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ON PRODUCTS OF ALMOST STRONG LIFTINGS

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Abstract

Dealing with a problem posed by Kupka we give results concerning the permanence of the almost strong lifting property (respectively of the universal strong lifting property) under finite and countable products of topological probability spaces. As a basis we prove a theorem on the existence of liftings compatible with products for general probability spaces, and in addition we use this theorem for discussing finite products of lifting topologies.

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Introduction

In this paper we are concerned with hereditary properties of almost strong liftings for finite and countable products of general topological probability spaces (respectively Baire probability spaces) (see Section 1 for terminology), with the existence of liftings compatible with products (see Section 2, Theorem 4 and Theorem 5), and with finite products of lifting topologies. The Baire-almost strong lifting property (Baire-ASLP for short) of a finite product of topological probability spaces implies the almost strong lifting property (ASLP for short) of the factors (by Theorem 1 of Section 2). This generalizes the result of [22, 3.2], but the converse is not true for the ordinary product of probability spaces by Section 3, Remark 5(b) even if we have Radon measures on the factors. The crucial point in the construction is that for factors with strong liftings, products of open sets are measurable in the product probability space (see Theorem 1 (respectively Lemma 1) of Section 3). But [8] gives a hyperstonian space where the latter fails (see Remark 5 of Section 3 for details).

Talagrand's paper [28] seems to be the first one where a certain compatibility for products and liftings appears, but it is only for products in which all factors are

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equal, so only applies in a very restricted situation. To be precise the condition (P) of Theorem 4 of Section 2 (respectively (iii) of Lemma 1, Section 3) holds true. In the latter lemma we give equivalent conditions in terms of lifting topologies. As a consequence we find that the product of lifting topologies is never a lifting topology if one assumes the non-existence of measurable cardinals (see Theorem 2 of Section 3). This result is a consequence of a theorem of Curtis, Hendriksen, and Isbell on products of extremally disconnected spaces (see [11, p. 53]).

If we replace the ASLP by the stronger '*universal strong lifting property*', USLP for short, which has proved useful in connection strong lifting compactness (see [1, 23 and 25]), the situation becomes more pleasant: If a finite product has the Baire-USLP each factor has the USLP, (see Theorem 2 of Section 2), and conversely the finite product and by Theorem 4 of Section 3, also the countable product, where in both cases the completion regularity also carries over to the product; and has the ASLP if each factor has the USLP, with the exception of probably one which has only the ASLP. This is based on Theorem 4 of Section 2, where for given lifting in one factor we can find in the other factor, and on the product, liftings such that compatibility with products holds true. By induction, we extend this result to products with countably many factors in Theorem 5 of Section 2. All factors of our products may be different, while Talagrand's construction of the consistent liftings works only if all factors are equal.

In the forthcoming papers [24, 25 and 26] we study the same problems for uncountable products and general projective limits. But this requires different techniques.

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1. Preliminaries

Throughout, a quadruple $(\Omega, \mathscr{T}, \Sigma, \mu)$ will be called a topological probability space (respectively Baire probability space) if and only if $(\Omega, \mathscr{T})$ is a completely regular Hausdorff topological space and (Ω, Σ, μ) is a complete probability space such that $\mathscr{B}(\Omega) \subseteq \Sigma$ where $\mathscr{B}(\Omega)$, the σ-field generated by $\mathscr{T}$, is the Borel σ-field of $(\Omega, \mathscr{T})$ (respectively $\mathscr{B}_0(\Omega) \subseteq \Sigma$ where $\mathscr{B}_0(\Omega)$, the σ-field generated by all bounded continuous functions on Ω, is the Baire σ-field of $(\Omega, \mathscr{T})$). Therefore any topological probability space is a Baire probability space but not vice versa.

We use the notion of lifting (respectively lower density) in the sense of [16, Chapter III, Section 1, Definition 3 (respectively Definition 4)] and for any complete probability space (Ω, Σ, μ) we denote by $\Lambda(\mu)$ the system of all liftings. For any $\rho \in \Lambda(\mu)$ there exists exactly one (multiplicative) lifting $\tilde{\rho}$ (in the sense of [16, Chapter III, Section 1, Definition 2]) on $\mathscr{L}^\infty(\mu)$, the space of all bounded Σ-measurable functions on Ω, such that $\tilde{\rho}(\chi_A) = \chi_{\rho(A)}$ for all $A \in \Sigma$ (χ_A denotes the characteristic function of

A) and vice versa (see [16, pp. 35,36]). For simplicity we write $\rho = \tilde{\rho}$ throughout what follows. The linear lifting is defined by [16, Chapter III, Section 1, Definition 1]. For a Baire probability space (in particular for a topological probability space) $(\Omega, \mathscr{T}, \Sigma, \mu)$ a lifting $\rho \in \Lambda(\mu)$ is called *strong* (respectively *almost strong*) if and only if $\rho(f) = f$ for all $f \in C_b(\Omega)$, the space of all bounded continuous functions on $(\Omega, \mathscr{T})$ (respectively there exists $N \in \Sigma$ such that $\mu(N) = 0$ and $\rho(f)(\omega) = f(\omega)$ for all $f \in C_b(\Omega)$ and all $\omega \in \Omega \setminus N$).

For a topological probability space it is equivalent with $\rho(G) \supseteq G$ for all $G \in \mathscr{T}$ (respectively there exists $N \in \Sigma$ such that $\mu(N) = 0$ and $G \subseteq \rho(G) \cup N$ for all $G \in \mathscr{T}$) (see [16, chapter VIII]). $(\Omega, \mathscr{T}, \Sigma, \mu)$ has the *almost strong lifting property*, ASLP for short (respectively the *Baire-almost strong lifting property*, Baire-ASLP for short), if and only if $(\Omega, \mathscr{T}, \Sigma, \mu)$ is a topological probability space (respectively a Baire probability space) and there exists $\rho \in \Lambda(\mu)$ which is almost strong. $(\Omega, \mathscr{T}, \Sigma, \mu)$ has the *universal strong lifting property*, USLP for short (respectively the *Baire-universal strong lifting property*, Baire-USLP for short), if and only if $(\Omega, \mathscr{T}, \Sigma, \mu)$ is a topological probability space (respectively a Baire probability space) and any $\rho \in \Lambda(\mu)$ is almost strong. Compare [25] for a list of spaces having the $USLP$ from which it becomes obvious that all spaces appearing in applications have the $USLP$.

For a given probability space (Ω, Σ, μ) a set $N \in \Sigma$ with $\mu(N) = 0$ is called a *μ-null set* and for $f, g \in \mathscr{L}^\infty(\mu)$ and $A, B \in \Sigma$ we write $f = g$ a.e. (μ) respectively $A = B$ a.e. (μ) if $\{\omega \in \Omega : f(\omega) \neq g(\omega)\}$ (respectively $A \triangle B$, the symmetric difference of A and B) is a μ-null set.

A Borel measure μ on Ω is said to be *completion regular* if and only if for any Borel set B there exist $A_1, A_2 \in \mathscr{B}_0(\Omega)$ such that $A_1 \subseteq B \subseteq A_2$ and $\mu(A_2 \setminus A_1) = 0$.

We denote by $(\Omega_1 \times \cdots \times \Omega_n, \Sigma_1 \otimes \cdots \otimes \Sigma_n, \mu_1 \otimes \cdots \otimes \mu_n)$ the product probability space of the probability spaces $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 1, \ldots, n)$ and by $(\Omega_1 \times \cdots \times \Omega_n, \Sigma_1 \hat{\otimes} \cdots \hat{\otimes} \Sigma_n, \mu_1 \hat{\otimes} \cdots \hat{\otimes} \mu_n)$ its (Carathéodory) completion. By $\mathbb{N}$ we denote the set $\{1, 2, 3, \ldots\}$ of the natural numbers.

2. Liftings and products

Before stating the first theorem we mention that the product of Baire probability spaces is in general not a Baire probability space (see [2]), and the same is true for topological probability spaces (see [8]), where the situation is even much worse (see [18 and 19]). For this reason we assume in the next theorem only the Baire-ASLP for the product which is more likely satisfied than the ASLP.

THEOREM 1. *If the completed product* $(\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathscr{T}_i, (\otimes_{i=1}^n \Sigma_i)^\wedge, (\otimes_{i=1}^n \mu_i)^\wedge)$ *of the topological probability spaces* $(\Omega_i, \mathscr{T}_i, \Sigma_i, \mu_i)$ *has the Baire-*ASLP *then each*

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$(\Omega_i, \mathscr{T}_i, \Sigma_i, \mu_i)$ has the ASLP *$(i = 1, \dots, n)$.*

PROOF. By induction it is sufficient to give the proof only for $n = 2$. If $(\Omega_1 \times \Omega_2, \mathscr{T}_1 \otimes \mathscr{T}_2, \Sigma_1 \hat{\otimes} \Sigma_2, \mu_1 \hat{\otimes} \mu_2)$ has an almost strong lifting then it has a strong lifting ρ by [16, p. 127] (there the blanket assumption of local compactness is not needed).

Let $p_i : \Omega_1 \times \Omega_2 \longrightarrow \Omega_i$ denote the canonical projections $(i = 1, 2)$ and define the partial maps $f_{\omega_2}(\omega_1) := f(\omega_1, \omega_2)$ for $f \in \mathscr{L}^\infty(\mu_1 \hat{\otimes} \mu_2)$, $\omega_i \in \Omega_i$ $(i = 1, 2)$. Then put

$$\theta(g)(\omega_2) := \int \rho(g \circ p_2)_{\omega_2}(\omega_1)\, d\mu_1(\omega_1)$$

for $g \in \mathscr{L}^\infty(\mu_2)$. Since $\rho(g \circ p_2) = g \circ p_2$ a.e. $(\mu_1 \hat{\otimes} \mu_2)$ we have for all $A \in \Sigma_2$ by Fubini

$$\begin{aligned} \int_A \int \rho(g \circ p_2)(\omega_1, \omega_2) d\mu_1(\omega_1)\, d\mu_2(\omega_2) \\ &= \iint \chi_{\Omega_1 \times A}(g \circ p_2)(\omega_1, \omega_2)\, d(\mu_1 \hat{\otimes} d\mu_2)(\omega_1, \omega_2) \\ &= \int_A g(\omega_2)\, d\mu_2(\omega_2), \end{aligned}$$

that is

$$\int_A \left[\int \rho(g \circ p_2)(\omega_1, \omega_2) d\mu_1(\omega_1) - g(\omega_2) \right] d\mu_2(\omega_2) = 0 \quad \text{for all} \quad A \in \Sigma_2$$

from which it follows that $\theta(g) = g$ a.e. (μ_2).

Since $g = h$ a.e. (μ_2) for $g, h \in \mathscr{L}^\infty(\mu_2)$ implies $\rho(g \circ p_2) = \rho(h \circ p_2)$ it follows that $\theta(g) = \theta(h)$ and it is easy to verify that θ is a linear lifting for $\mathscr{L}^\infty(\mu_2)$. Since ρ is strong and $g \circ p_2 \in C_b(\Omega_1 \times \Omega_2)$ for $g \in C_b(\Omega_2)$ we have

$$\theta(g)(\omega_2) = \int g \circ p_2(\omega_1, \omega_2)\, d\mu_1(\omega_1) = \int g(\omega_2)\, d\mu_1(\omega_1) = g(\omega_2) \quad \text{for} \quad \omega_2 \in \Omega_2,$$

that is, θ is strong. By [16, Chapter VIII, Theorem 2] (the proof of this theorem works for every topological probability space) there exists a strong lifting ρ' for $(\Omega_2, \mathscr{T}_2, \Sigma_2, \mu_2)$. The same reasoning gives us a strong lifting for $(\Omega_1, \mathscr{T}_1, \Sigma_1, \mu_1)$.

Theorem 3.2 in [22] is a special case of Theorem 1.

THEOREM 2. *If the completed product $\left(\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathscr{T}_i, (\otimes_{i=1}^n \Sigma_i)^\wedge, (\otimes_{i=1}^n \mu_i)^\wedge\right)$ of the topological probability spaces $(\Omega_i, \mathscr{T}_i, \Sigma_i, \mu_i)$ has the Baire-*USLP *then each factor $(\Omega_i, \mathscr{T}_i, \Sigma_i, \mu_i)$ has the* USLP *$(i = 1, \dots, n)$.*

PROOF. By induction it is again sufficient to give the proof for $n = 2$ and for simplicity we may put $i = 1$. We denote by $p_1(\omega) := \omega_1$ for $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ the canonical projection p_1 from $\Omega_1 \times \Omega_2$ onto Ω_1. For given $\rho_1 \in \Lambda(\mu_1)$ put

$$\rho(f_1 \circ p_1) = \rho_1(f_1) \circ p_1 \quad \text{for} \quad f_1 \in \mathscr{L}^\infty(\mu_1).$$

Then ρ is extendable to a lifting $\rho \in \Lambda(\mu)$, the so called inverse lifting of ρ_1, by an argument given in [1, proof of Theorem 2.3] (see also [23]). Since ρ is almost strong by assumption there exists a $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) = 1$ such that

$$(f_1 \circ p_1)(\omega) = \rho(f_1 \circ p_1)(\omega) = (\rho_1(f_1) \circ p_1)(\omega),$$

that is,

$$f_1(\omega_1) = \rho_1(f_1)(\omega_1) \quad \text{for all} \quad f_1 \in C_b(\Omega_1) \quad \text{and all} \quad \omega = (\omega_1, \omega_2) \in \Omega_0.$$

By Fubini we have $1 = \int \mu_1(\Omega_{0\omega_2}) d\mu_2(\omega_2)$, if $\Omega_{0\omega_2} := \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in \Omega_0\}$ is the section of Ω_0 for $\omega_2 \in \Omega_2$. This implies $\mu_1(\Omega_{0\omega_2}) = 1$ for almost all $\omega_2 \in \Omega_2$. Since $\mu_2(\Omega_2) > 0$ we can choose $\omega_2 \in \Omega_2$ such that $\mu_1(\Omega_{0\omega_2}) = 1$, $\Omega_{0\omega_2} \in \Sigma_1$, and $f_1(\omega_1) = \rho_1(f_1)(\omega_1)$ for all $\omega_1 \in \Omega_{0\omega_2}$, that is ρ_1 is almost strong.

For $f_i : \Omega_i \longrightarrow \mathbb{R}(i = 1, 2)$ we define $f_1 \otimes f_2 := (f_1 \circ p_1) \cdot (f_2 \circ p_2)$ if $p_i : \Omega_1 \times \Omega_2 \longrightarrow \Omega_i$ $(i = 1, 2)$ are the canonical projections. If (Ω, Σ, μ) is a probability space and η is a σ-subalgebra of Σ we write $E_\eta(f)$ for a version of the conditional expectation of $f \in \mathscr{L}^\infty(\mu)$ with respect to η. For the proof of Theorem 3 we need two lemmas which are more or less known but we could not find a suitable reference.

LEMMA 1. *Let there be given probability spaces $(\Omega_i, \Sigma_i, \mu_i)$ and σ-subalgebras η_i of Σ_i $(i = 1, 2)$. Then for $f_i \in \mathscr{L}^\infty(\mu_i)$ $(i = 1, 2)$ it holds true that*

$$E_{\eta_1 \otimes \eta_2}(f_1 \otimes f_2) = E_{\eta_1}(f_1) \otimes E_{\eta_2}(f_2) \text{ a.e. } ((\mu_1 \otimes \mu_2) \mid \eta_1 \otimes \eta_2).$$

PROOF. By the definition of the conditional expectation and by Fubini's theorem we have for $B_i \in \eta_i$ $(i = 1, 2)$

$$\begin{aligned}
\int_{B_1 \times B_2} E_{\eta_1 \otimes \eta_2}(f_1 \otimes f_2)\, d(\mu_1 \otimes \mu_2) &= \int_{B_1 \times B_2} f_1 \otimes f_2\, d(\mu_1 \otimes \mu_2) \\
&= \int (f_1 \cdot \chi_{B_1}) \otimes (f_2 \cdot \chi_{B_2})\, d(\mu_1 \otimes \mu_2) \\
&= \int_{B_1} f_1\, d\mu_1 \cdot \int_{B_2} f_2\, d\mu_2 \\
&= \int_{B_1} E_{\eta_1}(f_1)\, d\mu_1 \cdot \int_{B_2} E_{\eta_2}(f_2)\, d\mu_2 \\
&= \int_{B_1 \times B_2} E_{\eta_1}(f_1) \otimes E_{\eta_2}(f_2)\, d(\mu_1 \otimes \mu_2),
\end{aligned}$$

that is the measures with density on $\eta_1 \otimes \eta_2$ given by means of

$$P \in \eta_1 \otimes \eta_2 \longrightarrow \int_P E_{\eta_1 \otimes \eta_2}(f_1 \otimes f_2)\, d(\mu_1 \otimes \mu_2),$$

$$P \in \eta_1 \otimes \eta_2 \longrightarrow \int_P E_{\eta_1}(f_1) \otimes E_{\eta_2}(f_2)\, d(\mu_1 \otimes \mu_2)$$

coincide on the semi-ring $\mathscr{E} = \{B_1 \times B_2 : B_i \in \eta_i \ (i = 1, 2)\}$ which generates $\eta_1 \otimes \eta_2$. Hence they coincide on the ring $\mathscr{R}$ generated by $\mathscr{E}$ since $\mathscr{R}$ is the set of all finite disjoint unions of elements in $\mathscr{E}$ (see for example [14, 1.5, Satz 6]). But from $\mathscr{R}$ these measures are uniquely extendable to $\eta_1 \otimes \eta_2$ by [4, 5.6]. Now the uniqueness provision of the Radon-Nikodým theorem implies the assertion.

LEMMA 2. *Let there be given probability spaces $(\Omega_i, \Sigma_i, \mu_i)$ and σ-subalgebras η_i of Σ_i $(i = 1, 2)$. Then $(\Omega_1 \times \Omega_2, \eta_1 \otimes \eta_2, (\mu_1 \otimes \mu_2) \mid \eta_1 \otimes \eta_2)$ is identical with the product of the probability spaces $(\Omega_i, \eta_i, \mu_i \mid \eta_i)$ $(i = 1, 2)$.*

PROOF. Note that the semi-ring $\mathscr{E} = \{B_1 \times B_2 : B_i \in \eta_i \ (i = 1, 2)\}$ generates $\eta_1 \otimes \eta_2$. Now for $B_i \in \eta_i$ $(i = 1, 2)$ we have

$$\begin{aligned}(\mu_1 \mid \eta_1) \otimes (\mu_2 \mid \eta_2)(B_1 \times B_2) &= \mu_1(B_1)\mu_2(B_2)\\ &= (\mu_1 \otimes \mu_2)(B_1 \times B_2)\\ &= ((\mu_1 \otimes \mu_2) \mid \eta_1 \otimes \eta_2)(B_1 \times B_2),\end{aligned}$$

that is the measures $(\mu_1 \mid \eta_1) \otimes (\mu_2 \mid \eta_2)$ and $(\mu_1 \otimes \mu_2) \mid \eta_1 \otimes \eta_2$ coincide on $\mathscr{E}$. Hence the same reasoning as in the proof of Lemma 1 shows that they coincide on $\eta_1 \otimes \eta_2$.

THEOREM 3. *Let there be given two probability spaces (Ω, Σ, μ) and (Θ, T, ν) with product space $(\Omega \times \Theta, \Sigma \otimes T, \mu \otimes \nu)$. Then for any lower density φ for μ there exists a lower density ψ for ν and a lower density β for $\mu \otimes \nu$ such that $\beta(A \times B) = \varphi(A) \times \psi(B)$ for all $A \in \Sigma$ and $B \in T$.*

PROOF. Let there be given a lower density φ for μ (such a lower density exists by [12, Theorem 1]). Let $\mathscr{B}$ denote the system of all triples $(\eta, \psi_\eta, \beta_\eta)$ such that η is a σ-subalgbra of T containing all ν-null sets, ψ_η is a lower density for $\nu \mid \eta$ and β_η is a lower density for $\mu \otimes \nu \mid \Sigma \otimes \eta$ such that

$$\beta_\eta(A \times B) = \varphi(A) \times \psi_\eta(B) \quad \text{for all} \quad A \in \Sigma,\ B \in \eta.$$

For $(\eta, \psi_\eta, \beta_\eta),\ (\eta', \psi_{\eta'}, \beta_{\eta'}) \in \mathscr{B}$ write

$$(\eta, \psi_\eta, \beta_\eta) \leq (\eta', \psi_{\eta'}, \beta_{\eta'})$$

if and only if

$$\eta \subseteq \eta', \ \psi_{\eta'} \mid \eta = \psi_\eta, \quad \text{and} \quad \beta_{\eta'} \mid \Sigma \otimes \eta = \beta_\eta.$$

Then $\leq$ is a partial order on $\mathscr{B}$ and $\mathscr{B}$ is non-empty. For the latter define $(u, \varphi_u, \beta_u) \in \mathscr{B}$ as follows: u is a σ-subalgebra of T generated by $T_0 := \{N \in T : \nu(N) = 0\}$. Then $u = T_0 \cup T_c$ if $T_c := \{\Theta \setminus N : N \in T_0\}$. Define $\psi_u(B) := \emptyset$ if $B \in T_0$ and $\psi_u(B) := \Theta$ if $B \in T_c$. Clearly ψ_u is a lower density on u. On the other hand $\mu \otimes \nu(E) = \int \nu(E_\omega) d\mu(\omega)$ for $E \in \Sigma \otimes u$ if $E_\omega := \{\theta \in \Theta : (\omega, \theta) \in E\}$ for $\omega \in \Omega$, and $\widetilde{E} := \{\omega \in \Omega : \nu(E_\omega) = 1\} \in \Sigma$ since $\omega \in \Omega \longrightarrow \nu(E_\omega)$ is Σ-measurable (see [15, (21.4) and (21.8)]). Since $\nu(E_\omega) = 0$ or $\nu(E_\omega) = 1$ for all $\omega \in \Omega$ we find $E = \widetilde{E} \times \Theta$ a.e. $(\mu \otimes \nu)$. Hence we can define a lower density β_u for all $E \in \Sigma \otimes u$ by means of

$$\beta_u(E) := \varphi(\widetilde{E}) \times \Theta \quad \text{if} \quad E = \widetilde{E} \times \Theta \text{ a.e. } (\mu \otimes \nu)$$

which satisfies $\beta_u(A \times B) = \varphi(A) \times \psi_u(B)$ for all $A \in \Sigma,\ B \in u$.

We show that the partial order $\leq$ is inductive. For this let $\mathscr{K}$ be a chain in $\mathscr{B}$ and put for simplicity $\mathscr{K}_1 := \{\eta : (\eta, \psi_\eta, \beta_\eta) \in \mathscr{K}\}$. We have to distinguish two cases.

(A) There is no countable cofinal part in $\mathscr{K}$. Then $\eta_{\mathscr{K}} := \cup \mathscr{K}_1$ is a σ-subalgebra of T and by means of $\psi_{\mathscr{K}}(A) := \psi_\eta(A)$ if $A \in \eta \in \mathscr{K}_1$ is defined unambiguously a lower density $\psi_{\mathscr{K}}$ on $\eta_{\mathscr{K}}$ such that $\psi_{\mathscr{K}} \mid \eta = \psi_\eta$ for all $\eta \in \mathscr{K}_1$. It is immediate that $\Sigma \otimes \eta_{\mathscr{K}} = \cup\{\Sigma \otimes \eta : \eta \in \mathscr{K}_1\}$. If we put $\beta_{\mathscr{K}}(E) := \beta_\eta(E)$ if $E \in \Sigma \otimes \eta$, $\eta \in \mathscr{K}_1$ then $\beta_{\mathscr{K}}$ is a well-defined lower density on $\Sigma \otimes \eta_{\mathscr{K}}$ such that $\beta_{\mathscr{K}} \mid \Sigma \otimes \eta = \beta_\eta$ for $\eta \in \mathscr{K}_1$. For all $A \in \Sigma,\ B \in \eta_{\mathscr{K}}$ there exists a $\eta \in \mathscr{K}_1$ such that $B \in \eta$. This implies that

$$\beta_{\mathscr{K}}(A \times B) = \beta_\eta(A \times B) = \varphi(A) \times \psi_\eta(B) = \varphi(A) \times \psi_{\mathscr{K}}(B),$$

that is $(\eta_{\mathscr{K}}, \psi_{\mathscr{K}}, \beta_{\mathscr{K}}) \in \mathscr{B}$ is an upper bound for $\mathscr{K}$ in $\mathscr{B}$.

(B) There is a countable cofinal part $((\eta_n, \psi_{\eta_n}, \beta_{\eta_n}))_{n \in \mathbb{N}}$ in $\mathscr{K}$. Then put for simplicity $\psi_n := \psi_{\eta_n}$, $\beta_n := \beta_{\eta_n}$ for all $n \in \mathbb{N}$ and denote by $\eta_{\mathscr{K}}$ the σ-subalgebra of T generated by $\bigcup_{n \in \mathbb{N}} \eta_n$.

Claim 1. *$\Sigma \otimes \eta_{\mathscr{K}}$ is the σ-subalgebra Σ^x of $\Sigma \otimes T$ generated by $\bigcup_{n \in \mathbb{N}} \Sigma \otimes \eta_n$.*

Proof. Clearly $\Sigma^x \subseteq \Sigma \otimes \eta_{\mathscr{K}}$. Since $\Sigma \otimes \eta_{\mathscr{K}}$ is generated by $\mathscr{E} := \{A \times B : A \in \Sigma,\ B \in \eta_{\mathscr{K}}\}$ it will for the converse inclusion be sufficient to show $\mathscr{E} \subseteq \Sigma^x$. We have $\mathscr{E} \subseteq \Sigma^x$ if and only if $\mathscr{E}_A = \eta_{\mathscr{K}}$ for all $A \in \Sigma$, where $\mathscr{E}_A = \{B \in \eta_{\mathscr{K}} : A \times B \in \Sigma^x\}$. But $\bigcup_{n \in \mathbb{N}} \eta_n \subseteq \mathscr{E}_A$, and $\mathscr{E}_A$ is a σ-algebra, so the result follows.

By [12, Lemma] it is well-known that by means of

$$\psi_\infty(B) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \psi_m(\{E_{\eta_m}(\chi_B) > 1 - 1/k\})$$

for $B \in \eta_{\mathscr{X}}$ is defined a lower density for $\nu \mid \eta_{\mathscr{X}}$ such that $\psi_\infty \mid \eta_n = \psi_n$ for $n \in \mathbb{N}$. And accordingly (using claim 1)

$$\beta_\infty(P) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \beta_m(\{E_{\Sigma\otimes\eta_m}(\chi_P) > 1 - 1/k\})$$

for $P \in \Sigma \otimes \eta_{\mathscr{X}}$ defines a lower density for $\mu \otimes \nu \mid \Sigma \otimes \eta_{\mathscr{X}}$ such that $\beta_\infty \mid \Sigma \otimes \eta_m = \beta_m$ if again $E_{\Sigma\otimes\eta_m}(\chi_P)$ denotes a version of the conditional expectation of χ_P with respect to $\Sigma \otimes \eta_m$ for $m \in \mathbb{N}$. It is obvious that $\psi_\infty(B)$ and $\beta_\infty(P)$ are independent of the choice of the particular version of the conditional expectation $E_{\eta_m}(\chi_B)$ respectively $E_{\Sigma\otimes\eta_m}(\chi_P)$ in the defining formulas for $\psi_\infty(B)$ respectively $\beta_\infty(P)$. Now for $A \in \Sigma$ and $B \in \eta_{\mathscr{X}}$ we have

$$\beta_\infty(A \times B) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \beta_m(\{E_{\Sigma\otimes\eta_m}(\chi_{A\times B}) > 1 - 1/k\}).$$

But

$$\begin{aligned}\{E_{\Sigma\otimes\eta_m}(\chi_{A\times B}) > 1 - 1/k\} &= \{E_{\Sigma\otimes\eta_m}(\chi_A \otimes \chi_B) > 1 - 1/k\} \\ &\overset{(1)}{=} \{(E_\Sigma(\chi_A) \otimes E_{\eta_m}(\chi_B)) > 1 - 1/k\} \\ &= \{(\chi_A \otimes E_{\eta_m}(\chi_B)) > 1 - 1/k\} \\ &\overset{(2)}{=} A \times \{E_{\eta_m}(\chi_B) > 1 - 1/k\} \text{ a.e. } (\mu \otimes \nu \mid \Sigma \otimes \eta_m)\end{aligned}$$

for $m, k \in \mathbb{N}$ where (1) holds true a.e. $(\mu \otimes \nu \mid \Sigma \otimes \eta_m)$ by Lemma 1. Equation (2) holds true since we have

$$(\omega, \theta) \in M := \{(\chi_A \circ p_1) \cdot (E_{\eta_m}(\chi_B) \circ p_2) > 1 - 1/k\}$$

if and only if

$$\chi_A(\omega) \cdot E_{\eta_m}(\chi_B)(\theta) > 1 - 1/k.$$

For $\omega \notin A$ the latter does not hold true, but for $\omega \in A$ it holds true if and only if $E_{\eta_m}(\chi_B)(\theta) > 1 - 1/k$, that is $(\omega, \theta) \in M$ if and only if $\omega \in A$ and $E_{\eta_m}(\chi_B)(\theta) > 1 - 1/k$, that is if and only if $(\omega, \theta) \in A \times \{E_{\eta_m}(\chi_B)(\theta) > 1 - 1/k\}$ $(k \in \mathbb{N})$.

This implies that

$$\begin{aligned}\beta_\infty(A\times B) &= \bigcap_{k\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n}\beta_m\left(A\times\{E_{\eta_m}(\chi_B)>1-1/k\}\right)\\ &= \bigcap_{k\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n}\left(\varphi(A)\times\psi_m(\{E_{\eta_m}(\chi_B)>1-1/k\})\right)\\ &= \varphi(A)\times\left(\bigcap_{k\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n}\psi_m\left(\{E_{\eta_m}(\chi_B)>1-1/k\}\right)\right)\\ &= \varphi(A)\times\psi_\infty(B),\end{aligned}$$

that is $\beta_\infty(A\times B)=\varphi(A)\times\psi_\infty(B)$ for all $A\in\Sigma,\ B\in\eta_{\mathscr{K}}$.

Again we have found an upper bound $(\eta_{\mathscr{K}},\psi_{\mathscr{K}},\beta_{\mathscr{K}})$ for $\mathscr{K}$ in $\mathscr{B}$ if we define $\psi_{\mathscr{K}}:=\psi_\infty,\ \beta_{\mathscr{K}}:=\beta_\infty$.

According to Zorn's lemma we choose a maximal element $(\eta_0,\psi_0,\beta_0)\in\mathscr{B}$. If we assume that η_0 is a strict subset of T then there exists a $B_0\in T\setminus\eta_0$. Let $\overline{\eta}_0$ denote the σ-subalgebra of T generated by $\eta_0\cup\{B_0\}$. Using the well-known equality

$$\overline{\eta}_0=\{(G\cap B_0)\cup(H\cap B_0^c):G,H\in\eta_0\}$$

we can check:

CLAIM 2. $\Sigma\otimes\overline{\eta}_0=\{(G\cap(\Omega\times B_0))\cup(H\cap(\Omega\times B_0^c)):G,H\in\Sigma\otimes\eta_0\}$.

Put

$$\begin{aligned}B_1&:=\operatorname{ess\,inf}\{B\in\eta_0:B_0\subseteq B\text{ a.e. }(\nu)\},\\ B_2&:=\operatorname{ess\,inf}\{B\in\eta_0:B_0^c\subseteq B\text{ a.e. }(\nu)\},\\ E_1&:=\operatorname{ess\,inf}\{E\in\Sigma\otimes\eta_0:\Omega\times B_0\subseteq E\text{ a.e. }(\mu\otimes\nu)\},\\ E_2&:=\operatorname{ess\,inf}\{E\in\Sigma\otimes\eta_0:\Omega\times B_0^c\subseteq E\text{ a.e. }(\mu\otimes\nu)\},\end{aligned}$$

and note that E_1 respectively E_2 is the essential infimum of sets $E\in\Sigma\otimes\eta_0$ with $\Omega\times B_0\subseteq E$ a.e. $(\mu\otimes\nu)$ respectively $\Omega\times B_0^c\subseteq E$ a.e. $(\mu\otimes\nu)$ modulo $\mu\otimes\nu$-null sets in $\Sigma\otimes\eta_0$, while the essential infima B_1,B_2 may be taken with respect to arbitrary ν-nullsets since η_0 contains all ν-null sets by definition of η_0 for $(\eta_0,\psi_{\eta_0},\beta_{\eta_0})\in\mathscr{B}$.

Then it is well-known by [12, Lemma 2] that by means of

$$\begin{aligned}&\overline{\psi}_0((G\cap B_0)\cup(H\cap B_0^c))\\ &\quad:=(B_0\cap\psi_0((G\cap B_1)\cup(H\cap B_1^c)))\cup(B_0^c\cap\psi_0((H\cap B_2)\cup(G\cap B_2^c)))\end{aligned}$$

for $G,\ H\in\eta_0$, and

$$\begin{aligned}&\overline{\beta}_0((K\cap(\Omega\times B_0))\cup(L\cap(\Omega\times B_0^c))\\ &\quad:=((\Omega\times B_0)\cap\beta_0((K\cap E_1)\cup(L\cap E_1^c)))\\ &\qquad\cup((\Omega\times B_0^c)\cap\beta_0((L\cap E_2)\cup(K\cap E_2^c)))\end{aligned}$$

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for $K, L \in \Sigma \otimes \eta_0$ are defined unambiguously lower densities $\overline{\psi}_0$ on $\overline{\eta}_0$ respectively $\overline{\beta}_0$ on $\Sigma \otimes \overline{\eta}_0$ such that $\overline{\psi}_0 \mid \eta_0 = \psi_0$ and $\overline{\beta}_0 \mid \Sigma \otimes \eta_0 = \beta_0$.

CLAIM 3. *$E_1 = \Omega \times B_1$ a.e. $(\mu \otimes \nu \mid \Sigma \otimes \eta_0)$ and $E_2 = \Omega \times B_2$ a.e. $(\mu \otimes \nu \mid \Sigma \otimes \eta_0)$.*

PROOF. $B_0 \subseteq B_1$ a.e. (ν) implies $\Omega \times B_0 \subseteq \Omega \times B_1$ a.e. $(\mu \otimes \nu \mid \Sigma \otimes \overline{\eta}_0)$. Since $\Omega \times B_1 \in \Sigma \otimes \eta_0$ this implies $\Omega \times B_1 \supseteq E_1$ a.e. $(\mu \otimes \nu \mid \Sigma \otimes \eta_0)$ by definition of E_1. For $\omega \in \Omega$ let $E_{1\omega} := \{\theta \in \Theta : (\omega, \theta) \in E_1\}$ be the ω-section of E_1. Note that by Lemma 2 the probability space $(\Omega \times \Theta, \Sigma \otimes \eta_0, \mu \otimes \nu \mid \Sigma \otimes \eta_0)$ is the product of the probability spaces (Ω, Σ, μ) and $(\Theta, \eta_0, \nu \mid \eta_0)$. Hence applying Fubini's theorem (see [15, (21.4) and (21.8)]) to this product, we have $E_{1\omega} \in \eta_0$ for all $\omega \in \Omega$ since $E_1 \in \Sigma \otimes \eta_0$,

$$\begin{aligned} 0 &= (\mu \otimes \nu)((\Omega \times B_0) \setminus E_1) \\ &= \int \nu(((\Omega \times B_0) \setminus E_1)_\omega) d\mu(\omega) \\ &= \int \nu(B_0 \setminus E_{1\omega}) d\mu(\omega), \end{aligned}$$

$0 \leq \nu(B_0 \setminus E_{1\omega})$ for all $\omega \in \Omega$ and $\omega \in \Omega \longrightarrow \nu(B_0 \setminus E_{1\omega})$ is Σ-measurable. This implies for $N := \{\omega \in \Omega : \nu(B_0 \setminus E_{1\omega}) > 0\}$ that $N \in \Sigma$ and $\mu(N) = 0$. If $\Omega_0 := \Omega \setminus N$ then $\Omega_0 \in \Sigma$, $\mu(\Omega_0) = 1$, $\nu(B_0 \setminus E_{1\omega}) = 0$ hence $B_0 \subseteq E_{1\omega}$ a.e. (ν) for all $\omega \in \Omega_0$. Let $B := \operatorname{ess\,inf}\{E_{1\omega} : \omega \in \Omega_0\}$ where the essential infimum is taken in η_0 with respect to ν-null sets. It follows that $B \in \eta_0$, and $B \supseteq B_0$ a.e. (ν), hence $B \supseteq B_1$a.e. (ν). Again by [15, (21.8)] we have

$$\begin{aligned} (\mu \otimes \nu)((\Omega \times B_1) \setminus E_1) &= \int \nu(((\Omega \times B_1) \setminus E_1)_\omega) d\mu(\omega) \\ &= \int \nu(B_1 \setminus E_{1\omega}) d\mu(\omega) \\ &= 0, \end{aligned}$$

the latter since $B_1 \subseteq E_{1\omega}$ a.e. (ν) for all $\omega \in \Omega_0$ and since $\mu(\Omega_0) = 1$, that is $\Omega \times B_1 \subseteq E_1$ a.e. $(\mu \otimes \nu \mid \Sigma \otimes \eta_0)$. The proof for $E_2 = \Omega \times B_2$ a.e. $(\mu \otimes \nu \mid \Sigma \otimes \eta_0)$ is similar.

For $A \in \Sigma$ and $B \in \overline{\eta}_0$ write $B = ((G \cap B_0) \cup (H \cap B_0^c))$ for $G, H \in \eta_0$. And then we have

$$\begin{aligned} F := A \times B &= A \times ((G \cap B_0) \cup (H \cap B_0^c)) \\ &= ((A \times G) \cap (\Omega \times B_0)) \cup ((A \times H) \cap (\Omega \times B_0^c)) \end{aligned}$$

together with $K := A \times G,\ L := A \times H \in \Sigma \otimes \eta_0$. For simplicity put $E_0 := \Omega \times B_0$. By definition we have

$$\overline{\beta}_0(F) = (E_0 \cap \beta_0((K \cap E_1) \cup (L \cap E_1^c))) \cup (E_0^c \cap \beta_0((L \cap E_2) \cup (K \cap E_2^c))).$$

By an application of claim 3 this can be rewritten as

$$\overline{\beta}_0(F) = (E_0 \cap \beta_0(A \times R)) \cup (E_0^c \cap \beta_0((A \times S))$$

if

$$R := (G \cap B_1) \cup (H \cap B_1^c), \quad S := (H \cap B_2) \cup (G \cap B_2^c).$$

Since $R, S \in \eta_0$ this implies

$$\overline{\beta}_0(F) = (E_0 \cap (\varphi(A) \times \psi_0(R))) \cup (E_0^c \cap (\varphi(A) \times \psi_0(S))).$$

By means of $E_0 = \Omega \times B_0$ the latter formula can be transformed into

$$\overline{\beta}_0(A \times B) = \varphi(A) \times \overline{\psi}_0(B) \quad \text{for all} \quad A \in \Sigma \quad \text{and} \quad B \in \overline{\beta}_0.$$

This implies that $(\overline{\eta}_0, \overline{\psi}_0, \overline{\beta}_0) \in \mathscr{B}$ and clearly $(\eta_0, \psi_0, \beta_0) < (\overline{\eta}_0, \overline{\psi}_0, \overline{\beta}_0)$, a contradiction. Therefore $\eta_0 = T$ and if we put $\psi = \psi_0,\ \beta = \beta_0$ we are done.

THEOREM 4. *Let (Ω, Σ, μ) and (Θ, T, ν) be complete probability spaces and $(\Omega \times \Theta, \Sigma\hat{\otimes}T, \mu\hat{\otimes}\nu)$ the completion of their product. Then for any $\rho \in \Lambda(\mu)$ there exist $\sigma \in \Lambda(\nu)$ and $\pi \in \Lambda(\mu\hat{\otimes}\nu)$ such that*

(P) $$\pi(A \times B) = \rho(A) \times \sigma(B) \quad \textit{for all} \quad A \in \Sigma \quad \textit{and} \quad B \in T.$$

In addition $\pi(f \otimes g) = \rho(f) \otimes \sigma(g)$ for all $f \in \mathscr{L}^\infty(\mu)$ and $g \in \mathscr{L}^\infty(\nu)$.

For the proof we denote for any σ-subalgebra η of T by $\Sigma\overline{\otimes}\eta$ the σ-subalgebra of $\Sigma\hat{\otimes}T$ consisting of all $F \in \Sigma\hat{\otimes}T$ for which there exists a set $E \in \Sigma \otimes \eta$ with $F = E$ a.e. $(\mu\hat{\otimes}\nu)$, in particular $\Sigma\overline{\otimes}\eta$ contains all $\mu\hat{\otimes}\nu$-null sets and clearly $\Sigma\overline{\otimes}T = \Sigma\hat{\otimes}T$.

The proof of Theorem 4 follows the same general pattern as that of Theorem 3 and with the exception of part (B) it consists in minor modifications of the proof of Theorem 3 so that we have only to replace 'lower density' by 'lifting', '$\Sigma \otimes \eta$' by '$\Sigma\overline{\otimes}\eta$', and '$\mu \otimes \nu$' by '$\mu\hat{\otimes}\nu$' throughout. In order to shorten the proof let this be done together with the following indications:

(i) The existence of a lifting φ for μ follows from ([16, IV, 2, Theorem 3]).
(ii) If $(\eta, \psi_\eta, \beta_\eta) \in \mathscr{B}$ then η is complete since η contains all ν-null sets.

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(iii) For $F \in \Sigma\hat{\otimes}u$ we find $E \in \Sigma \otimes u$ with $F = E$ a.e. $(\mu\hat{\otimes}\nu)$ and then we can define the lifting β_u for $\mu\hat{\otimes}\nu \mid \Sigma\overline{\otimes}u$ by means of $\beta_u(F) = \varphi(\tilde{E}) \times \Theta$ if $\tilde{E}$ and u are defined as in the proof of Theorem 3 (independent of the choice of E).

(iv) From $\Sigma \otimes \eta_{\mathscr{K}} = \bigcup\{\Sigma \otimes \eta : \eta \in \mathscr{K}_1\}$ follows follows immediately that $\Sigma\overline{\otimes}\eta_{\mathscr{K}} = \bigcup\{\Sigma\overline{\otimes}\eta : \eta \in \mathscr{K}_1\}$.

(v) Claim 2 in the proof of Theorem 3 implies

$$\Sigma\overline{\otimes}\eta_0 = \{(G \cap (\Omega \times B_0)) \cup (H \cap (\Omega \times B_0^c)) : G, H \in \Sigma\overline{\otimes}\eta_0\}.$$

(vi) For Claim 3 the essential infima E_1, E_2 can (and will) be chosen in $\Sigma \otimes \eta_0$. This allows us to take over the proof of Claim 3 from Theorem 3 unchanged.

(vii) Once we have $\eta_0 = T$ we define $\rho := \varphi$, $\sigma := \psi_0$, and $\pi := \beta_0$.

(viii) $\pi(f \otimes g) = \rho(f) \otimes \sigma(g)$ follows from (P) first for simple functions f, g and then for $f \in \mathscr{L}^\infty(\mu)$, $g \in \mathscr{L}^\infty(\nu)$ by approximation with simple functions.

We now turn to the proof of (B). Let there be given a countable cofinal part $((\eta_n, \psi_{\eta_n}, \beta_{\eta_n}))_{n\in\mathbb{N}}$ in $\mathscr{K}$. Again put $\psi_n := \psi_{\eta_n}$, $\beta_n := \beta_{\eta_n}$ for all $n \in \mathbb{N}$ and denote by η_∞ the σ-subalgebra of T generated by $\bigcup_{n\in\mathbb{N}} \eta_n$. Claim 1 of the proof of Theorem 3 implies $\Sigma\overline{\otimes}\eta_\infty = \sigma\left(\bigcup_{n\in\mathbb{N}} \Sigma\overline{\otimes}\eta_n\right)$. We now choose an ultrafilter $\mathscr{U}$ on $\mathbb{N}$ finer than the Fréchet filter and define by [16, IV, 1, Theorem 2] linear liftings for ν respectively $\mu\hat{\otimes}\nu$ by means of

$$\psi'(h) := \lim_{n\in\mathscr{U}} \psi_n(E_{\eta_n}(h)) \quad \text{for} \quad h \in \mathscr{L}^\infty(\nu \mid \eta_\infty),$$
$$\beta'(f) := \lim_{n\in\mathscr{U}} \beta_n(E_{\Sigma\bar{\otimes}\eta_n}(f)) \quad \text{for} \quad f \in \mathscr{L}^\infty(\mu\hat{\otimes}\nu \mid \Sigma\overline{\otimes}\eta_\infty)$$

which satisfy $\psi' \mid \mathscr{L}^\infty(\nu \mid \eta_n) = \psi_n$, respectively $\beta' \mid \mathscr{L}^\infty(\mu\hat{\otimes}\nu) \mid \Sigma\overline{\otimes}\eta_n) = \beta_n$ for $n \in \mathbb{N}$. For $g \in \mathscr{L}^\infty(\mu)$, $h \in \mathscr{L}^\infty(\nu)$, applying Lemma 1, we have

$$\begin{aligned}
\beta'(g \otimes h) &= \lim_{n\in\mathscr{U}} \beta_n(E_{\Sigma\overline{\otimes}\eta_n}(g \otimes h)) = \lim_{n\in\mathscr{U}} \beta_n(g \otimes E_{\eta_n}(h)) \\
&= \lim_{n\in\mathscr{U}}(\varphi(g) \otimes \psi_n(E_{\eta_n}(h))) = \varphi(g) \otimes \lim_{n\in\mathscr{U}} \psi_n(E_{\eta_n}(h)) \\
&= \varphi(g) \otimes \psi'(h),
\end{aligned}$$

that is

$$\beta'(g \otimes h) = \varphi(g) \otimes \psi'(h) \quad \text{for all} \quad g \in \mathscr{L}^\infty(\mu),\ h \in \mathscr{L}^\infty(\nu). \tag{1}$$

(Let us remember that according to Section 1 we write φ for the lifting of functions $\tilde{\varphi}$ defined by $\tilde{\varphi}(\chi_A) = \chi_{\varphi(A)}$, $A \in \Sigma$.)

According to [16, III, 1, page 36] we then define lower densities ψ_d for $\nu \mid \eta_\infty$ respectively β_d for $\mu\hat{\otimes}\nu \mid \Sigma\overline{\otimes}\eta_\infty$ by means of

$$\psi_d(B) := \{\psi'(\chi_B) = 1\} \text{ for } B \in \eta_\infty, \qquad \beta_d(E) := \{\beta'(\chi_E) = 1\} \text{ for } E \in \Sigma\bar{\otimes}\eta_\infty.$$

For $\omega \in \Omega, \theta \in \Theta$ let us define filterbases $\mathscr{F}(\omega) := \{A \in \Sigma : \omega \in \varphi(A)\}$ on Ω, $\mathscr{F}(\theta) := \{B \in \eta_\infty : \theta \in \psi_d(B)\}$ on Θ, $\mathscr{F}(\omega, \theta) := \{E \in \Sigma\overline{\otimes}\eta_\infty : (\omega, \theta) \in \beta_d(E)\}$ on $\Omega \times \Theta$ and choose ultrafilters $\mathscr{U}(\omega)$ in Σ, $\mathscr{U}(\theta)$ in η_∞ finer than $\mathscr{F}(\omega)$ respectively $\mathscr{F}(\theta)$. Note that $\mathscr{U}(\omega) = \mathscr{F}(\omega)$.

CLAIM. *$E \cap (A \times B) \neq \emptyset$ for $E \in \mathscr{F}(\omega, \theta)$, $A \in \mathscr{U}(\omega)$, and $B \in \mathscr{U}(\theta)$ if $(\omega, \theta) \in \Omega \times \Theta$.*

PROOF. Assuming that $E \cap (A \times B) = \emptyset$ it follows by (1) that

$$0 = 1 - \beta'(\chi_E)(\omega, \theta) \geq \beta'(\chi_{A\times B})(\omega, \theta) = \varphi(\chi_A)(\omega)\psi'(\chi_B)(\theta) = \psi'(\chi_B)(\theta) \geq 0,$$

hence $\psi'(\chi_B)(\theta) = 0$. Since $\psi'(\chi_B) + \psi'(\chi_{B^c}) = 1$ this implies $\theta \in \psi_d(B^c)$, a contradiction.

By the Claim we find an ultrafilter $\mathscr{U}(\omega, \theta) \subseteq \Sigma\overline{\otimes}\eta_\infty$ finer than $\mathscr{F}(\omega, \theta)$ and such that

$$A \times B \in \mathscr{U}(\omega, \theta) \quad \text{for all} \quad A \in \mathscr{U}(\omega), B \in \mathscr{U}(\theta).$$

According to [30] we define liftings $\bar{\varphi}$ for μ, ψ_∞ for $\nu \mid \eta_\infty$, and β_∞ for $\mu\hat{\otimes}\nu \mid \Sigma\overline{\otimes}\eta_\infty$ respectively by means of

$$\begin{aligned}
&\bar{\varphi}(A) := \{\omega \in \Omega : A \in \mathscr{U}(\omega)\} \quad \text{with} \quad \varphi(A) \subseteq \bar{\varphi}(A) \subseteq \varphi(A^c)^c \quad \text{for} \quad A \in \Sigma,\\
&\psi_\infty(B) := \{\theta \in \Theta : B \in \mathscr{U}(\theta)\} \quad \text{with} \quad \psi_d(B) \subseteq \psi_\infty(B) \subseteq \psi_d(B^c)^c \text{ for } B \in \eta_\infty,\\
&\beta_\infty(E) := \{(\omega, \theta) \in \Omega \times \Theta : E \in \mathscr{U}(\omega, \theta)\} \quad \text{with} \quad \beta_d(E) \subseteq \beta_\infty(E) \subseteq \beta_d(E^c)^c
\end{aligned}$$

for $E \in \Sigma\overline{\otimes}\eta_\infty$. Note that $\bar{\varphi} = \varphi$ and for $n \in \mathbb{N}$, $B \in \eta_n$ we have $\psi'(\chi_B) = \psi_n(\chi_B)$, hence $\psi_d(B) = \{\psi'(\chi_B) = 1\} = \psi_n(B)$, $\psi_n(B) = \psi_d(B) \subseteq \psi_\infty(B) \subseteq \psi_d(B^c)^c = \psi_n(B)$, so $\psi_\infty \mid \eta_n = \psi_n$. In the same way we find that $\beta_\infty \mid \Sigma\overline{\otimes}\eta_n = \beta_n$ for $n \in \mathbb{N}$.

For $(\omega, \theta) \in \Omega \times \Theta$, $A \in \Sigma$, $B \in \eta_\infty$ we have $(\omega, \theta) \in \varphi(A) \times \psi_\infty(B)$ if and only if $A \in \mathscr{U}(\omega)$ and $B \in \mathscr{U}(\theta)$, hence $A \times B \in \mathscr{U}(\omega, \theta)$ by the claim, so $(\omega, \theta) \in \beta_\infty(A \times B)$, that is $\varphi(A) \times \psi_\infty(B) \subseteq \beta_\infty(A \times B)$.

If $p_1 : \Omega \times \Theta \longrightarrow \Omega$, $p_2 : \Omega \times \Theta \longrightarrow \Theta$ are the canonical projections, then $p_1(\mathscr{U}(\omega, \theta)) \cap \Sigma$ is again an ultrafilter in Σ and so is $\mathscr{U}(\omega) = \mathscr{F}(\omega)$, $p_2(\mathscr{U}(\omega, \theta)) \cap \eta_\infty$ is an ultrafilter in η_∞ and so is $\mathscr{U}(\theta)$. By the claim $\mathscr{U}(\omega) \subseteq p_1(\mathscr{U}(\omega, \theta)) \cap \Sigma$, $\mathscr{U}(\theta) \subseteq p_2(\mathscr{U}(\omega, \theta)) \cap \eta_\infty$. This implies $p_1(\mathscr{U}(\omega, \theta)) \cap \Sigma = \mathscr{F}(\omega)$, $p_2(\mathscr{U}(\omega, \theta)) \cap \eta_\infty = \mathscr{U}(\theta)$. So if $(\omega, \theta) \in \beta_\infty(A \times B)$, that is $A \times B \in$

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$\mathscr{U}(\omega, \theta)$ we have $A \in \mathscr{F}(\omega)$ and $B \in \mathscr{U}(\theta)$, that is $\omega \in \varphi(A)$ and $\theta \in \psi_\infty(B)$, hence $\beta_\infty(A \times B) \subseteq \varphi(A) \times \psi_\infty(B)$, so $\beta_\infty(A \times B) = \varphi(A) \times \psi_\infty(B)$.

Again $(\eta_\infty, \psi_\infty, \beta_\infty)$ is an upper bound for $\mathscr{K}$ in $\mathscr{B}$.

If for $\rho \in \Lambda(\mu)$, $\sigma \in \Lambda(\nu)$ and $\pi \in \Lambda(\mu\hat{\otimes}\nu)$ the equation (P) of Theorem 4 is true we write $\pi = \rho \otimes \sigma$ and call π a *product lifting* of ρ and σ. A lifting $\pi \in \Lambda(\mu\hat{\otimes}\nu)$ is called *decomposable* if it is a product lifting.

For a sequence $((\Omega_n, \Sigma_n, \mu_n))_{n\in\mathbb{N}}$ of complete probability spaces we define recursively for $n \in \mathbb{N}$ a lifting ρ_n^x on $(\otimes_{i=1}^n(\Omega_i, \Sigma_i, \mu_i))^\wedge$ in the following way: Choose $\rho \in \Lambda(\mu_1)$ and put $\rho_1^x := \rho$. If $\rho_n^x \in \Lambda(\mu_1\hat{\otimes}\cdots\hat{\otimes}\mu_n)$ has already been constructed, we find by Theorem 4 a $\rho_{n+1} \in \Lambda(\mu_{n+1})$ such that $\rho_{n+1}^x := \rho_n^x \otimes \rho_{n+1} \in \Lambda(\mu_1\hat{\otimes}\cdots\hat{\otimes}\mu_{n+1})$ for $n \in \mathbb{N}$. Then put

$$\rho_n^x := (((\rho_1 \otimes \rho_2) \otimes \rho_3) \cdots \otimes \rho_{n-1}) \otimes \rho_n,$$

but we may not rearrange the brackets. For this product the following recursion formulas hold true.

(i) $\rho_n^x(A \times A_n) = \rho_{n-1}^x(A) \times \rho_n(A_n)$ for all $A \in (\otimes_{i=1}^{n-1}\Sigma_i)^\wedge$ and $A_n \in \Sigma_n$, $n > 1$.

(ii) If $\leq m \leq n$ then $\rho_n^x(A \times A_{m+1} \times \cdots \times A_n) = \rho_m^x(A) \times \rho_{m+1}(A_{m+1}) \times \cdots \times \rho_n(A_n)$ for all $A \in (\otimes_{i=1}^m \Sigma_i)^\wedge$, $A_j \in \Sigma_j$ $(j = m+1, \ldots, n)$, that is

$$\rho_n^x := ((\rho_m^x \otimes \rho_{m+1}) \otimes \cdots \otimes \rho_{n-1}) \otimes \rho_n.$$

In particular we have

(iii) $\rho_n^x(A_1 \times \cdots \times A_n) = \rho_1(A_1) \times \cdots \times \rho_n(A_n)$ for all $A_i \in \Sigma_i$ $(i = 1, \ldots, n)$.

(iv) If $1 \leq m \leq n$ then $\rho_n^x(A \times \Omega_{m+1} \times \cdots \times \Omega_n) = \rho_m^x(A) \times \Omega_{m+1} \times \cdots \times \Omega_n$ for all $A \in (\otimes_{i=1}^m \Sigma_i)^\wedge$.

The following is a generalization of theorem 4 for countable products.

THEOREM 5. *Let there be given a sequence $((\Omega_n, \Sigma_n, \mu_n))_{n\in\mathbb{N}}$ of complete probability spaces with completed product (Ω, Σ, μ). Then for any $\rho_1 \in \Lambda(\mu_1)$ there exist $\rho_n \in \Lambda(\mu_n)$ $(n \geq 2)$ and $\rho_\infty \in \Lambda(\mu)$ such that*

(i) $\rho_\infty(A \times \prod_{i=n+1}^\infty \Omega_i) = (\rho_1 \otimes \cdots \otimes \rho_n)(A) \times \prod_{i=n+1}^\infty \Omega_i$ *for* $A \in \Sigma_1\hat{\otimes}\cdots\hat{\otimes}\Sigma_n$, *and*

(ii) $\rho_\infty(A_1 \times \cdots \times A_n \times \prod_{i=n+1}^\infty \Omega_i) = \rho_1(A_1) \times \cdots \times \rho_n(A_n) \times \prod_{i=n+1}^\infty \Omega_i$ *for* $A_i \in \Sigma_i$ $(i = 1, \ldots, n)$.

PROOF. For any $n \in \mathbb{N}$ consider the σ-algebra $\Sigma_n^* := p_{[n]}^{-1}(\Sigma_1\hat{\otimes}\cdots\hat{\otimes}\Sigma_n)$ where $p_{[n]}$ is the canonical projection from Ω onto $\prod_{i=1}^n \Omega_i$ and $[n] := \{1, 2, \ldots, n\}$. Clearly it holds true that $\Sigma_m^* \subseteq \Sigma_n^*$ for any $1 \leq m \leq n$. Let $(\Sigma_n^*)_\mu$ be the σ-subfield of Σ generated by $\Sigma_n^* \cup u$ for $u := \{N \in \Sigma : \mu(N) = 0\}$, $\hat{\mu}_n := \mu \mid (\Sigma_n^*)_\mu$ and $\Sigma^* =$

$\bigcup_{n\in\mathbb{N}}(\Sigma_n^*)_\mu$. Then Σ^* is a field of subsets of Ω and Σ is equal to the σ-field generated by Σ^*. Moreover $\mu\circ p_{[n]}^{-1} = \mu_1\hat{\otimes}\cdots\hat{\otimes}\mu_n$ for all $n\in\mathbb{N}$ (cf. [17, VI, Proposition 5.4]). Define a lifting ρ_n^* for $(\Omega, (\Sigma_n^*)_\mu, \hat{\mu}_n)$ by means of $\rho_n^*(A^*) := \rho_n^x(A)\times\prod_{i=n+1}^\infty \Omega_i$ where $A^*\in(\Sigma_n^*)_\mu$ and $A\in\Sigma_n$ with $A^* = A\times\prod_{i=n+1}^\infty\Omega_i$ a.e. $(\hat{\mu}_n)$.

It then holds true that $\rho_n^* \mid (\Sigma_m^*)_\mu = \rho_m^*$ for any m with $1\le m\le n$.

Indeed for $A^*\in(\Sigma_m^*)_\mu$ there exists a set $A\in\Sigma_m$ such that $A^* = A\times\prod_{i=m+1}^\infty\Omega_i$ a.e. $(\hat{\mu}_m)$, $m\in\mathbb{N}$. Using recursion formula (iv) following the proof of theorem 4 we get

$$\begin{aligned}
\rho_n^*(A^*) &= \rho_n^*(A\times\prod\nolimits_{i=m+1}^\infty\Omega_i)\\
&= \rho_n^*(A\times\Omega_{m+1}\times\cdots\times\Omega_n\times\prod\nolimits_{i=n+1}^\infty\Omega_i)\\
&= \rho_n^x(A\times\Omega_{m+1}\times\cdots\times\Omega_n)\times\prod\nolimits_{i=n+1}^\infty\Omega_i\\
&= \rho_m^x(A)\times\Omega_{m+1}\times\cdots\times\Omega_n\times\prod\nolimits_{i=n+1}^\infty\Omega_i\\
&= \rho_m^x(A)\times\prod\nolimits_{i=m+1}^\infty\Omega_i\\
&= \rho_m^*(A^*).
\end{aligned}$$

Thus there exists a lifting $\rho_\infty\in\Lambda(\mu)$ with property (i) (cf. for example [17, XVI, Proposition 1.8]). Relation (ii) follows immediately from relation (i) and Theorem 4.

3. Products of lifting topologies

For a complete probability space (Ω, Σ, μ) and a lifting $\rho\in\Lambda(\mu)$ can be associated two so-called lifting topologies

$$\mathscr{T}_\rho := \left\{\bigcup_{i\in I}\rho(A_i) : A_i\in\Sigma\right\} \quad\text{and}\quad \mathscr{T}_\rho^* := \{A\in\Sigma : A\subseteq\rho(A)\}.$$

Both $\mathscr{T}_\rho$ and $\mathscr{T}_\rho^*$ are extremally disconnected topologies such that $\mathscr{T}_\rho\subseteq\mathscr{T}_\rho^*$ and $C_b(\Omega,\mathscr{T}_\rho) = C_b(\Omega,\mathscr{T}_\rho^*) = \{f\in\mathscr{L}^\infty(\mu) : f = \rho(f)\}$ (see [16, Chapter V]). Here again we use the same notation for ρ and its uniquely associated lifting for functions in $\mathscr{L}^\infty(\mu)$ as in Section 2.

LEMMA 1. *Let $(\Omega_i, \Sigma_i, \mu_i)$ be a complete probability space, $\rho_i\in\Lambda(\mu_i)$ $(i = 1,\dots,n)$, and denote by $(\Omega_1\times\cdots\times\Omega_n, \Sigma, \mu)$ a complete probability space such that $\Sigma_1\hat{\otimes}\cdots\hat{\otimes}\Sigma_n\subseteq\Sigma$ and $\mu\mid\Sigma_1\hat{\otimes}\cdots\hat{\otimes}\Sigma_n = \mu_1\hat{\otimes}\cdots\hat{\otimes}\mu_n$. Then for a $\pi\in\Lambda(\mu)$ the following conditions are all equivalent.*

(i) $\mathscr{T}_{\rho_1}\times\cdots\times\mathscr{T}_{\rho_n}\subseteq\Sigma$ *and π is strong with respect to* $\mathscr{T}_{\rho_1}\times\cdots\times\mathscr{T}_{\rho_n}$.

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(ii) $\mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n} \subseteq \Sigma$ *and* π *is strong with respect to* $\mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n}$.
(iii) $\pi(A_1 \times \cdots \times A_n) = \rho_1(A_1) \times \cdots \times \rho_n(A_n)$ *for all* $A_i \in \Sigma_i$ $(i = 1, \ldots, n)$.
(iv) $\mathcal{T}_{\rho_1} \times \cdots \times \mathcal{T}_{\rho_n} \subseteq \mathcal{T}_\pi$.
(v) $\mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n} \subseteq \mathcal{T}^*_\pi$.

PROOF. Either (i) or (ii) implies (iii). For $h_i \in C_b(\Omega_i, \mathcal{T}_{\rho_i}) = C_b(\Omega_i, \mathcal{T}^*_{\rho_i})$, $\Omega := \Omega_1 \times \cdots \times \Omega_n$, $p_i : \Omega \longrightarrow \Omega_i$, the canonical projection, we have $h_i \circ p_i \in C_b(\Omega, \mathcal{T}_{\rho_1} \times \cdots \times \mathcal{T}_{\rho_n}) \subseteq C_b(\Omega, \mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n})$ for $i = 1, \ldots, n$). Let $\mathcal{T}_{\rho_1} \times \cdots \times \mathcal{T}_{\rho_n} \subseteq \Sigma$ respectively $\mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n} \subseteq \Sigma$ and let $\pi \in \Lambda(\mu)$ be strong with respect to $\mathcal{T}_{\rho_1} \times \cdots \times \mathcal{T}_{\rho_n}$ respectively $\mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n}$. Then it follows that

$$(*) \qquad \pi\left(\prod_{i=1}^n (h_i \circ p_i)\right) = \prod_{i=1}^n (h_i \circ p_i).$$

For $A_i \in \Sigma_i$ $(i = 1, \ldots, n)$ the sets $A_1 \times \cdots \times A_n$ and $\rho_1(A_1) \times \cdots \times \rho_n(A_n)$ differ only by a set of μ-measure zero. Therefore $\pi(A_1 \times \cdots \times A_n) = \pi(\rho_1(A_1) \times \cdots \times \rho_n(A_n)) = \rho_1(A_1) \times \cdots \times \rho_n(A_n)$, where the latter equality follows from $(*)$ for $h_i = \chi_{\rho_i}(A_i) \in C_b(\Omega_i, \mathcal{T}_{\rho_i}) = C_b(\Omega_i, \mathcal{T}^*_{\rho_i})$ $(i = 1, \ldots, n)$.

(iii) implies (iv). Since $\eta_i := \{\rho_i(A_i) : A_i \in \Sigma_i\}$ is a basis for the topology $\mathcal{T}_{\rho_i}$ $(i = 1, \ldots, n)$. Therefore $\tilde{\eta} := \{\rho_1(A_1) \times \cdots \times \rho_n(A_n) : A_i \in \Sigma_i \ (i = 1, \ldots, n)\}$ is a basis for the topology $\mathcal{T}_{\rho_1} \times \cdots \times \mathcal{T}_{\rho_n}$. By (iii) we have $\tilde{\eta} \subseteq \mathcal{T}_\pi$, $\mathcal{T}_{\rho_1} \times \cdots \times \mathcal{T}_{\rho_n} \subseteq \mathcal{T}_\pi$.

(iii) implies (v). $\eta^* := \{A_1 \times \cdots \times A_n : A_i \in \mathcal{T}^*_{\rho_i} (i = 1, \ldots, n)\}$ is a basis for the topology $\mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n}$. But $A_1 \times \cdots \times A_n \subseteq \rho_1(A_1) \times \cdots \times \rho_n(A_n) = \pi(A_1 \times \cdots \times A_n)$ by (iii) for $A_1 \times \cdots \times A_n \in \eta^*$, that is $\eta^* \subseteq \mathcal{T}^*_\pi$, hence $\mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n} \subseteq \mathcal{T}^*_\pi$.

The equivalences (i) if and only if (iv) and (ii) if and only if (v) hold true by [16, Theorem 3, p. 64].

THEOREM 1. *Let* $(\Omega_i, \mathcal{T}_i, \Sigma_i, \mu_i)$ *be a complete topological probability space with a lifting* $\rho_i \in \Lambda(\mu_i)$, $\Omega_i \neq \emptyset$ $(i = 1, \ldots, n)$, *let* $(\Omega_1 \times \cdots \times \Omega_n, \Sigma, \mu)$ *be a complete probability space such that* $\Sigma_1 \hat{\otimes} \cdots \hat{\otimes} \Sigma_n \subseteq \Sigma$ *and* $\mu \mid \Sigma_1 \hat{\otimes} \cdots \hat{\otimes} \Sigma_n = \mu_1 \hat{\otimes} \cdots \hat{\otimes} \mu_n$, *and let* $\pi \in \Lambda(\mu)$ *satisfy*

$$\pi(A_1 \times \cdots \times A_n) = \rho_1(A_1) \times \cdots \times \rho_n(A_n) \quad \textit{for all} \quad A_i \in \Sigma_i \ (i = 1, \ldots, n).$$

Then we have $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n \subseteq \Sigma$ *for a* $\pi \in \Lambda(\mu)$ *and* π *is strong with respect to* $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n$ *if and only if all* ρ_i *are strong for* $i = 1, \ldots, n$.

Moreover if $\Sigma_i = \hat{\mathcal{B}}(\Omega_i)$, μ_i *is completion regular for* $i = 1, \ldots, n$, $\Sigma = \Sigma_1 \hat{\otimes} \cdots \hat{\otimes} \Sigma_n$, *and* $\mathcal{B}_0(\Omega) = \mathcal{B}_0(\Omega_1) \otimes \cdots \otimes \mathcal{B}_0(\Omega_n)$ *then* $\mu_1 \hat{\otimes} \cdots \hat{\otimes} \mu_n$ *is in addition completion regular.*

PROOF. If the liftings ρ_i $(i = 1, \ldots, n)$ are strong we have $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n \subseteq \mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n}$. By Lemma 1, (iii) if and only if (v) it follows that $\mathcal{T}^*_{\rho_1} \times \cdots \times \mathcal{T}^*_{\rho_n} \subseteq \mathcal{T}^*_\pi$.

Since clearly $\mathcal{T}_\pi^* \subseteq \Sigma$ we have $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n \subseteq \mathcal{T}_\pi^*$, that is π is strong by [16, Theorem 3, p. 64]. On the other hand if $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n \subseteq \Sigma$ and π is strong then $G_1 \times \Omega_2 \times \cdots \times \Omega_n \in \mathcal{T}_1 \times \cdots \times \mathcal{T}_n$ for $G_1 \in \mathcal{T}_1$ and hence $G_1 \times \Omega_2 \times \cdots \times \Omega_n \subseteq \pi(G_1 \times \Omega_2 \times \cdots \times \Omega_n) = \rho_1(G_1) \times \Omega_2 \times \cdots \times \Omega_n$. This implies $G_1 \subseteq \rho_1(G_1)$ for $G_1 \in \mathcal{T}_1$, that is ρ_1 is strong. In the same way it follows that $\rho_2, \dots, \rho_n$ are strong.

To show that μ is completion regular let $A = A_1 \times \cdots \times A_n$, $A_i \in \Sigma_i$ $(i = 1, \dots, n)$. The completion regularity of μ_i implies that $A_i \in \hat{\mathcal{B}}_0(\Omega_i)$ $(i = 1, \dots, n)$. So there exist $E_i, F_i \in \mathcal{B}_0(\Omega_i)$ such that $E_i \subseteq A_i \subseteq F_i$ and $\mu_i(F_i \setminus E_i) = 0$ for any $i = 1, \dots, n$. Hence $E := E_1 \times \cdots \times E_n$, $F := F_1 \times \cdots \times F_n \in \mathcal{B}_0(\Omega)$, $E \subseteq A \subseteq F$, and $\mu(F \setminus E) = 0$. Therefore $A \in \hat{\mathcal{B}}_0(\Omega)$ and hence $\Sigma \subseteq \hat{\mathcal{B}}_0(\Omega)$. But from $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n \subseteq \Sigma$ it follows that $\hat{\mathcal{B}}(\Omega) = \Sigma$ and consequently $\hat{\mathcal{B}}_0(\Omega) = \hat{\mathcal{B}}(\Omega)$.

REMARK 1. If $\rho_i \in \Lambda(\mu_i)$ for a complete probability space $(\Omega_i, \Sigma_i, \mu_i)$ then ρ_i is strong with respect to $\mathcal{T}_i = \mathcal{T}_{\rho_i}$ respectively $\mathcal{T}_i = \mathcal{T}_{\rho_i}^*$ $(i = 1, \dots, n)$. If we choose $\rho_i \in \Lambda(\mu_i)$ $(i = 1, \dots, n)$ and $\rho_n^x \in \Lambda(\mu_1 \hat{\otimes} \cdots \hat{\otimes} \mu_n)$ according to the discussion following Theorem 4 in Section 2 then by Theorem 1 for $\mathcal{T}_i = \mathcal{T}_{\rho_i}$ respectively $\mathcal{T}_i = \mathcal{T}_{\rho_i}^*$ $(i = 1, \dots, n)$ it follows that ρ_n^x is strong with respect to $\mathcal{T}_{\rho_1} \times \cdots \times \mathcal{T}_{\rho_n}$ respectively $\mathcal{T}_{\rho_1}^* \times \cdots \times \mathcal{T}_{\rho_n}^*$, that is $\mathcal{T}_{\rho_1} \times \cdots \times \mathcal{T}_{\rho_n} \subseteq \mathcal{T}_{\rho_n^x}$ respectively $\mathcal{T}_{\rho_1}^* \times \cdots \times \mathcal{T}_{\rho_n}^* \subseteq \mathcal{T}_{\rho_n^x}^*$. This raises the question whether $\mathcal{T}_{\rho_n^x} = \mathcal{T}_{\rho_1} \times \cdots \times \mathcal{T}_{\rho_n}$ respectively $\mathcal{T}_{\rho_n^x}^* = \mathcal{T}_{\rho_1}^* \times \cdots \times \mathcal{T}_{\rho_n}^*$? Barring measurable cardinals, Curtis, Hendricksen, and Isbell proved (see [11, p. 53]) that a product of two topological spaces is extremally disconnected if and only if one factor is extremally disconnected and the other one is discrete. This implies the following result whose blanket assumptions are satisfied for example by the Lebesgue measure space on [0, 1].

THEOREM 2. *If we are barring measurable cardinals then for two complete probability spaces* $(\Omega_i, \Sigma_i, \mu_i)$ *and* $\rho_i \in \Lambda(\mu_i)$ *with non-discrete lifting topologies* $\mathcal{T}_{\rho_i}$, $\mathcal{T}_{\rho_i}^*$ $(i = 1, 2)$ *the product topologies* $\mathcal{T}_{\rho_1} \times \mathcal{T}_{\rho_2}$, $\mathcal{T}_{\rho_1}^* \times \mathcal{T}_{\rho_2}^*$ *are not extremally disconnected. If* $(\Omega_1 \times \Omega_2, \Sigma, \mu)$ *is a complete probability space such that* $\Sigma_1 \hat{\otimes} \Sigma_2 \subseteq \Sigma$, $\mu \mid \Sigma_1 \hat{\otimes} \Sigma_2 = \mu_1 \hat{\otimes} \mu_2$, $\mathcal{T}_{\rho_1} \times \mathcal{T}_{\rho_2}$, $\mathcal{T}_{\rho_1}^* \times \mathcal{T}_{\rho_2}^* \subseteq \Sigma$ *then in particular*

$$\mathcal{T}_{\rho_1} \times \mathcal{T}_{\rho_2} \neq \mathcal{T}_\pi, \mathcal{T}_\pi^* \quad \text{and} \quad \mathcal{T}_{\rho_1}^* \times \mathcal{T}_{\rho_2}^* \neq \mathcal{T}_\pi, \mathcal{T}_\pi^*$$

for any $\pi \in \Lambda(\mu)$, *and if* π *is strong with respect to* $\mathcal{T}_{\rho_1} \times \mathcal{T}_{\rho_2}$ *respectively* $\mathcal{T}_{\rho_1}^* \times \mathcal{T}_{\rho_2}^*$ *(for example if* $\pi = \rho_2^x$*) then it holds true that*

$$\mathcal{T}_{\rho_1} \times \mathcal{T}_{\rho_2} \subsetneqq \mathcal{T}_\pi \qquad \mathcal{T}_{\rho_1}^* \times \mathcal{T}_{\rho_2}^* \subsetneqq \mathcal{T}_\pi^*.$$

REMARK 2. By [8] there exists a compact Radon measure probability space $F := (X, \mathcal{T}, \Sigma, \mu)$ such that $\mathcal{T} \times \mathcal{T} \not\subseteq \Sigma \hat{\otimes} \Sigma$ and this situation occurs for complete topological probability spaces such that $(X, \mathcal{T})$ is compact extremally disconnected,

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μ is a diffuse Radon measure, and $\overline{G} \in \mathscr{T}$, $\mu(\overline{G}) = \mu(G)$ for all $G \in \mathscr{T}$. Such spaces are given by the hyperstonian space derived from a diffuse probability space, for example the hyperstonian space of the Lebesgue measure space on [0, 1] will do (see [8]). It is well-known that such spaces have a unique strong lifting σ. Since $\overline{G}$ is clopen for $G \in \mathscr{T}$ we have $\chi_{\overline{G}} \in C(X)$, $\sigma(\chi_{\overline{G}}) = \chi_{\overline{G}}$, that is $\sigma(\overline{G}) = \overline{G}$. But then $\mu(\overline{G}) = \mu(G)$ implies $\sigma(G) = \sigma(\overline{G}) = \overline{G}$. Moreover for any $A \in \Sigma$ we find a $G \in \mathscr{T}$ such that $\mu(A \Delta G) = 0$ by [7, p. 27] or [20, p.533 note 12] since F is hyperstonian, and this implies $\sigma(A) = \sigma(G) = \overline{G}$, hence we get $\mathscr{T}_\sigma \subseteq \mathscr{T} \subseteq \mathscr{T}_\sigma^*$. Since $(X, \mathscr{T})$ is compact $(X, \mathscr{T}_\sigma)$ must be too, and any $\mathscr{T}$-compact subset of X is $\mathscr{T}_\sigma$-compact; therefore μ is a Radon measure for $\mathscr{T}_\sigma$ too. If we assume $\mathscr{T}_\sigma$ Hausdorff (this is true for example for the hyperstonian space of the Lebesgue space on [0, 1]) then $\mathscr{T}$ and $\mathscr{T}_\sigma$ are completely regular and $C(X, \mathscr{T}_\sigma) = C(X, \mathscr{T}) = C(X, \mathscr{T}_\sigma^*)$ since $\mathscr{T}_\sigma \subseteq \mathscr{T} \subseteq \mathscr{T}_\sigma^*$. This implies $\mathscr{T} = \mathscr{T}_\sigma$. So from $\mathscr{T} \times \mathscr{T} \nsubseteq \Sigma \hat{\otimes} \Sigma$ we infer $\mathscr{T}_\sigma \times \mathscr{T}_\sigma$, $\mathscr{T}_\sigma^* \times \mathscr{T}_\sigma^* \nsubseteq \Sigma \hat{\otimes} \Sigma$.

REMARK 3. Nevertheless it is a standard procedure (see for example [5]) to construct for compact Radon measure spaces $(\Omega_i, \mathscr{T}_i, \Sigma_i, \mu_i)$, $i = 1, \dots, n$ a complete probability space $(\Omega_1 \times \cdots \times \Omega_n, \Sigma, \mu)$ satisfying the assumption of Lemma 1 and Theorems 1 and 2. For $h \in C(\Omega_1 \times \cdots \times \Omega_n)$ define $R(h) := \int \cdots \int h(\omega_1, \dots, \omega_n)\, d\mu_1(\omega_1) \cdots d\mu_n(\omega_n)$. Then R defines a positive Radon measure $\lambda_R = \mu$ on the complete σ-algebra $\Sigma := m(\lambda_R)$ of all λ_R-measurable subsets of $\Omega_1 \times \cdots \times \Omega_n$.

For another procedure to construct for more general spaces $(\Omega_i, \mathscr{T}_i, \Sigma_i, \mu_i)$, $i = 1, \dots, n$ a complete probability space $(\Omega_1 \times \cdots \Omega_n, \Sigma, \mu)$ as above see [27, Theorem 1].

REMARK 4. In [28] is proved the existence of a so-called *consistent lifting* ρ for a complete probability space (Ω, Σ, μ), that is of a $\rho \in \Lambda(\mu)$ such that for all $n \in \mathbb{N}$ there exists a $\rho^n \in \Lambda(\mu^n)$, $\mu^n := \mu \hat{\otimes} \cdots \hat{\otimes} \mu$ (n factors) with

$$\rho^n(A_1 \times \cdots \times A_n) = \rho(A_1) \times \cdots \times \rho(A_n) \quad \text{for all} \quad A_i \in \Sigma \ (i = 1, \dots, n).$$

Let $F := (X, \mathscr{T}, \Sigma, \mu)$ be defined as in Remark 2 and choose a consistent lifting for this space. If we assume $\rho = \sigma$, σ the only strong lifting for F, then Theorem 1 implies $\mathscr{T}^n \subseteq \Sigma^n$, a contradiction for $n = 2$ according to [8]. Therefore the unique strong lifting for F is not consistent and any consistent lifting for F is not strong. In [29] Talagrand proves the existence of non-consistent liftings under the continuum hypothesis. The above construction does not need this hypothesis. In particular F is a space with the SLP but not with the USLP.

REMARK 5. We consider in more detail hyperstonian spaces $(\Omega_1, \mathscr{T}_1, \Sigma_1, \mu_1)$ for $\Omega_1 \neq \emptyset$ which are derived from a diffuse probability space. Throughout σ denotes the unique strong lifting of such a space and we assume the lifting topology $\mathcal{T}_\sigma$ Hausdorff (see Remark 2). Such spaces provide additional simplifications. Write $\Omega = \Omega_1 \times \Omega_1$, $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_1$ and let $(\Omega, \mathcal{T}, \Lambda_R, \lambda_R)$ denote the Radon product of $(\Omega_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ taken with itself according to Remark 3. Clearly $\mathcal{T} \subseteq \Lambda_R$.

For a measure space (Ω, Σ, μ) such that $\Sigma_1 \hat{\otimes} \Sigma_1 \subseteq \Sigma$ and $\mu \mid \Sigma_1 \hat{\otimes} \Sigma_1 = \mu_1 \hat{\otimes} \mu_1$ (which holds true for $\Sigma = \Lambda_R$, $\mu = \lambda_R$) we have not in general $\mathcal{T} \subseteq \Sigma$ and hence we define $\pi \in \Lambda(\mu)$ to be $\mathcal{T} \cap \Sigma$-strong if $G \subseteq \pi(G)$ for all $G \in \mathcal{T} \cap \Sigma$. Such liftings are uniquely determined by σ on the generator of $\Sigma_1 \otimes \Sigma_1$.

(a) *If $\pi \in \Lambda(\mu)$ is $\mathcal{T} \cap \Sigma$-strong then $\pi(A \times B) = \sigma(A) \times \sigma(B)$ for $A, B \in \Sigma_1$, moreover $\mathcal{T} \subseteq \Sigma$ and π is in fact strong; this is in particular true for any strong lifting of λ_R.*

Indeed we have $\sigma(A) \times \sigma(B) \in \mathcal{T}$ for $A, B \in \Sigma_1$ since $\mathcal{T}_1 = \mathcal{T}_\sigma$ by Remark 2. This implies $\sigma(A) \times \sigma(B) \subseteq \pi(\sigma(A) \times \sigma(B))$. Since

$$\mu((\sigma(A) \times \sigma(B))\Delta(A \times B)) = \mu_1 \hat{\otimes} \mu_1((\sigma(A) \times \sigma(B))\Delta(A \times B)) = 0$$

we have $\sigma(A) \times \sigma(B) \subseteq \pi(A \times B)$. From this inclusion follows the converse inclusion by considering the complement of $A \times B$. Now Theorem 1 implies $\mathcal{T} \subseteq \Sigma$ since σ is strong. In particular for $\Sigma = \Sigma_1 \hat{\otimes} \Sigma_1$ we have $\mathcal{T} \not\subseteq \Sigma$ by [8] (see Remark 2) and (a) implies

(b) *$\mu_1 \hat{\otimes} \mu_1$ does not have $\mathcal{T} \cap (\Sigma_1 \hat{\otimes} \Sigma_1)$-strong lifting.*

If we choose by Theorem 4 of Section 2 liftings $\rho \in \Lambda(\mu_1)$ and $\pi \in \Lambda(\mu_1 \hat{\otimes} \mu_1)$ such that $\pi(A \times B) = \sigma(A) \times \rho(B)$ for $A, B \in \Sigma_1$ then we have

(c) *π is not $\mathcal{T} \cap (\Sigma_1 \hat{\otimes} \Sigma_1)$-strong, $\rho \neq \sigma$ and ρ is neither strong nor consistent.*

Indeed if π would be $\mathcal{T} \cap (\Sigma_1 \hat{\otimes} \Sigma_1)$-strong then (a) would imply $\pi(A \times B) = \sigma(A) \times \sigma(B)$ for $A, B \in \Sigma_1$, in particular $\Omega_1 \times \rho(B) = \pi(\Omega_1 \times B) = \Omega_1 \times \sigma(B)$ therefore $\rho(B) = \sigma(B)$ for $B \in \Sigma_1$, hence $\mathcal{T} \subseteq \Sigma_1 \hat{\otimes} \Sigma_1$ by Theorem 1, a contradiction according to (b).

(d) *Assuming the continuum hypothesis and weight$(\Omega_1, \mathcal{T}_1) \leq \mathbf{c}$, $\mathbf{c}$ the cardinality of the continuum (for example take the hyperstonian space of the Lebesgue measure space on* [0, 1]*) then $(\Omega, \mathcal{T}, \Lambda_R, \lambda_R)$ has the* ASLP *by* [9] *but $(\Omega, \mathcal{T}, \Sigma_1 \hat{\otimes} \Sigma_1, \mu_1 \hat{\otimes} \mu_1)$ does not.*

This means that $(\Omega, \mathcal{T}, \Lambda_R, \lambda_R)$ has a unique strong lifting $\pi \in \Lambda(\lambda_R)$ satisfying $\pi(A \times B) = \sigma(A) \times \sigma(B)$ by (a) while for $(\Omega, \mathcal{T}, \Sigma_1 \hat{\otimes} \Sigma_1, \mu_1 \hat{\otimes} \mu_1)$ does not exist such a lifting by (b).

A natural question to ask is: Given complete probability spaces (Ω, Σ, μ),(Θ, T, ν) and liftings $\rho \in \Lambda(\mu), \sigma \in \Lambda(\nu)$ does there always exist a 'product lifting' $\pi = \rho \otimes \sigma$? It follows from (c) above that the answer is to the negative and this shows that Theorem 4 of Section 2 is the best possible result in the direction of product liftings.

The first author is indebted to Dr Grekas and Dr Gryllakis for finding a gap in an earlier version of Remark 5.

THEOREM 3. *Let there be given topological probability spaces* $(\Omega_i, \mathscr{T}_i, \Sigma_i, \mu_i)$ $(i = 1, \dots, n)$ *such that one of these spaces has the* ASLP *and all the other ones have the* USLP. *Then the completed product* $(\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathscr{T}_i, (\otimes_{i=1}^n \Sigma_i)^\wedge, (\otimes_{i=1}^n \mu_i)^\wedge)$ *is a topological probability space which has the* ASLP.

PROOF. By induction on n we may assume $n = 2$. Let us choose an almost strong lifting ρ_1 for μ_1 and then according to Theorem 4 of Section 2 a lifting ρ_2 for μ_2 and a lifting π for $\mu_1 \hat{\otimes} \mu_2$ such that $\pi(A_1 \times A_2) = \rho_1(A_1) \times \rho_2(A_2)$ for $A_i \in \Sigma_i$ $(i = 1, 2)$. Since $(\Omega_2, \mathscr{T}_2, \Sigma_2, \mu_2)$ has the USLP ρ_2 must be almost strong and there exist $N_i \in \Sigma_i$ such that $\mu_i(N_i) = 0$, $G_i \subseteq \rho_i(G_i) \cup N_i$ for $G_i \in \mathscr{T}_i$ $(i = 1, 2)$. Putting $N := (N_1 \times \Omega_2) \cup (\Omega_1 \times N_2)$ we have $(\mu_1 \hat{\otimes} \mu_2)(N) = 0$ and

$$G_1 \times G_2 \subseteq (\rho_1(G_1) \times \rho_2(G_2)) \cup N = \pi(G_1 \times G_2) \cup N.$$

Since the set $\mathscr{G}$ of all $G_1 \times G_2$, $G_i \in \mathscr{T}_i$ $(i = 1, 2)$ is a generator for the product topology $\mathscr{T}_1 \times \mathscr{T}_2$ it follows from [25, Lemma 4.1] that $\mathscr{T}_1 \times \mathscr{T}_2 \subseteq \Sigma_1 \hat{\otimes} \Sigma_2$ and π is almost strong.

The following is a generalization of Theorem 3 for countable products.

THEOREM 4. *Let* $((\Omega_i, \mathscr{T}_i, \Sigma_i, \mu_i))_{i \in \mathbb{N}}$ *be a sequence of complete probability spaces with completed product* $(\Omega, \mathscr{T}, \Sigma, \mu)$. *Suppose that one of the spaces* $(\Omega_i, \mathscr{T}_i, \Sigma_i, \mu_i)$ *for* $i \in \mathbb{N}$ *has the* ASLP *and the other ones have the* USLP. *Then* $\mathscr{T} \subseteq \Sigma$ *and* $(\Omega, \mathscr{T}, \Sigma, \mu)$ *has the* ASLP. *Moreover, if* $\Sigma_i = \hat{\mathscr{B}}(\Omega_i)$, μ_i *is completion regular for any* $i \in \mathbb{N}$, *and* $\mathscr{B}_0(\Omega) = \bigotimes_{i=1}^\infty \mathscr{B}_0(\Omega_i)$ *then* μ *is completion regular.*

PROOF. We may assume that $(\Omega_1, \mathscr{T}_1, \Sigma_1, \mu_1)$ has the ASLP and all the other spaces have the USLP. Then choose an almost strong lifting ρ_1 for $(\Omega_1, \mathscr{T}_1, \Sigma_1, \mu_1)$ and by Theorem 4 of Section 2, for any $n \in \mathbb{N}$ and $i \in \{2, \dots, n\}$ a lifting $\rho_i \in \Lambda(\mu_i)$ which is almost strong with respect to $\mathscr{T}_i$ by assumption, and a lifting $\rho_n^x \in \Lambda(\mu_1 \hat{\otimes} \cdots \hat{\otimes} \mu_n)$ such that for each $A_i \in \Sigma_i$ $(i = 1, \dots, n)$

$$\rho_n^x(A_1 \times \cdots \times A_n) = \rho_1(A_1) \times \cdots \times \rho_n(A_n).$$

Then there exists a μ_i-null set $N_i \in \Sigma_i$ such that for each $G_i \in \mathscr{T}_i$, $G_i \subseteq \rho_i(G_i) \cup N_i$ $(i = 1, \dots, n)$. Put

$$N_{[n]} := (N_1 \times \Omega_2 \times \cdots \times \Omega_n) \cup \cdots \cup (\Omega_1 \times \cdots \times \Omega_{n-1} \times N_n).$$

In view of Theorem 3 it holds true that $(\mu_1 \hat{\otimes} \cdots \hat{\otimes} \mu_n)(N_{[n]}) = 0$ and for any $G \in \mathscr{T}_1 \times \cdots \times \mathscr{T}_n$

$$G \subseteq \rho_n^x(G) \cup N_{[n]}. \tag{1}$$

Now let $N_n^* := p_{[n]}^{-1}(N_{[n]})$ and $N := \bigcup_{n\in\mathbb{N}} N_n^*$. Then it follows $\mu(N) = 0$. By Theorem 5 of Section 2 there exists a lifting $\rho_\infty \in \Lambda(\mu)$ such that

$$(2) \qquad \rho_\infty(A \times \prod_{i=n+1}^{\infty} \Omega_i) = \rho_n^x(A) \times \prod_{i=n+1}^{\infty} \Omega_i \quad \text{for} \quad A \in \Sigma_1 \hat{\otimes} \cdots \hat{\otimes} \Sigma_n.$$

For $i, n \in \mathbb{N}$ let $\Omega_i^0 := \Omega_i \setminus N_i$, $\mathcal{T}_i^0 := \mathcal{T}_i \cap \Omega_i^0$, $\Sigma_i^0 := \Sigma_i \cap \Omega_i^0$, $\mu_i^0 := \mu_i \mid \Sigma_i^0$, and let ρ_i^0 be the lifting on Σ_i^0 defined by

$$\rho_i^0(A_i \cap \Omega_i^0) := \rho_i(A_i) \cap \Omega_i^0 \quad \text{for any} \quad A_i \in \Sigma_i.$$

Denote by $(\Omega^0, \mathcal{T}^0, \Sigma^0, \mu^0)$ the completed product of the family $((\Omega_i^0, \mathcal{T}_i^0, \Sigma_i^0, \mu_i^0))_{i\in\mathbb{N}}$, and let ρ_∞^0 be the lifting defined by

$$\rho_\infty^0(A \cap \Omega^0) := \rho_\infty(A) \cap \Omega^0 \quad \text{for any} \quad A \in \Sigma.$$

It is easily seen that relation (2) implies

$$\rho_\infty(p_i^{-1}(A_i)) = p_i^{-1}(\rho_i(A_i)) \quad \text{for all} \quad A_i \in \Sigma_i \quad \text{and} \quad i \in \mathbb{N},$$

and hence it holds true that

$$\rho_\infty^0(p_i^{-1}(A_i)) = p_i^{-1}(\rho_i^0(A_i)) \quad \text{for all} \quad A_i \in \Sigma_i^0 \quad \text{and} \quad i \in \mathbb{N}.$$

The last relation and the fact that each ρ_i^0 is strong imply the $\mathcal{T}_{\rho_\infty^0}^* - \mathcal{T}_{\rho_i^0}^*$-continuity of p_i for any $i \in \mathbb{N}$. Thus

$$\mathcal{T}^0 \subseteq \mathcal{T}_{\rho_\infty^0}^* \subseteq \Sigma^0 \subseteq \Sigma.$$

From the above relations it follows that $\mathcal{T} \subseteq \Sigma$. Indeed for $G \in \mathcal{T}$ it holds true that $G = (G \cap \Omega^0) \cup (G \cap (\Omega \setminus \Omega^0))$, $G \cap \Omega^0 \in \mathcal{T}^0 \subseteq \Sigma$ because of $G \cap (\Omega \setminus \Omega^0) \subseteq \Omega \setminus \Omega^0$ and $\mu(\Omega \setminus \Omega^0) = 0$. Hence $G \in \Sigma$.

To show that ρ_∞ is almost strong let $G_n^* := p_{[n]}^{-1}(G_1 \times \cdots \times G_n)$ for $G_i \in \mathcal{T}_i$ $(i = 1, \ldots, n)$. It follows from (1) and (2) that

$$\begin{aligned} \rho_\infty(G_n^*) \cup N \supseteq \rho_\infty(G_n^*) \cup N_n^* &= p_{[n]}^{-1}(\rho_n^x(G_1 \times \cdots \times G_n) \cup N_{[n]}) \\ &\supseteq p_{[n]}^{-1}(G_1 \times \cdots \times G_n) = G_n^*. \end{aligned}$$

Hence for $G \in \mathcal{T}$ we get $G \subseteq \rho_\infty(G) \cup N$. Using now the same arguments with those in the proof of Theorem 1 we conclude the completion regularity of μ.

COROLLARY. *Let $((\Omega_i, \mathcal{T}_i, \hat{\mathcal{B}}_0(\Omega_i), \mu_i))_{i\in\mathbb{N}}$ be a sequence of Baire probability spaces with completed product $(\Omega, \mathcal{T}, \Sigma, \mu)$. Suppose that one of the spaces $(\Omega_i, \mathcal{T}_i, \hat{\mathcal{B}}_0(\Omega_i), \mu_i)$ for $i \in \mathbb{N}$ has the Baire-*ASLP *and the other ones have the Baire-*USLP*, and $\mathcal{B}_0(\Omega) = \bigotimes_{i=1}^{\infty} \mathcal{B}_0(\Omega_i)$. Then $(\Omega, \mathcal{T}, \Sigma, \mu)$ has the Baire-*ASLP.

PROOF. From [3, Proposition 3] each μ_i is completion regular. We may suppose that $(\Omega_1, \mathscr{T}_1, \hat{\mathscr{B}}_0(\Omega_1), \mu_1)$ has the Baire-ASLP and all the other ones have the Baire-USLP. So $(\Omega_1, \mathscr{T}_1, \hat{\mathscr{B}}_0(\Omega_1), \mu_1)$ has the ASLP and all the other spaces have the USLP. Applying now Theorem 4 we conclude that μ has the ASLP and is completion regular. Hence μ has the Baire-ASLP.

Property $\mathscr{B}_0(\Omega) = \bigotimes_{i=1}^{\infty} \mathscr{B}_0(\Omega_i)$ holds true for compact or Polish spaces Ω_i, $i \in \mathbb{N}$ (see for example [17, VI, Theorem 5.5 and Theorem 5.6]). For more general spaces with the above property, see [21].

The product of two completion regular even Radon measures is not in general completion regular (see [8, p. 288]). For positive results in special cases compare also [13]. In [24] we study the same problem for uncountable products and general projective limits.

Finally we remark that since the space F of [8] described in Remark 2 is completion regular (see [6, p. 85]) and has the ASLP it has the Baire-ASLP too. Assuming that the probability space $(X \times X, \mathscr{T} \times \mathscr{T}, \Sigma\hat{\otimes}\Sigma, \mu\hat{\otimes}\mu)$ has the Baire-ASLP then it would be completion regular according to [3, Proposition 3], a contradiction to [8]. So $(X \times X, \mathscr{T} \times \mathscr{T}, \Sigma\hat{\otimes}\Sigma, \mu\hat{\otimes}\mu)$ does not have the Baire-ASLP and hence the above property is not in general invariant under the formation of products of Baire probability spaces.

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