical morphism $c_+ : \partial M_{gm}(X_n^r)^e \to M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e$, and an exact triangle

$$C_- \longrightarrow \partial M_{gm}(X_n^r)^e \xrightarrow{c_+} M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e \longrightarrow C_-[1],$$

where C_- equals the shift by $[-1]$ of a cone of

$$j_n^r : M_{gm}(X_n^r)^e \longrightarrow M_{gm}(\overline{\overline{X}}_n^r)^e.$$

By Corollary 3.4(b) and (c),

$$C_- \cong M_{gm}(S_n^\infty)(r+1)[r+1]$$

is pure of weight $-(r+1)$. It follows from this and from Corollary 1.14(a) that the exact triangle

$$C_- \longrightarrow \partial M_{gm}(X_n^r)^e \xrightarrow{c_+} M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e = M_{gm}^c(\overline{\overline{X}}_n^{r,\infty})^e \longrightarrow C_-[1]$$

is a weight filtration of $\partial M_{gm}(X_n^r)^e$ avoiding weights $-r, \ldots, -1$, and hence, in particular, avoiding weight minus one (since $r \geqslant 1$). Our claim then follows from Proposition 4.15(b) (observe that e is symmetric).

(Alternatively, use claim (C) of the proof of Theorem 3.3, to see directly that the last term of the weight filtration of $\partial M_{gm}(X_n^r)^e$,

$$M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e \cong M_{gm}(S_n^\infty)[r]$$

is pure of weight $r \geqslant 1$.)

(e) As a by-product of the above identification of the weight filtration of $\partial M_{gm}(X_n^r)^e$, we see that the cohomological Betti realization $H^i(\partial M_{gm}(X_n^r)^e)$ (see [Lec08]), tensored with $\mathbb{Z}[1/(2n \cdot r!)]$, is without torsion for all integers i. We thus recover a result of Hida's (see [Gha02, Proposition 3]): the odd primes p dividing the torsion of the (e-part of the) *boundary cohomology* of X_n^r (see [Gha02, § 3.2]) satisfy $p \leqslant r$ or $p \mid n$.

Remark 4.17. Let us agree to forget the results from § 3, and see what the theory developed in the present section implies in the situation studied in Example 4.16(d). We only use the validity of Assumption 4.2.

(a) It follows formally from Theorem 4.3(a)–(c) that $\operatorname{Gr}_0 M_{gm}(X_n^r)^e$ and $\operatorname{Gr}_0 M_{gm}^c(X_n^r)^e$ are defined, and canonically isomorphic, and that there are exact triangles

$$C_- \longrightarrow M_{gm}(X_n^r)^e \xrightarrow{\pi_0} \operatorname{Gr}_0 M_{gm}(X_n^r)^e \xrightarrow{\delta_-} C_-[1]$$

and

$$M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e \xrightarrow{\delta_+} \operatorname{Gr}_0 M_{gm}^c(X_n^r)^e \xrightarrow{i_0} M_{gm}^c(X_n^r)^e \longrightarrow M_{gm}(\overline{\overline{X}}_n^{r,\infty})^e[1].$$

(b) It follows formally from Corollary 4.6 that $\operatorname{Gr}_0 M_{gm}(X_n^r)^e$ is a direct factor of $_n^r\mathcal{W} = M_{gm}(\overline{\overline{X}}_n^r)^e$, with a canonical complement. Call this complement N. It follows formally from Theorems 4.7(b) and 4.8(b) that the realizations of the motive $\operatorname{Gr}_0 M_{gm}(X_n^r)^e$ equal interior cohomology, that is, the image of the morphism

$$H_c^n(X_n^r)^e \longrightarrow H^n(X_n^r)^e.$$

By [Sch90, § 1.2.0, Theorem 1.2.1 and § 1.3.4], the same is true for $_n^r\mathcal{W}$. Therefore, the complement N has trivial realizations. Thus, its underlying Grothendieck motive is trivial. This means that the construction of the Grothendieck motives for modular forms $M(f)$ can be done by replacing the Chow motive $_n^r\mathcal{W}$ by $\operatorname{Gr}_0 M_{gm}(X_n^r)^e$.

(c) As recalled in [And04, § 11.5.2], Voevodsky's *nilpotence conjecture* implies that homological equivalence equals smash-nilpotent equivalence. Therefore [And04, Corollary 11.5.1.2], this conjecture implies that the forgetful functor from Chow motives to Grothendieck motives is conservative. Thus, the nilpotence conjecture gives a hypothetical abstract reason for the complement N of $\operatorname{Gr}_0 M_{gm}(X_n^r)^e$ in $_n^r\mathcal{W}$ to be zero, which would mean that $\operatorname{Gr}_0 M_{gm}(X_n^r)^e = {_n^r\mathcal{W}}$.

(d) Recall that Scholl's motives $M(f)$ are constructed out of $_n^r\mathcal{W}$ using cycles, which are only known to be idempotent after passage to the Grothendieck motive underlying $_n^r\mathcal{W}$: indeed, the Eichler–Shimura isomorphism allows for a control of the action of the Hecke algebra on the relevant cohomology group. In particular, there are only finitely many eigenvalues, a fact which is needed for the construction of the projector to the eigenspace corresponding to f. The nilpotence conjecture implies [And04, Corollary 11.5.1.2] that idempotents can be lifted from Grothendieck to Chow motives. Hence, its validity would mean that Scholl's cycles can be modified by terms homologically equivalent to zero, to give idempotent endomorphisms of $_n^r\mathcal{W}$. This would produce Chow motives, whose underlying Grothendieck motives are the $M(f)$.

(e) By [Kim05, Corollary 7.8], the conclusion from part (d) holds under an assumption, which is *a priori* weaker than the nilpotence conjecture: the *finite dimensionality* [Kim05, Definition 3.7] of the Chow motive $_n^r\mathcal{W}$. By [Kim05, Theorem 4.2 and Corollary 5.11], finite dimensionality is known for motives of curves and motives of Abelian varieties. Unfortunately, the methods of [Kim05] do not seem to admit an immediate generalization to families of Abelian varieties over curves, or degenerations of such families (such as $\overline{\overline{X}}_n^r$).

Remark 4.18. We make the following remarks.

(a) Of course, none of the implications listed in parts (a)–(c) of the preceding remark are new: they are all consequences of Theorem 3.3, Corollaries 3.4 and 3.6, whose proof involves the geometry of the boundary of the smooth compactification $\overline{\overline{X}}_n^r$ of X_n^r. Observe that some of these results were even used in our proof in Example 4.16(d) of the validity of Assumption 4.2. In other words, we applied the strategy from Remark 4.5, and proved Assumption 4.2 via parts (a) and (c) of Theorem 4.3.

(b) For Shimura varieties of higher dimension, Hecke-equivariant smooth compactifications (such as $\overline{\overline{X}}_n^r$ in the case of powers of the universal elliptic curve over a modular curve) are not known (and maybe not reasonable to expect) to exist. In this more general setting, a different strategy of proof of Assumption 4.2 'avoiding geometry as far as possible' would therefore be of interest. Such a strategy, disjoint from that of Remark 4.5, will be developed in [Wil09].

Acknowledgements

References

And04   Y. André, *Une introduction aux motifs*, Panoramas et Synthèses, vol. 17 (Société Mathématique de France, Paris, 2004).

BS01   P. Balmer and M. Schlichting, *Idempotent completion of triangulated categories*, J. Algebra **236** (2001), 819–834.

Bei86   A. A. Beilinson, *Higher regulators of modular curves*, in *Applications of algebraic K-theory to algebraic geometry and number theory. Proceedings of the AMS–IMS–SIAM joint summer research conference held June 12–18, 1983*, Contemporary Mathematics, vol. 55, eds S. J. Bloch, R. Keith Dennis, E. M. Friedlander and M. R. Stein (American Mathematical Society, Providence, RI, 1986), 1–34.

BBD82   A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, in *Analyse et topologie sur les espaces singuliers (I)*, Astérisque, vol. 100, eds B. Teissier and J. L. Verdier (Société Mathématique de France, Paris, 1982).

Bon07   M. V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, Preprint (2007), http://arXiv.org/abs/0704.4003.

Bon09   M. V. Bondarko, *Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura*, J. Inst. Math. Jussieu **8** (2009), 39–97.

1224

Del69   P. Deligne, *Formes modulaires et représentations ℓ-adiques*, in *Sém. Bourbaki, Exposé 355*, Lecture Notes in Mathematics, vol. 179 (Springer, Berlin, 1969), 139–172.

Del72   P. Deligne, *Résumé des premiers exposés de A. Grothendieck*, in *Groupes de Monodromie en Géométrie Algébrique (SGA 7_I)*, Lecture Notes in Mathematics, vol. 288 (Springer, Berlin, 1972), 1–24.

DG05   P. Deligne and A. B. Goncharov, *Groupes fondamentaux motiviques de Tate mixte*, Ann. Sci. École Norm. Sup. (4) **38** (2005), 1–56.

FV00   E. M. Friedlander and V. Voevodsky, *Bivariant cycle cohomology*, in *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol. 143 (Princeton University Press, Princeton, NJ, 2000).

Gha02   E. Ghate, *Adjoint L-values and primes of congruence for Hilbert modular forms*, Compositio Math. **132** (2002), 243–281.

GS96   H. Gillet and C. Soulé, *Descent, motives and K-theory*, J. Reine Angew. Math. **478** (1996), 127–176.

GN02   F. Guillén and V. Navarro Aznar, *Un critère d'extension des foncteurs définis sur les schémas lisses*, Publ. Math. Inst. Hautes Études Sci. **95** (2002), 1–91.

Hub04   A. Huber, *Realization of Voevodsky's motives*, J. Algebraic Geom. **9** (2000), 755–799, *Corrigendum*, **13** (2004), 195–207.

Kim05   S.-I. Kimura, *Chow groups are finite dimensional, in some sense*, Math. Ann. **331** (2005), 173–201.

Lec08   F. Lecomte, *Réalisation de Betti des motifs de Voevodsky*, C. R. Acad. Sci. Paris Ser. I Math. **346** (2008), 1083–1086.

Niz08   W. Nizioł, *Semistable conjecture via K-theory*, Duke Math. J. **141** (2008), 151–178.

RZ82   M. Rapoport and Th. Zink, *Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), 21–101.

Sch88   P. Schneider, *Introduction to the Beilinson conjectures*, in *Beilinson's conjectures on special values of L-functions*, Perspectives in Mathematics, vol. 4, eds M. Rapoport, N. Schappacher and P. Schneider (Academic Press, New York, 1988), 1–35.

Sch90   A. J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), 419–430.

Sch97   A. J. Scholl, *L-functions of modular forms and higher regulators*, in preparation (the version dated May 15, 1997 is available at http://www.dpmms.cam.ac.uk/~ajs1005/mono/index.html).

Ser91   J.-P. Serre, *Motifs*, in *Journées Arithmétiques (Luminy, 1989)*, Astérisque, vol. 198–200 (Société Mathématique de France, Paris, 1991), 333–349.

Tsu99   T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), 233–411.

Voe00   V. Voevodsky, *Triangulated categories of motives*, in *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol. 143 (Princeton University Press, Princeton, NJ, 2000).

Voe02   V. Voevodsky, *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*, Int. Math. Res. Not. **2002** (2002), 351–355.

VSF00   V. Voevodsky, A. Suslin and E. M. Friedlander, *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol. 143 (Princeton University Press, Princeton, NJ, 2000).

Wil06   J. Wildeshaus, *The boundary motive: definition and basic properties*, Compositio. Math. **142** (2006), 631–656.

Wil09   J. Wildeshaus, *On the interior motive of certain Shimura varieties: the case of Hilbert–Blumenthal varieties*, Preprint (2009), http://arXiv.org/abs/0906.4239, submitted.

Jörg Wildeshaus wildesh@math.univ-paris13.fr
LAGA, UMR 7539, Institut Galilée, Université Paris 13, Avenue Jean-Baptiste Clément,
F-93430 Villetaneuse, France