ANOTHER APPROACH TO EXTENSIONS
OF CONTINUOUS MAPPINGS

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First, we give a more general extension theorem about continuous mappings. It is shown
that the Taimanov-Eilenberg-Steenrod extension theorem and the Engelking extension
theorem are special cases and that this theorem implies the Dugunji extension theorem.
The localic version of this theorem also generalises Joyal's extension theorem for locales.
Then, using the same technique, we obtain another more interesting extension theorem
and its applications. In particular, we sharpen a well-known result due to Wallman.

Throughout this paper, all notation is standard. Undefined notions may be found
in [3] and [7] (topological notions in [3] and lattice-theoretic notions in [7]).

To determine a continuous extension, a standard method is to use the convergence
of a filter or a net. All the proofs in 1, 2, 3, 4, 6 depend on this method. However, we
provide another technique here, which seems to be new, to construct an extension of
a continuous mapping. This technique can be used in cases where the image space is
not the Hausdorff. Our second main result (Theorem 4) will illustrate the point. As an
interesting application of the theorem, we shall construct well-compactifications for any
T_0-spaces such that this well-compactification has a weaker universal property similar
to that possessed by Stone-Čech compactifications. In particular, we shall sharpen a
well-known result due to Wallman.

The following theorem is our first main result.

THEOREM 1. *Let A be a dense subspace of a topological space X, and f a
continuous mapping of A to a T_3-space Y. The mapping f has a continuous extension
over X if and only if for every open cover $\{V_\alpha\colon \alpha \in \Lambda\}$ of Y, there exists an open
cover $\{U_i\colon i \in I\}$ of X such that the open cover $\{U_i \cap A\colon i \in I\}$ of the subspace A is
a refinement of $\{f^{-1}(V_\alpha)\colon \alpha \in \Lambda\}$.*

PROOF: The necessity is obvious. To show the converse, first note that the assumed
condition is equivalent to the condition that $\{\mathrm{Ex}f^{-1}(V_\alpha)\colon \alpha \in \Lambda\}$ is an open cover of
X, where $\mathrm{Ex}D = X \setminus \overline{(A \setminus D)}$. Now let lattices of the open sets of Z be denoted by
$O(Z)$. Since f is continuous, we have a frame mapping $f^{-1}\colon O(Y) \to O(A)$, that is
f^{-1} preserves finite meets and arbitrary joins. Now note that $O(A)$ is isomorphic to

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the set $\{\mathrm{Ex}U\colon U \in O(A)\}$ as a meet-sublattice of $O(X)$, that is $\mathrm{Ex}\colon O(A) \to O(X)$ is injective and preserves finite meets. Thus we define a mapping $g^*\colon O(Y) \to O(X)$ as follows:

$$g^*(V) = \bigcup\{\mathrm{Exf}^{-1}(V')\colon \overline{V}' \subseteq V\}, \text{ for each } V \in O(Y).$$

Clearly, we have $g^*(\emptyset) = \emptyset$ and $g^*(Y) = X$ by the regularity.

Now we want to show that g^* preserves finite meets and arbitrary joins. Let $V_1, V_2 \in O(Y)$, then

$$g^*(V_1) \cap g^*(V_2) = \left(\bigcup\{\mathrm{Exf}^{-1}(V')\colon \overline{V}' \subseteq V_1\}\right) \cap \left(\bigcup\{\mathrm{Exf}^{-1}(V''')\colon \overline{V}'' \subseteq V_2\}\right)$$

$$= \bigcup\{\mathrm{Exf}^{-1}(V') \cap \mathrm{Exf}^{-1}(V'')\colon \overline{V}' \subseteq V_1, \overline{V}'' \subseteq V_2\}$$

$$= \bigcup\{\mathrm{Exf}^{-1}(W)\colon \overline{W} \subseteq V_1 \cap V_2\} = g^*(V_1 \cap V_2).$$

So g^* preserves finite meets since $\overline{V}' \subseteq V_1$ and $\overline{V}'' \subseteq V_2$ imply $\overline{V' \cap V''} \subseteq V_1 \cap V_2$.

To show $g^*\left(\bigcup_{\alpha \in \Lambda} V_\alpha\right) = \bigcup_{\alpha \in \Lambda} f^*(V_\alpha)$, we consider the union $\bigcup_{\alpha \in \Lambda} V_\alpha$ in $O(Y)$; by regularity, we also have $\bigcup_{\alpha \in \Lambda} V_\alpha = \bigcup T$, where $T = \{W_i \in O(Y)\colon \overline{W}_i \subseteq V_{\alpha_i} \text{ for some } \alpha_i \in \Lambda\}$. Thus we have $\mathrm{Exf}^{-1}(W_i) \subseteq g^*(V_{\alpha_i})$ for each $W_i \in T$.

If $\overline{V}' \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha = \bigcup T$, that is, $\left(Y \setminus \overline{V}'\right) \cup (\bigcup W_i) = Y$, then, by the assumed condition, we have $\bigcup\{\mathrm{Exf}^{-1}(W)\colon W \in T\} \cup \left(\mathrm{Exf}^{-1}\left(Y \setminus \overline{V}'\right)\right) = X$, hence $\mathrm{Exf}^{-1}(V') \subseteq \bigcup\{\mathrm{Exf}^{-1}(W)\colon W \in T\} \cup \left(\mathrm{Exf}^{-1}\left(Y \setminus \overline{V}'\right)\right)$. Since $\mathrm{Exf}^{-1}(V') \cap \mathrm{Exf}^-\left(Y \setminus \overline{V}'\right) \neq \emptyset$, so we have $\mathrm{Exf}^-(V') \subseteq \bigcup\{\mathrm{Exf}^{-1}(W)\colon W \in T\} \subseteq \bigcup_{\alpha \in \Lambda} g^*(V_\alpha)$, therefore

$$g^*\left(\bigcup_{\alpha \in \Lambda} V_\alpha\right) = \bigcup\{\mathrm{Exf}^{-1}(V')\colon \overline{V}' \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha\} \subseteq \bigcup_{\alpha \in \Lambda} g^*(V_\alpha).$$

But the reverse inequality is trivial since g^* is order-preserving.

Now we can define $g_*\colon O(X) \to O(Y)$ as follows:

$$g_*(U) = \bigcup\{V \in O(Y)\colon g^*(V) \subseteq U\} \text{ for each } U \in O(X).$$

Since g^* preserves arbitrary joins, we have, for each pair $U \in O(X)$, $V \in O(Y)$,

(*) $V \subseteq g_*(U)$ if and only if $g^*(V) \subseteq U$.

Now we claim that for each $x \in X$, $g_*\left(X \setminus \overline{\{x\}}\right) = Y \setminus \{y\}$ for some $y \in Y$.(**). If $g_*\left(X \setminus \overline{\{x\}}\right) = Y$ for some $x \in X$, by (*), $g^*(Y) \subseteq X \setminus \{X\}$, but $g^*(Y) = X$, so we have a contradiction.

If $g_*\left(X \setminus \overline{\{x\}}\right) = Y \setminus F$, for some $x \in X$, where F is not a single point set, let $y_1, y_2 \in F$ $y_1 \neq y_2$. Then we have $V_1, V_2 \in O(Y)$ such that $V_1 \cap V_2 = \emptyset$, and $x_i \in V_i$, $i = 1, 2$, hence $V_1 \cap V_2 \subseteq g_*\left(X \setminus \overline{\{x\}}\right) g^*(V_1 \cap V_2) \subseteq X \setminus \overline{\{x\}}$, by finite meet-preservation $g^*(V_1) \cap g^*(V_2) \subseteq X \setminus \overline{\{x\}}$, so we have some $i \in \{1, 2\}$, such that $g^*(V_i) \subseteq X \setminus \overline{\{x\}}$, $V_i \subseteq g_*\left(X \setminus \overline{\{x\}}\right) = Y \setminus F$. But $x_i \in V_i \cap F$, so again we have a contradiction. Thus formula (**) is established. By (**), we can define a mapping $g \colon X \to Y$ as follows:

$$\text{for each } x \in X, g(x) = y \text{ if and only if } g_*\left(X \setminus \overline{\{x\}}\right) = Y \setminus \{y\}.$$

We claim that g is a continuous extension of f.

To show that g is continuous, it suffices to show $g^{-1} = g^*$. For each $V \in O(Y)$, we have $g^{-1}(V) = \{x \in X : g(x) \in V\} = \{x : V \not\subseteq Y \setminus \{g(x)\}\} = \{x : V \not\subseteq g_*\left(X \setminus \overline{\{x\}}\right)\} = \{x : g^*(V) \not\subseteq X \setminus \overline{\{x\}}\} = \{x : x \in g^*(V)\} = g^*(V) \in O(X)$.

To show that g is an extension of f, we first show $g^{-1}(V) \cap A = f^{-1}(V)$ for each $V \in 0(Y)$. In fact, $g^{-1}(V) \cap A = g^*(V) \cap A = \left(\cup\{\mathrm{Exf}^{-1}(V') : \overline{V}' \subseteq V\}\right) \cap A = \cup\{\mathrm{Exf}^{-1}(V') \cap A : \overline{V}' \subseteq V\} = \cup\{f^{-1}(V') : \overline{V}' \subseteq V\} = f^{-1}(V)$. Now we can show that g is as required. If $a \in A$, $f(a) \neq g(a)$, there exists a $V \in O(Y)$ such that $f(a) \in V$ and $g(a) \notin V$, but $a \in f^{-1}(V) = g^{-1}(V) \cap A$ implies $g(a) \in V$. It is a contradiction. This also completes the proof. ∎

COROLLARY 1. (Eilenberg–Steenrod–Taimanov [2, p.280]). *Let A be a dense subspace of a topological space and f a continuous mapping of A to a compact Hausdorff space Y. Then the mapping f has a continuous extension over X if and only if for each open cover $\{V_i\}_{i=1}^k$ of Y there is an open cover $\{U_i\}_{i=1}^m$ of X such that the open cover $\{U_i \cap A\}_{i=1}^m$ of A is a refinement of $\{f^{-1}(V_i)\}_{i=1}^k$.*

COROLLARY 2. (Englking–Vulih [4]). *Let A be a dense subspace of a topological space X and f a continuous mapping of A to a real compact space Y. Then the mapping f has a continuous extension over X if and only if for every sequence $F_1, F_2, \ldots$ of closed subsets of Y such that $\bigcap\limits_{i=1}^\infty F_i = \emptyset$, we have $\bigcap\limits_{i=1}^\infty \overline{f^{-1}(F_i)} = \emptyset$.*

PROOF: By Theorem 1, it is clear if Y is a Lindelof space. Then by virtue of the lemma ([3, p.176]), the proof is complete (note that a space is real compact if and only if it is homeomorphic to a closed subspace of a cartesian product of copies of the real line). ∎

COROLLARY 3. (Dugundji). *Let D be a dense subspace of X. Y a regular space and $f \colon D \to Y$ continuous. Then f has a continuous extension $F \colon X \to Y$ if and only*

if the filterbase $f(D \cap \mathcal{U}(x))$ *converges for each* $x \in X$, *where* $\mathcal{U}(x)$ *is the neighbourhood system of* x. *(see* [**1**, *p.216*]*)*

PROOF: Let $\mathcal{V}$ be open cover of Y such that $\bigcup\{\mathrm{Exf}^{-1}(V): V \in \mathcal{V}\} \neq X$. Then there is an $x \in X$ such that $x \notin \bigcup\{\mathrm{Exf}^{-1}(V): V \in \mathcal{V}\}$, that is, for each $U \in \mathcal{U}(x)$, $V \in \mathcal{V}$, $U \not\subseteq \mathrm{Exf}^{-1}(V)$, $U \cap D \not\subseteq f^{-1}(V)$, hence $f(U \cap D) \not\subseteq V$. But $f(D \cap \mathcal{U}(x))$ converges to some $y \in Y$; so there is a $V_0 \in \mathcal{V}$ such that $y \in V_0$. Hence there exists an $U_0 \in \mathcal{U}(x)$ such that $f(U_0 \cap D) \subseteq V_0$. Thus we have a contradiction. ∎

Mimicking the proof of Theorem 1, we have

THEOREM 2. *Let* A *be a dense subspace of a topological space* X *and* F *be a continuous mapping of* A *to a locally compact space* y. *The mapping* F *has a continuous extension over* X *if and only if there is an open cover* $\{V_\alpha: \alpha \in \Lambda\}$ *of* Y *in which each element has a compact closure such that* $\{\mathrm{Exf}^{-1}(V_\alpha): \alpha \in \Lambda\}$ *is an open cover of* X *and for every finite family* $\{F_i: i \in k\}$ *of compact subsets of* Y *with* $\bigcap\limits_{i=1}^{k} F_i = \emptyset$ *we have* $\bigcap\limits_{i=1}^{k} \overline{f^{-1}(F_i)} = \emptyset$.

The localic version of Theorem 1 is the following

THEOREM 3. *Let* A_j *be a dense sublocale of a locale* A, B *a regular locale, then continuous map* $f: A_j \to B$ *has a unique extension to a continuous map* $g: A \to B$ *if* f *satisfies that for each family* $\{b_\alpha: \alpha \in \Lambda\}$ *with* $\bigvee\{b_\alpha: \alpha \in \Lambda\} = 1$, *there exists a family* $\{a_i: i \in I\}$ *such that* $\bigvee\{a_i: i \in I\} = 1$ *and* $\{j(a_i): i \in I\}$ *is a refinement of* $\{f^*(b_\alpha): \alpha \in \Lambda\}$.

COROLLARY. (A. Joyal) *Let* A_j *be a flat sublocale of a locale* A, B *a compact regular locale; then every continuous map* $f: A_j \to B$ *has a unique extension to a continuous map* $g: A \to B$. *(see* [**7**, *p.91*]*)*

Remark. Note that our proof was inspired to some extent by Joyal's proof.

Before we discuss further non-separated spaces, we need to introduce a property, which arises very naturally in algebraic geometry and in the spectral theory of rings and lattices and which partially makes up for the loss of separation. Firstly, a closed subset A of a space X is called *irreducible*, if it is not the union of two closed proper subsets. Now we say a space is *sober* if each nonvoid closed irreducible subset has a unique dense point. It is clear that every Hausdorff space is sober and every sober space is T_0.

For open sets U, V of a space X we write $V \ll U$ if for each open covering $\mathcal{U}$ of U there is some finite part of $\mathcal{U}$ which covers V. Thus $V \ll U$ implies $V \subseteq U$ and $U \ll U$ implies that U is quasi-compact. Following [**5**], we say a space X is core-compact if for each $x \in X$ and $U \in O(X)$ with $x \in U$, there is some $V \in O(X)$

with $x \in V \ll U$. The relation $\ll$ is known as the way below relation and is one of the fundamental attributes of a continuous lattice. A full discussion of core-compactness can be found in [5]; in particular, it is noted there that a space X is core-compact exactly when $O(X)$ is a continuous lattice and that a sober space X is core-compact exactly when it is locally quasicompact.

Finally let us say a space X is stable if for each open sets U_1, U_2, V_1, V_2 of X, $V_1 \ll U_1$ together with $V_2 \ll U_2$ implies $V_1 \cap V_2 \ll U_1 \cap U_2$. Following [9], we say a space X is well-compacted if

  (i) X is quasi-compact sober;
  (ii) X is core-compact, (equivalently, locally quasicompact);
  (iii) X is stable.

Clearly, every compact Hausdorff space is well-compacted.

Recall that a space is spectral if it is quasicompact sober and has a base of quasi-compact open sets which is closed under finite intersections. Spectral spaces are exactly the spaces occurring as spectra of distributive lattices with 0 and 1, or of commutative rings with 1. In [9], it is shown that a space is well-compacted if and only if it is a retract of a spectral space; in particular, every spectral space is well-compacted.

Now we turn to the second main result in this paper. In the sequel, all space are assumed only to have T_0-separation unless otherwise specified.

To show the following theorem, we need another property. We say a dense subset A of a space X is flat if there is a lattice homomorphism $s \colon O(A) \to O(X)$ such that $s(U) \cap A = U$ for each $U \in O(A)$.

Example. Every T_1 space is a flat subset of its Wallman compactification; in particular, each discrete (normal) space is a flat subset of its Ston-Čech compactification.

THEOREM 4. *Let A be a flat subset of a space X and f a continuous mapping of A to Y, where Y is well-compacted; then f has a continuous extension over X.*

PROOF: As the proof of Theorem 1, we define $g^* \colon O(Y) \to O(X)$ as follows:

$$\text{for each } V \in O(Y),\ g^*(V) = \bigcup \{s(f^{-1}(\operatorname{Int}Q)) \colon Q \subseteq V,\ Q \text{ quasicompact}\}$$

where s is the lattice homomorphism. We can show that g^* preserves finite meets and arbitrary joins. In particular, we have $g^*(Y) = X$ and $g^*(\emptyset) = \emptyset$.

Now we define $g_* \colon O(X) \to O(Y)$ as follows:

$$\text{for each } U \in OX,\ g_*(U) = \bigcup \{V \in OY \colon g^*(V) \subseteq U\}.$$

Thus we also have, for each $U \in O(X)$, $V \in O(Y)$, $V \subseteq g_*(U)$ if and only if $g^*(V) \subseteq U$.

Hence we can claim that for each $x \in X$, $g_* \left(X \setminus \overline{\{x\}} \right) = Y \setminus \overline{\{y\}}$ for some $y \in Y$. It suffices to observes that the set $Y \setminus g_* \left(X \setminus \overline{\{x\}} \right)$ is an irreducible closed set and that Y is sober. On the other hand, for each $x \in X$, the point y with $g_* \left(X \setminus \overline{\{x\}} \right) = Y \setminus \overline{\{y\}}$ is unique since Y is sober.

So we can define a mapping $g \colon X \to Y$ as follows:

$$\text{for each } x \in X,\ g(x) = y \text{ if and only if } g_* \left(X \setminus \overline{\{x\}} \right) = Y \setminus \overline{\{y\}}.$$

Mimicing the proof of Theorem 1, we can show that g is as required.   ∎

Remark. The localic version of this theorem was essentially obtained by Johnstone [7]. But our notion of flatness is more general than the one of the spatial version of localic flatness. In fact, it suffices to note that $O(A)$ is a flat sublocale of $O(X)$ if and only if Ex: $O(A) \to O(X)$ is lattice homomorphism, where $\mathrm{Ex}(V) = X \setminus \overline{A \setminus V}$, and the following lemma.

   LEMMA. *The following conditions are equivalent for any space X;*
   - (i)   *A is a flat subset of X such that for each $x \in X$ with $x \notin \overline{F}$, where F is a closed subset of A, there is an $U \in O(A)$ with $x \in s(U)$ and $s(U) \cap F = \emptyset$;*
   - (ii) *the mapping Ex: $O(A) \to O(X)$ preserves finite unions;*
   - (iii) *for each pair F_1, F_2 of closed subsets of A, we have $\overline{F_1} \cap \overline{F_2} = \overline{F_1 \cap F_2}$.*

   PROOF: (ii) $\Leftrightarrow$ (iii) and (ii) $\Rightarrow$ (i) are clear. Now we show (i) $\Rightarrow$ (iii). Suppose that $x \in \overline{F_1} \cap \overline{F_2} \setminus \overline{F_1 \cap F_2}$, where F_1 and F_2 are closed sets of A; then there is some $U \in OA$ with $x \in s\emptyset(U)$ and $s(U) \cap F_1 \cap F_2 = \emptyset$ by assumption, hence $U \cap F_1 F_2 = \emptyset$ and $U = (A \setminus F_1) \cup (A \setminus F_2)$. So we have $s(U) = s(A \setminus F_1) \cup s(A \setminus F_2)$ and either $x \in s(A \setminus F_1)$ or $x \in s(A \setminus F_2)$, which contracts with the assumption $x \in \overline{F_1} \cap \overline{F_2}$.   ∎

   The following example shows that the additional condition in the above Lemma (i) cannot be omitted.

Example. Let X be the right half of the real line $[0, \infty)$ and the topology $\theta = \{[0, x) \setminus A : A$ countable whenever $x < 1$ and A finite whenever $x \geqslant 1\}$ then (X, θ) is T_1. Now let Y be the space obtained from X by adding a new point p such that X is open and the neighbourhoods system $\mathcal{B}(p)$ of the point p is $\{\{p\} \cup [0, 1) \setminus B : B \subseteq X$ is finite$\}$. Thus X is an open dense subset of Y and y is also T_1. Now we show that Ex: $O(X) \to O(Y)$ does not preserve binary unions: Let A_1, A_2 be two disjoint infinite countable subsets of $[0, 1)$; then the sets $([0, 1) \setminus A_1)$ and $([0, 1) \setminus A_2)$ are open in Y, morever, $\mathrm{Ex}([0, 1) \setminus A_i) = [0, 1) \setminus A_i$, $i = 1, 2$. But $\mathrm{Ex}(([0, 1) \setminus A_1) \cup ([0, 1) \setminus A_2)) = \mathrm{Ex}[0, 1) = [0, 1) \cup \{p\} \neq [0, 1)$.

On the other hand, we can define a lattice homomorphism $s \colon O(X) \to O(Y)$ such that $s(U) \cap X = U$ for each $U \in O(X)$, that is, X is a flat subset of Y. In fact, s is defined by $s(U) = U$ is infinite, and $s(U) = U \cup \{p\}$ if $X \setminus U$ is finite. It is easy to check that s is as required.

Remark. No such space is regular since the assumption that A is a flat subset of X implies the one that each pair F_1, F_2 of disjoint closed sets of A have disjoint closure in X which is equivalent to the one that $\mathrm{Ex} \colon O(A) \to O(X)$ is a lattice homomorphism in the realm of regular spaces.

In the sequel, we shall give some applications of Theorem 4.

The next theorem will generalise the following known result due to Wallman. 'Every continuous mapping from a T_1 space to a compact T_2 space has a unique continuous extension over its Wallman compactification'.

One might have conjectured that the uniqueness would be dependent on the T_2-separation of the image space in the above result of Wallman, as usual.

To show that the T_2-separation can be omitted, let us note that, for each T_1-space X, each continuous mapping f from its Wallman compactification ωX to a compact T_2-space Y has the following property,

(*)   for each pair $V_1, V_2 \in OY$, there is an open set U of X with $f^{-1}(V_1) \subseteq U^* \subseteq f^{-1}(V_2)$ if $V_1 \ll V_2$.

To see this, it suffices to note that $V_1 \ll V_2 \Rightarrow f^{-1}(V_1) \ll f^{-1}(V_2)$ and that $\{U^* \colon U \in OX\}$ forms a base closed under finite unions.

THEOREM 5. *Let X be a T_1 space and $f \colon X \to Y$ a continuous mapping, where Y is well-compacted; then f has a unique continuous extension satisfying (*) over its Wallman compactification ωX.*

PROOF: For each $U \in O(X)$, let $s(U) = U^*$; then s is a lattice homomorphism. It follows from Theorem 4 that f has a continuous extension g over ωX. Noting that, for each $V \in O(Y)$, $g^{-1}(V) = g^*(V) = \bigcup\{(f^{-1}(\mathrm{Int}Q))^* \colon Q \subseteq V, Q \text{ quasicompact}\}$, we have $f^{-1}(V_1)^* \subseteq g^{-1}(V_2)$ whenever $V_1 \ll V_2$. On the other hand, we clearly have $g^{-1}(V_1) \subseteq (f^{-1}(V_1))^*$. Thus we have shown that the extension g satisfies (*).

Now we prove the uniqueness. Let $X \overset{\subset}{\longrightarrow} \omega X \overset{h_1}{\underset{h_2}{\rightrightarrows}} Y$ such that $h_1 i = h_2 i$, where both h_1 and h_2 satisfy (*). If $h_1 \neq h_2$, that is, $h_1(p) \neq h_2(p)$ for some $p \in \omega X$, we can assume there are $V_1, V_2, U \in O(Y)$ with $h_1(p) \in V_1 \ll V_2 \ll U$ and $h_2(p) \notin U$ by Y being well-compacted. Furthermore, we find $W_1, W_2 \in O(X)$, since both h_1 and h_2 satisfy (*), such that $p \in h_1^{-1}(V_1) \subseteq W_1^* \subseteq h_1^{-1}(V_2)$ and $h_2^{-1}(V_2) \subseteq W_2^* \subseteq h_2^{-1}(U)$, in particular, $p \in W_1^* \setminus W_2^*$ and $W_1 \setminus W_2 \neq \emptyset$. Thus we can choose some $x \in X$ with $x \in W_1 \setminus W_2$, hence $x \in W_1^* \setminus W_2^*$; so $h_1(x) \in V_2$ and $h_2(x) \notin V_2$ which contradicts

with $h_1 i = h_2 i$.   ∎

Remark. This theorem generalises the above result of Wallman.

As another interesting application, we consider the well-compactifications of T_0-spaces (it is just the second part of Stone representation theorem; it was also mentioned in [**9**].

Let L be a distributive lattice with 0 and 1 and σL denote the prime spectra of L (that is, the set of all prime ideals of L with the Stone–Zariski topology). Then L is a spectral space. Now let X be a T_0-space and $O(X)$ be the topology of X. Then $O(X)$ is a distributive lattice with 0 and 1. Let $\Sigma X = \sigma O(X)$ then ΣX is well-compacted (indeed, a spectral space). It is easy to show that X is homeomorphic to the subspace $\{I_x : x \in X\}$, where $I_x = \{U \in O(X): x \notin U\}$, of X, that is, the mapping $I \colon X \to \Sigma X$ defined by:

$$\text{for each } x \in X,\ I(X) = I_x \text{ is continuous,}$$
$$\text{relatively open and } I(X) \text{ is dense in } \Sigma X.$$

Thus we have shown that ΣX is a well-compactification of X. Now we want to establish some counterparts of Theorem 5. To do this, we define $s \colon O(X) \to O\Sigma X$ as follows:

$$\text{for each } U \in O(X), \qquad s(U) = \{p \in \Sigma X.\ U \notin p\}.$$

Then it is easy to check that s is a lattice homomorphism such that $s(U) \cap X = U$, that is, X is a flat subset of X.

Following Simmons, we say a continuous map $f \colon S \to T$ is well compacted if for each open sets U, V of T, the implication

$$V \ll U \Rightarrow f^{-1}(V) \ll f^{-1}(U) \quad \text{holds.}$$

As another application of our Theorem 4, we give a new proof of the following theorem essentially due to Simmons ([**9**]).

THEOREM 6. *Let X be a T_0 space and $f \colon X \to Y$ a continuous mapping, where Y is well compacted. Then f has a unique continuous well-compacted extension over ΣX.*

PROOF: By virtue of Theorem 4, we have a continuous extension g of f such that $g^{-1}(V) = g^*(V) = \bigcup\{sf^{-1}(\text{Int}Q): Q \subseteq V,\ Q \text{ quasicompact}\}$. Noting each $s(U)$ is quasicompact open, so we see that g is well-compacted. The proof of the uniqueness is similar to that in Theorem 5.   ∎

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