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THE STRICT TOPOLOGY ON THE DISCRETE LEBESGUE SPACES

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Abstract

Let Σ be a set and σ be a positive function on Σ. We introduce and study a locally convex topology $\beta^1(\Sigma, \sigma)$ on the space $\ell^1(\Sigma, \sigma)$ such that the strong dual of $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ can be identified with the Banach space $(c_0(\Sigma, 1/\sigma), \|\cdot\|_{\infty,\sigma})$. We also show that, except for the case where Σ is finite, there are infinitely many such locally convex topologies on $\ell^1(\Sigma, \sigma)$. Finally, we investigate some other properties of the locally convex space $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$, and as an application, we answer partially a question raised by A. I. Singh ['$L_0^\infty(G)^*$ as the second dual of the group algebra $L^1(G)$ with a locally convex topology', *Michigan Math. J.* **46** (1999), 143–150].

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1. Introduction

Throughout this paper, let Σ be an arbitrary set and σ be a positive function on Σ. We denote by $\ell^1(\Sigma, \sigma)$ the space of all complex-valued functions φ on Σ such that $\sigma\varphi \in \ell^1(\Sigma)$, the usual Lebesgue space of the discrete space Σ. Then $\ell^1(\Sigma, \sigma)$ with the norm $\|\cdot\|_{1,\sigma}$ defined by

$$\|\varphi\|_{1,\sigma} := \|\sigma\varphi\|_1$$

is a Banach space. For each $x \in \Sigma$, we denote by δ_x the function defined on Σ by $\delta_x(t) = 1$ for $t = x$ and $\delta_x(t) = 0$ otherwise. Also, let $\ell^\infty(\Sigma, 1/\sigma)$ denote the space of all complex-valued functions f on Σ with $f/\sigma \in \ell^\infty(\Sigma)$, the space of all bounded functions on Σ. Then $\ell^\infty(\Sigma, 1/\sigma)$ with the norm $\|\cdot\|_{\infty,\sigma}$ defined by

$$\|f\|_{\infty,\sigma} = \|f/\sigma\|_\infty$$

is a Banach space. Moreover, $\ell^\infty(\Sigma, 1/\sigma)$ is the dual of $\ell^1(\Sigma, \sigma)$ by the pairing

$$\langle\theta(f), \varphi\rangle := \sum_{x\in\Sigma} f(x)\varphi(x) \quad (f \in \ell^\infty(\Sigma, 1/\sigma), \varphi \in \ell^1(\Sigma, \sigma)).$$

Denote by $c_0(\Sigma, 1/\sigma)$ the subspace of $\ell^\infty(\Sigma, 1/\sigma)$ consisting of all functions f on Σ with $f/\sigma \in c_0(\Sigma)$, the space of all functions on Σ vanishing at infinity, and note that $\ell^1(\Sigma, \sigma)$ is the dual of $c_0(\Sigma, 1/\sigma)$ under the above duality.

The study of the strict topology on $C(X)$, the space of continuous functions on the topological space X, began with Buck's work in [1]. There is an extensive literature on this subject; see, for example, [10, 11]. Also, for such a study in another context, see [4, 14, 16], for example. For a generalization of the strict topology and/or strict topology in a more general setting, see, for example, [2, 7].

In this paper, we introduce and study a locally convex topology $\beta^1(\Sigma, \sigma)$ on $\ell^1(\Sigma, \sigma)$ such that $c_0(\Sigma, 1/\sigma)$ can be identified with the strong dual of $\ell^1(\Sigma, \sigma)$. We then show that, except for the trivial case where Σ is finite, there are infinitely many such locally convex topologies τ on $\ell^1(\Sigma, \sigma)$, and hence $c_0(\Sigma, 1/\sigma)$ can be considered as the strong dual of $\ell^1(\Sigma, \sigma)$. We study, among other things, some locally convex space properties of the space $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$. Finally, we give a partial answer to a question raised by Singh in [12].

2. A locally convex topology on $\ell^1(\Sigma, \sigma)$

Let Σ be a set and $\sigma : \Sigma \longrightarrow (0, \infty)$. The set of increasing sequences of finite subsets of Σ is denoted by $\mathcal{F}$ and the set of increasing sequences (r_n) of real numbers in $(0, \infty)$ with $r_n \to \infty$ by $\mathcal{R}$. For any $(F_n) \in \mathcal{F}$ and $(r_n) \in \mathcal{R}$, set

$$U((F_n), (r_n)) := \Big\{\varphi \in \ell^1(\Sigma, \sigma) : \sum_{x \in F_n} |\varphi(x)|\sigma(x) \le r_n \text{ for all } n \ge 1\Big\},$$

and note that $U((F_n), (r_n))$ is a convex balanced absorbing set in the space $\ell^1(\Sigma, \sigma)$. It is easy to see that the family $\mathcal{U}$ of all sets $U((F_n), (r_n))$, for $(F_n) \in \mathcal{F}$ and $(r_n) \in \mathcal{R}$, is a base of neighbourhoods of zero for a locally convex topology on $\ell^1(\Sigma, \sigma)$; see, for example, [13, Theorem 1.18]. We denote this topology by $\beta^1(\Sigma, \sigma)$ and call it the *strict topology* on $\ell^1(\Sigma, \sigma)$. Note that the strict topology can be generated by the family $\{\mathcal{P}_U : U \in \mathcal{U}\}$ of seminorms on $\ell^1(\Sigma, \sigma)$, where

$$\mathcal{P}_U(\varphi) := \sup\Big\{r_n^{-1} \sum_{x \in F_n} |\varphi(x)|\sigma(x) : n \ge 1\Big\}$$

for all $\varphi \in \ell^1(\Sigma, \sigma)$ and $U := U((F_n), (r_n)) \in \mathcal{U}$. We denote the norm topology on $\ell^1(\Sigma, \sigma)$ by $n(\Sigma, \sigma)$; note that $\beta^1(\Sigma, \sigma) \le n(\Sigma, \sigma)$.

PROPOSITION 2.1. *Let Σ be an infinite set and σ be a positive function on Σ. Then a subset of $\ell^1(\Sigma, \sigma)$ is $n(\Sigma, \sigma)$-bounded if and only if it is $\beta^1(\Sigma, \sigma)$-bounded.*

PROOF. Let B be a $\beta^1(\Sigma, \sigma)$-bounded set in $\ell^1(\Sigma, \sigma)$, and suppose that B is not $n(\Sigma, \sigma)$-bounded. Then there is a sequence $(\varphi_n) \subseteq B$ such that $\|\varphi_n\|_{1,\sigma} > n$ for all $n \ge 1$. For each $n \ge 1$, choose a finite set F_n in Σ such that

$$\sum_{x \in F_n} |\varphi_n(x)|\sigma(x) \ge n$$

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and note that $(F_n) \in \mathcal{F}$. Let (r_n) be a sequence in $\mathcal{R}$ with $r_n^2 \geq n$. Since B is $\beta^1(\Sigma, \sigma)$-bounded, there is a constant $s > 0$ such that

$$B \subseteq sU((F_n), (r_n))$$

for all $n \geq 1$. We therefore have

$$n \leq \sum_{x \in F_n} |\varphi_n(x)|\sigma(x) < r_n s$$

which is a contradiction. The converse is clear. □

We denote by $\tau_b(\Sigma, \sigma)$ the strong topology on $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^*$; that is, the topology of uniform convergence on bounded subsets of $\ell^1(\Sigma, \sigma)$ with respect to the weak topology $\sigma(\ell^1(\Sigma, \sigma), (\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^*)$. We also denote the topology given on $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^*$ by the norm

$$\|f\| = \sup\{|f(\varphi)| : \varphi \in \ell^1(\Sigma, \sigma), \|\varphi\|_{1,\sigma} = 1\},$$

by $\tau_n(\Sigma, \sigma)$. An immediate consequence of Proposition 2.1 is that on $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^*$ the strong topology $\tau_b(\Sigma, \sigma)$ coincides with the topology $\tau_n(\Sigma, \sigma)$.

PROPOSITION 2.2. *Let Σ be a set and σ be a positive function on Σ. On $\ell^1(\Sigma, \sigma)$ the norm topology $n(\Sigma, \sigma)$ coincides with the strict topology $\beta^1(\Sigma, \sigma)$ if and only if Σ is finite.*

PROOF. Consider the set

$$U := \{\varphi \in \ell^1(\Sigma, \sigma) : \|\varphi\|_{1,\sigma} < 1\}$$

and note that U is $n(\Sigma, \sigma)$-open, and thus $\beta^1(\Sigma, \sigma)$-open. It follows that there is a sequence $((F_n), (r_n))$ in $\mathcal{F} \times \mathcal{R}$ such that $U((F_n), (r_n)) \in U$. Suppose that Σ is not finite, so we can choose n_0 such that $r_{n_0} > 1$ and $x_{n_0} \in \Sigma \setminus F_{n_0}$. Let

$$\varphi := \sigma(x_{n_0})^{-1}\delta_{x_{n_0}}.$$

We then have $\varphi \in U((F_n), (r_n))$, but $\varphi \notin U$. □

3. Dual of $\ell^1(\Sigma, \sigma)$ with the strict topology

We commence this section with the following key result.

THEOREM 3.1. *Let Σ be a set and σ be a positive function on Σ. Let τ be a locally convex topology on $\ell^1(\Sigma, \sigma)$ with $\sigma_0(\Sigma, \sigma) \leq \tau \leq \beta^1(\Sigma, \sigma)$. Then the dual of $(\ell^1(\Sigma, \sigma), \tau)$ endowed with the strong topology can be identified with $c_0(\Sigma, 1/\sigma)$ endowed with $\|\cdot\|_{\infty,\sigma}$-topology.*

PROOF. It is sufficient to prove the theorem for the case $\tau = \beta^1(\Sigma, \sigma)$. To this end we first show that

$$\theta(c_0(\Sigma, 1/\sigma)) \subseteq (\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^*.$$

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Let f be in $c_0(\Sigma, 1/\sigma)$ and $\varepsilon > 0$ be given. Choose an element $((F_n), (r_n))$ of $\mathcal{F} \times \mathcal{R}$ with $r_n \longrightarrow \infty$ and $r_1 \geq 2$ such that

$$|f(x)| \leq \varepsilon r_n^{-2} \sigma(x) \quad (n \geq 1)$$

for $x \in \Sigma \setminus F_n$. We show that

$$|\langle \theta(f), \varphi \rangle| \leq \varepsilon \quad \text{for all } \varphi \in U((F_n), (r_n))$$

from which it follows that $\theta(f) \in (\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^*$.

To this end, let $\varphi \in U((F_n), (r_n))$, and set $F_0 = \emptyset$ and $r_0 = 2$. Since $f(x) = 0$ for all $x \in \Sigma \setminus \bigcup_{n=2}^{\infty} F_n$, it follows from

$$\bigcup_{n=2}^{\infty} F_n = \bigcup_{n=0}^{\infty} (F_{n+1} \setminus F_n)$$

that

$$\begin{aligned} |\langle \theta(f), \varphi \rangle| &= \Big| \sum_{x \in \Sigma} g(x) \varphi(x) \Big| \\ &\leq \sum_{x \in \bigcup_{n=2}^{\infty} F_n} |f(x)| |\varphi(x)| \\ &\leq \sum_{n=0}^{\infty} \Big(\sum_{x \in F_{n+1} \setminus F_n} |f(x)| |\varphi(x)| \Big) \\ &\leq \sum_{n=0}^{\infty} \varepsilon r_n^{-2} \Big(\sum_{x \in F_{n+1} \setminus F_n} |\varphi(x)| \sigma(x) \Big). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{n=0}^{m} r_n^{-2} \Big(\sum_{x \in F_{n+1} \setminus F_n} |\varphi(x)| \sigma(x) \Big) &= \sum_{n=0}^{m} (r_n^{-2} - r_{n+1}^{-2}) \Big(\sum_{x \in F_{n+1} \setminus F_1} |\varphi(x)| \sigma(x) \Big) \\ &\quad + r_{m+1}^{-2} \sum_{x \in F_{m+1} \setminus F_1} |\varphi(x)| \sigma(x) \\ &\leq \sum_{n=0}^{m} 2(r_n^{-1} - r_{n+1}^{-1}) r_n^{-1} \Big(\sum_{x \in F_{n+1}} |\varphi(x)| \sigma(x) \Big) \\ &\quad + r_{m+1}^{-2} \sum_{x \in F_{m+1}} |\varphi(x)| \sigma(x) \\ &\leq \sum_{n=0}^{m} 2(r_n^{-1} - r_{n+1}^{-1}) + r_{m+1}^{-1}. \end{aligned}$$

Thus,

$$|\langle \theta(f), \varphi \rangle| \leq \varepsilon (2r_0^{-1} - r_{m+1}^{-1}) < \varepsilon.$$

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This shows that

$$\theta(f) \in (\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^*.$$

Now, let H be a $\beta^1(\Sigma, \sigma)$-continuous functional on $\ell^1(\Sigma, \sigma)$. Then there is an element $((F_n), (r_n))$ in $\mathcal{F} \times \mathcal{R}$ such that

$$|\langle H, \varphi\rangle| < 1 \quad \text{for all } \varphi \in U((F_n), (r_n)).$$

It is clear that H is also norm continuous on $\ell^1(\Sigma, \sigma)$. It follows that $H = \theta(f)$ for some $f \in \ell^\infty(\Sigma, 1/\sigma)$. We show that $f \in c_0(\Sigma, 1/\sigma)$. It suffices to prove that

$$|f(x)| \le \sigma(x) r_n^{-1}$$

for all $n \ge 1$ and all $x \in \Sigma \setminus F_n$.

To this end, suppose on the contrary that there exist $m \ge 1$ and $x_0 \in \Sigma \setminus F_m$ such that

$$|f(x_0)| > \sigma(x_0) r_m^{-1}.$$

Thus, there is a function $g \in \ell^\infty(\Sigma, 1/\sigma)$ such that $gf = |f|\sigma$ and $\|g\|_{\infty,\sigma} \le 1$. Let φ be a function in $\ell^1(\Sigma, \sigma)$ with

$$\sigma^2\varphi = r_m g \delta_{x_0}.$$

Then

$$\begin{aligned}\Big|\sum_{x\in\Sigma} f(x)\varphi(x)\Big| &= \Big|\sum_{x\in\Sigma} \frac{r_m g f \delta_{x_0}}{\sigma^2}\Big| \\ &= r_m \frac{|f(x_0)|}{\sigma(x_0)} \\ &> 1.\end{aligned}$$

That is, $|\langle H, \varphi\rangle| > 1$ which contradicts the fact that $\varphi \in U((F_n), (r_n))$. Therefore,

$$\theta(c_0(\Sigma, 1/\sigma)) = (\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^*.$$

Moreover, $\|f\|_{\infty,\sigma} = \|\theta(f)\|$ for all $f \in c_0(\Sigma, 1/\sigma)$. Now, invoke Proposition 2.1 to conclude that θ is an identification from $c_0(\Sigma, 1/\sigma)$ endowed with the $\|\cdot\|_{\infty,\sigma}$-topology onto $(\ell^1(\Sigma, \sigma)), \beta^1(\Sigma, \sigma))^*$ endowed with the norm topology. □

We denote by $\sigma_0(\Sigma, \sigma)$ the weak topology $\sigma(\ell^1(\Sigma, \sigma), \theta(c_0(\Sigma, 1/\sigma)))$. Let us remark that

$$\sigma_0(\Sigma, \sigma) \le \beta^1(\Sigma, \sigma) \le n(\Sigma, \sigma).$$

PROPOSITION 3.2. *Let Σ be a set and σ be a positive function on Σ. Then the weak topology $\sigma_0(\Sigma, \sigma)$ on $\ell^1(\Sigma, \sigma)$ coincides with the strict topology $\beta^1(\Sigma, \sigma)$ if and only if Σ is finite.*

PROOF. Suppose that Σ is infinite. Let $(F_n) \in \mathcal{F}$ be an increasing sequence with $F_0 = \emptyset$. So, if $r_n = n$, then $U((F_n), (r_n))$ is a $\beta^1(\Sigma, \sigma)$-neighbourhood of zero.

Let E be the subspace of $\ell^1(\Sigma, \sigma)$ consisting of all $\varphi \in \ell^1(\Sigma, \sigma)$ with

$$\sum_{x \in F_n} \varphi(x)\sigma(x) = 0 \quad \text{for all } n \geq 1,$$

and note that $\varphi_n \notin E$, where $\varphi_n = \chi_{F_n \setminus F_{n-1}}$. Then E has infinite codimension in $\ell^1(\Sigma, \sigma)$. It follows that any subspace F of $\ell^1(\Sigma, \sigma)$ contained in $U((F_n), (r_n))$ has infinite codimension; this is because $F \subset E$. Since any $\sigma_0(\Sigma, \sigma)$-neighbourhood of zero contains a subspace of $\ell^1(\Sigma, \sigma)$ with finite codimension, $U((F_n), (r_n))$ is not a $\sigma_0(\Sigma, \sigma)$-neighbourhood of zero, whereas it is a $\beta^1(\Sigma, \sigma)$-neighbourhood. □

COROLLARY 3.3. *Let Σ be an infinite set and σ be a positive function on Σ. Then there exist uncountably many locally convex topologies τ on $\ell^1(\Sigma, \sigma)$ such that $\sigma_0(\Sigma, \sigma) \leq \tau \leq \beta^1(\Sigma, \sigma)$.*

PROOF. Since Σ is infinite, Proposition 3.2 implies that $\sigma_0(\Sigma, \sigma) < \beta^1(\Sigma, \sigma)$. We now only need to recall from [8] that the only case in which the dual pair generates a finite number of polar topologies is when all polar topologies are equal to the weak topology. □

4. Some properties of the strict topology

In this section, we investigate the strict topology on $\ell^1(\Sigma, \sigma)$ as a locally convex topology.

PROPOSITION 4.1. *Let Σ be a set and σ be a positive function on Σ. The locally convex space $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ is complete.*

PROOF. Let (φ_α) be a $\beta^1(\Sigma, \sigma)$-Cauchy net in $\ell^1(\Sigma, \sigma)$. Obviously, we can find a function φ on Σ such that (φ_α) converges to φ in the pointwise topology. Suppose towards a contradiction that φ is not in $\ell^1(\Sigma, \sigma)$. Then we can find a sequence (x_n) in Σ such that

$$\sum_{i=1}^{k_n} |\varphi(x_i)|\sigma(x_i) \geq 2k_n$$

for all $n \geq 1$, where $1 < k_1 < k_2 < \cdots$. Let $F_n := \{x_{k_1}, x_{k_2}, \ldots, x_{k_n}\}$ and $r_n := k_n$. There exists α_0 such that

$$\sum_{x \in F_n} |\varphi_\alpha(x) - \varphi_\beta(x)|\sigma(x) < k_n \quad (\alpha, \beta \geq \alpha_0).$$

Taking the limit over β we get

$$\sum_{x \in F_n} |\varphi_{\alpha_0}(x) - \varphi(x)|\sigma(x) < k_n \quad \text{for all } n \geq 1,$$

and so

$$\sum_{x \in F_n} |\varphi_{\alpha_0}(x)|\sigma(x) \geq k_n,$$

which contradicts the fact that φ_0 is in $\ell^1(\Sigma, \sigma)$. Hence $\varphi \in \ell^1(\Sigma, \sigma)$. Since $\beta^1(\Sigma, \sigma)$ has a base at zero consisting of a pointwise closed set, it follows easily that (φ_α) converges to φ in the strict topology. □

We denote the topology of pointwise convergence on $\ell^1(\Sigma, \sigma)$ by $\pi(\Sigma, \sigma)$.

Proposition 4.2. *Let Σ be a set and σ be a positive function on Σ. On $\ell^1(\Sigma, \sigma)$ the topology $\pi(\Sigma, \sigma)$ coincides with the topology $\beta^1(\Sigma, \sigma)$ if and only if Σ is finite.*

Proof. Suppose that Σ is infinite and let A be an infinite countable subset of Σ, say

$$A = \{x_1, x_2, \ldots\},$$

such that $x_i \neq x_j$ for $i \neq j$. Then $A \setminus F \neq \emptyset$ for all finite subsets F of Σ. Let F be a finite subset of Σ and choose $x_F \in A \setminus F$. For each natural number n, define the function $\varphi_{(F,n)} \in \ell^1(\Sigma, \sigma)$ by

$$\varphi_{(F,n)}(x_F) = n!\sigma^{-1}(x_F)$$

and zero otherwise. Consider the set

$$\Gamma = \{(F, n) : F \subset \Sigma \text{ is finite and } n \geq 1\}$$

directed by $(F, n) \leq (F', n')$ if and only if $F \subset F'$ and $n \leq n'$. Then $(\varphi_\gamma)_{\gamma \in \Gamma}$ converges to zero in the $\pi(\Sigma, \sigma)$-topology.

Define $F_n := \{x_1, x_2, \ldots, x_n\}$ and $r_n := n!$. For any $\gamma := (F, n) \in \Gamma$, the chosen x_F is an element of F_{n_0} for some $n_0 \geq n$ and hence

$$\sup\Big\{\frac{1}{r_n} \sum_{x \in F_n} |\varphi_{(F,n_0)}(x)|\sigma(x) : n \geq 1\Big\} \geq 1.$$

In other words, $\mathcal{P}_U(\varphi_{(F,n_0)}) \geq 1$, where $U := U((F_n), (r_n))$. Therefore $(\varphi_\gamma)_{\gamma \in \Gamma}$ could not converge to zero in the $\beta^1(\Sigma, \sigma)$-topology. □

Proposition 4.3. *Let Σ be a set and σ be a positive function on Σ. The topologies $\pi(\Sigma, \sigma)$ and $\beta^1(\Sigma, \sigma)$ coincide on all norm bounded subsets of $\ell^1(\Sigma, \sigma)$.*

Proof. We only need to prove that if (φ_α) is uniformly bounded and $\varphi_\alpha \longrightarrow 0$ in the $\pi(\Sigma, \sigma)$-topology, then $\varphi_\alpha \longrightarrow 0$ in the $\beta^1(\Sigma, \sigma)$-topology. Assume that $\|\varphi_\alpha\|_{1,\sigma} \leq M$ for all α, and let $((F_n), (r_n)) \in \mathcal{F} \times \mathcal{R}$. Let $\varepsilon > 0$ and $n_0 \geq 1$ be such that $\varepsilon r_{n_0} \geq M$. Then

$$\sup\Big\{\frac{1}{r_n} \sum_{x \in F_n} |\varphi_\alpha(x)|\sigma(x) : n \geq n_0\Big\} \leq \varepsilon.$$

Since $\varphi_\alpha \longrightarrow 0$ in the $\pi(\Sigma, \sigma)$-topology, there exists α_0 such that

$$\sum_{x \in F_n} |\varphi_\alpha(x)|\sigma(x) < \varepsilon r_n$$

for all $n < n_0$ and $\alpha \geq \alpha_0$. So, $\mathcal{P}_U(\varphi_\alpha) \leq \varepsilon$ for all $\alpha \geq \alpha_0$ and $U \in \mathcal{U}$. The result now follows. □

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PROPOSITION 4.4. *Let Σ be a set and σ be a positive function on Σ. A sequence (φ_n) in $\ell^1(\Sigma, \sigma)$ is $\beta^1(\Sigma, \sigma)$-convergent if and only if it is $\pi(\Sigma, \sigma)$-convergent and norm bounded.*

PROOF. The 'if' part follows from Proposition 4.3. To prove the converse, suppose that (φ_n) is a sequence in $\ell^1(\Sigma, \sigma)$ which is not norm bounded. We show that (φ_n) does not also converge in the strict topology. We can assume that $\|\varphi_n\|_{1,\sigma} > 2^n$ for all $n \geq 1$. Select $K_n := \{x_1, x_2, \dots, x_n\}$ such that

$$\sum_{x \in F_n} |\varphi_n(x)|\sigma(x) \geq 2^n.$$

Setting $F_n := \bigcup_{i=1}^n K_i$ and $r_n := n$,

$$\begin{aligned} \mathcal{P}_U(\varphi_n) &:= \sup\left\{\frac{1}{r_n} \sum_{x \in F_n} |\varphi_n(x)|\sigma(x) : n \geq 1\right\} \\ &\geq \sup\left\{\frac{2^n}{n} : n \geq 1\right\}, \end{aligned}$$

where $U := U((F_n), (r_n))$. So, (φ_n) does not converge in the strict topology. □

Let us recall some definitions from the theory of locally convex spaces. A locally convex space (E, τ) is called a barrelled space if each barrel set (that is, a closed convex balanced absorbing set) in E is a neighbourhood of zero; it is called a bornological space when every convex balanced subset that absorbs bounded subsets in E is a neighbourhood of zero.

PROPOSITION 4.5. *Let Σ be a set and σ be a positive function on Σ. Let τ be a locally convex topology such that $\sigma_0(\Sigma, \sigma) \leq \tau \leq \beta^1(\Sigma, \sigma)$. Then the following statements are equivalent.*

(a) $(\ell^1(\Sigma, \sigma), \tau)$ *is bornological.*
(b) $(\ell^1(\Sigma, \sigma), \tau)$ *is barrelled.*
(c) $(\ell^1(\Sigma, \sigma), \tau)$ *is quasi-barrelled.*
(d) $(\ell^1(\Sigma, \sigma), \tau)$ *is reflexive.*
(e) $(\ell^1(\Sigma, \sigma), \tau)$ *is metrizable.*
(f) Σ *is finite.*

PROOF. We only need to show that (f) holds if (a) or (b) holds. This follows from the fact that any metrizable space is a bornological space, and any reflexive space is a quasi-barrelled and therefore barrelled space.

First, suppose that (a) holds and let I be the identity map from $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ into $(\ell^1(\Sigma, \sigma), n(\Sigma, \sigma))$. Then I is a bounded map by Proposition 2.1. Since by assumption $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ is a bornological space, I is continuous. Therefore $n(\Sigma, \sigma) = \beta^1(\Sigma, \sigma)$. This, together with Proposition 2.2, implies (f).

Next, suppose that (b) holds. Then the unit ball

$$\{\varphi \in \ell^1(\Sigma, \sigma) : \|\varphi\|_{1,\sigma} \leq 1\}$$

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is a $\beta^1(\Sigma, \sigma)$-closed barrel set in $\ell^1(\Sigma, \sigma)$, and by assumption it is a $\beta^1(\Sigma, \sigma)$-neighbourhood of zero. That is, $n(\Sigma, \sigma) \leq \beta^1(\Sigma, \sigma)$. Invoke Proposition 2.2 to infer that Σ is finite. □

Let us recall that the locally convex space (E, τ) is said to be a dual space if there exists a locally convex space (E_0, τ_0) such that (E, τ) coincides with the strong dual of (E_0, τ_0).

PROPOSITION 4.6. *Let Σ be a set and σ be a positive function on Σ. The space $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ is a dual space if and only if Σ is finite.*

PROOF. We only prove the 'only if' part. By Theorem 3.1,

$$(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^* = c_0(\Sigma, 1/\sigma).$$

So, if $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ is a dual space, then $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ must be normable; this follows from the fact that a dual space whose dual is normable, itself is normable; see [5, Lemma 3.2]. Hence Σ is finite. □

PROPOSITION 4.7. *Let Σ be a set, σ be a positive function on Σ and A be a subset of $c_0(\Sigma, 1/\sigma)$. Then the following statements are equivalent.*

(a) *A is $\beta^1(\Sigma, \sigma)$-equicontinuous.*
(b) *A is $\|\cdot\|_{\infty,\sigma}$-bounded and, for $\varepsilon > 0$, there exists a finite subset F of Σ such that $\langle |f|, |\varphi| \rangle < \varepsilon$ for all $f \in A$ and $\varphi \in \ell^1(\sigma, \Sigma)$ with $\|\varphi\|_{1,\sigma} \leq 1$ and $\mathrm{coz}(\varphi) \subset \Sigma \setminus F$.*
(c) *A is $\|\cdot\|_{\infty,\sigma}$-bounded and, for $\varepsilon > 0$, there exists a finite subset F of Σ such that $|f(x)| < \varepsilon$ for all $f \in A$ and $x \in \Sigma \setminus F$.*

PROOF. (a) $\Rightarrow$ (b). Norm boundedness of A follows easily from definition and $\beta^1(\Sigma, \sigma)$-boundedness of the unit ball of $\ell^1(\Sigma, \sigma)$. Now, choose a neighbourhood $U((F_n), (r_n))$ such that

$$|\langle f, \varphi \rangle| \leq 1$$

for $f \in A$ and $\varphi \in U((F_n), (r_n))$. For an arbitrary $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $\varepsilon r_{n_0} > 1$. Set

$$F := \bigcup_{n=1}^{n_0} F_n.$$

We then have $\varphi \in \varepsilon U((F_n), (r_n))$ for all $\varphi \in \ell^1(\Sigma, \sigma)$ with $\|\varphi\|_{1,\sigma} \leq 1$ and

$$\mathrm{coz}(\varphi) \subseteq \Sigma \setminus F.$$

So $|\langle f, \varphi \rangle| \leq \varepsilon$ for $f \in A$. This implies that $\langle |f|, |\varphi| \rangle \leq \varepsilon$.

(b) $\Rightarrow$ (c). Let $\varepsilon > 0$ and F be as in part (b) and note that, for any point $x \in \Sigma \setminus F$ and $f \in A$,

$$|f(x)| = \langle |f|, \delta_x \rangle < \varepsilon.$$

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(c) ⇒ (a). For $n \in \mathbb{N}$, choose a finite set F_n such that $|f(x)| \le 2^{-2n}$ for $x \in \Sigma \setminus F_n$. Setting $r_n := 2^n$ and $U := U((F_n), (r_n))$, for each $f \in A$,

$$
\begin{aligned}
|\langle f, \varphi\rangle| &= \sum_{x\in F_1} |f(x)||\varphi(x)| + \sum_{n=1}^{\infty}\Big(\sum_{x\in F_{n+1}\setminus F_n} |f(x)||\varphi(x)|\Big)\\
&\le \sum_{x\in F_1} |f(x)||\varphi(x)| + \sum_{n=1}^{\infty}\Big(2^{-2n}\sum_{x\in F_{n+1}\setminus F_n} |\varphi(x)|\Big)\\
&\le \sum_{x\in F_1} |f(x)||\varphi(x)| + \sum_{n=1}^{\infty} 2^{-2n}\binom{n+1}{2}\\
&\le 2\|f\|_{\infty,\sigma} + 2,
\end{aligned}
$$

for all $\varphi \in U$ and $f \in A$. So, f is bounded on U for all $f \in A$. This completes the proof. □

A locally convex space (E, τ) is called a Mackey space if τ coincides with the Mackey topology $\mu(E, E^*)$; also (E, τ) is called a DF space if E possesses a fundamental sequence of bounded sets (that is, a sequence of bounded sets (B_n) such that $B_n + B_n \subset B_{n+1}$), and if every strongly bounded countable union of equicontinuous subsets of E^* is equicontinuous; see [6] for more details.

PROPOSITION 4.8. *Let Σ be a set and σ be a positive function on Σ. Then the following statements are equivalent.*

(a) $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ *is Mackey space.*
(b) $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ *is DF space.*
(c) Σ *is finite.*

PROOF. (a) ⇒ (c). Let $\Delta = \{\delta_x : x \in \Sigma\} \subseteq c_0(\Sigma, 1/\sigma)$. Then an easy application of the Smulian–Eberlein and Krein theorems implies weak compactness of Δ and its closed convex hull. It follows from [15, Theorem 9.4.2] that Δ is equicontinuous. Now invoke Proposition 4.7 to conclude that Σ is finite.

(b) ⇒ (c). If there is a sequence (x_n) of distinct elements of Σ, then $\bigcup_{n=1}^{\infty}\{\delta_{x_n}\}$ is equicontinuous by (b). So, $\bigcup_{n=1}^{\infty}\{\delta_{x_n}\}$ satisfies condition (b) in Proposition 4.7, a contradiction. □

PROPOSITION 4.9. *Let Σ be a set and σ be a positive function on Σ. The strict topology is the finest locally convex topology that agrees with the strict topology on norm bounded subsets of $\ell^1(\Sigma, \sigma)$ if and only if Σ is countable.*

PROOF. Let $\beta^0(\Sigma, \sigma)$ denote the locally convex topology generated by seminorms

$$\mathcal{P}_f(\varphi) = \|f\varphi\|_1,$$

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where $f \in c_0(\Sigma, 1/\sigma)$. By [10], $\beta^0(\Sigma, \sigma)$ is the finest locally convex topology on $\ell^1(\Sigma, \sigma)$ that agrees with $\beta^0(\Sigma, \sigma)$ on bounded sets. It is clear that $\beta^1(\Sigma, \sigma) \leq \beta^0(\Sigma, \sigma)$. Suppose that Σ is uncountable. Let V_f be the $\beta^0(\Sigma, \sigma)$-neighbourhood

$$\{\varphi \in \ell^1(\Sigma, \sigma) : \|f\varphi\|_1 \leq 1\}$$

of zero, where $f \in c_0(\Sigma, 1/\sigma)$ and $f(x) \neq 0$ for all $x \in \Sigma$.

Then V_f is not a $\beta^1(\Sigma, \sigma)$-neighbourhood; indeed, if there exists $((F_n), (r_n)) \in \mathcal{F} \times \mathcal{R}$ such that $U((F_n), (r_n)) \subseteq V_f$, then $r_n\delta_x \in U((F_n), (r_n))$ for all $n \geq 1$ and $x \in F_n$, but $r_n\delta_x \notin V_f$ for some $n \geq 1$ and $x \in F_n$.

Conversely, let Σ be countable and

$$V_f = \{\varphi \in \ell^1(\Sigma, \sigma) : \|f\varphi\|_1 \leq 1\}.$$

For $n \in \mathbb{N}$, choose a finite set $F_n \subset \Sigma$ such that

$$n2^n f(x) < \sigma(x)$$

for all $x \in \Sigma \setminus F_n$. We show that

$$U((F_n), (n)) \subseteq V_f.$$

Let $\varphi \in U((F_n), (n))$. Then

$$\begin{aligned}\sum_{x \in F_n} |\varphi(x) f(x)| &= \sum_{m=1}^{n} \Big(\sum_{x \in F_m \setminus F_{m-1}} |f(x)\varphi(x)| \Big) \\ &\leq \sum_{m=1}^{n} \Big(\sum_{x \in F_m \setminus F_{m-1}} \frac{1}{m2^m} |\varphi(x)|\sigma(x) \Big) \\ &\leq \sum_{m=1}^{n} \frac{1}{2^m}.\end{aligned}$$

Since $f(x) = 0$ for all $x \in \Sigma \setminus \bigcup_{n=1}^{\infty} F_n$, it follows that $\|f\varphi\|_1 \leq 1$ as required. □

A locally convex space E is called quasi-normable if every open subset $U \subseteq E$ contains an open subset $V \subseteq E$ such that, for each $\alpha > 0$, we can find a bounded subset $B \subseteq E$ with $V \subseteq B + \alpha U$.

PROPOSITION 4.10. *Let Σ be a set and σ be a positive function on Σ. Then $\ell^1(\Sigma, \sigma)$ with the strict topology is always quasi-normable.*

PROOF. Let $U = U((F_n), (r_n))$ be an arbitrary $\beta^1(\Sigma, \sigma)$-neighbourhood of zero. Choose a sequence of positive numbers $(s_n) \in \mathcal{R}$ such that $s_n \leq r_n$ and (s_n/r_n) tends to zero. Define

$$V = V((F_n), (s_n)).$$

For a given $0 < \alpha < 1$, choose a natural number n_0 such that $s_n \leq \alpha r_n$ for all $n \geq n_0$. It is easy to see that each $\varphi \in V$ is the sum of two functions $\chi_{F_{n_0}}\varphi$ and $(1 - \chi_{F_{n_0}})\varphi$

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such that

$$\chi_{F_{n_0}}\varphi \in \{\varphi \in \ell^1(\Sigma, \sigma) : \|\varphi\|_{1,\sigma} \le s_{n_0}\}$$

and $(1 - \chi_{F_{n_0}})\varphi \in \alpha U$, and the proof is complete. $\square$

Let E be a locally convex space, and let $\mathcal{U}$ be a base at zero for E consisting of absolutely convex sets. The linear space of all sequences (x_n) in E such that $(\langle f, x_n\rangle)_n \in \ell^1(\mathbb{N})$ for all $f \in E^*$ is denoted by $\ell_1[E]$. The seminorms

$$\varepsilon_U((x_n)) := \sup\Big\{\sum_{n=1}^{\infty} |\langle f, x_n\rangle| : f \in U^\circ\Big\} \quad (U \in \mathcal{U})$$

generate a locally convex topology on $\ell_1[E]$. A sequence (x_n) in E is called absolutely Cauchy if

$$\pi_U((x_n)) := \sum_{n=1}^{\infty} q_U(x_n) < \infty$$

for all $U \in \mathcal{U}$, where q_U denotes the Minkowski functional of U. The linear space of all absolutely Cauchy sequences in E is denoted by $\ell_1\{E\}$ equipped with the topology given by the seminorms π_U. A locally convex space E is called nuclear if $\ell_1\{E\} = \ell_1[E]$ topologically and algebraically; for more details, see [6], for example.

The following theorem shows that $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ behaves as a Banach space with respect to nuclearity.

PROPOSITION 4.11. *Let Σ be a set and σ be a positive function on Σ. Then $\ell^1(\Sigma, \sigma)$ with strict topology is a nuclear space if and only if Σ is finite.*

PROOF. Proposition 4.7 implies that $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ is sequentially evaluable; recall that a locally convex space (E, τ) is said to be sequentially evaluable if every $\beta(E^*, E)$-convergent sequence in E^* is equicontinuous. Note also that $\ell^1(\Sigma, \sigma)$ has a fundamental sequence of bounded sets (take, for example,

$$B_n = \{\varphi : \|\varphi\|_{1,\sigma} \le n\}$$

for all $n \ge 1$). Now, if $(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))$ is nuclear, then [9, Theorem 2.14] implies that the Banach space

$$(\ell^1(\Sigma, \sigma), \beta^1(\Sigma, \sigma))^* = c_0(\Sigma, 1/\sigma)$$

is nuclear, and hence Σ must be finite. $\square$

5. An application to semigroup algebra

Let S be a semigroup and σ be a weight function on it; that is, a positive function with $\sigma(st) \le \sigma(s)\sigma(t)$ for all $s, t \in S$. The convolution product on $\ell^1(S, \sigma)$ is defined by

$$(\varphi * \psi)(x) = \sum_{st=x} \varphi(s)\psi(t)$$

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for $\varphi, \psi \in \ell^1(S, \sigma)$ and $x \in S$ when $st = x$ has a solution, and $(\varphi * \psi)(x) = 0$ otherwise.

Here, we consider the semigroup algebra $\ell^1(S, \sigma)$ with convolution as multiplication, and prove separate $\beta^1(S, \sigma)$-continuity of this multiplication for a large class of semigroups.

First, let us recall that a semigroup S is called finitely cancellative if

$$t^{-1}x = \{s \in S : ts = x\}$$

is finite for all $x, t \in S$.

PROPOSITION 5.1. *Suppose that S is a countable finitely cancellative semigroup. Then $(\ell^1(S, \sigma), \beta^1(S, \sigma))$ with convolution as multiplication is a complete semi-topological algebra.*

PROOF. Since S is countable, in view of Proposition 4.9, we only need to show that convolution on $(\ell^1(S, \sigma), \beta^1(S, \sigma))$ is $\beta^1(S, \sigma)$-continuous on $\beta(\Sigma, \sigma)$-bounded sets; see [6]. Let (φ_α) be a norm bounded net in $\ell^1(S, \sigma)$ convergent to zero in $\beta^1(S, \sigma)$. Let $\psi \in \ell^1(S, \sigma)$ and fix $x_0 \in S$. Choose a finite set $F \subseteq S$ such that

$$\sum_{t \in S \setminus F} |\psi(t)|\sigma(t) < \frac{\varepsilon\sigma(x_0)}{2M},$$

where M is a bound for the net (φ_α). Then $F^{-1}x_0$ is finite by the finite cancellativity of S. So, if we put

$$F_n := F^{-1}x_0 \quad \text{and} \quad r_n := \frac{\varepsilon n\sigma(x_0)}{2\|\psi\|_{1,\sigma}},$$

then $((F_n), (r_n)) \in \mathcal{F} \times \mathcal{R}$, and so there is α_0 such that $\varphi_\alpha \in U((F_n), (r_n))$ for all $\alpha \geq \alpha_0$. In particular,

$$\sum_{s \in F^{-1}x_0} |\varphi_\alpha(s)|\sigma(s) < \frac{\varepsilon\sigma(x_0)}{2\|\psi\|_{1,\sigma}}$$

for all $\alpha \geq \alpha_0$, where

$$F^{-1}x_0 := \{s \in S : ts = x_0 \text{ for some } t \in F\}.$$

Now, for each $\alpha \geq \alpha_0$,

$$\begin{aligned}
\Big|\sum_{st = x_0} \varphi_\alpha(s)\psi(t)\Big| &\leq \sigma(x_0)^{-1} \sum_{st = x_0} |\varphi_\alpha(s)||\psi(t)|\sigma(s)\sigma(t) \\
&\leq \sigma(x_0)^{-1} \sum_{t \in F} \sum_{s \in F^{-1}x_0} |\varphi_\alpha(s)|\sigma(s)|\psi(t)|\sigma(t) \\
&\quad + \sigma(x_0)^{-1} \sum_{t \in S \setminus F} \sum_{s \in (S \setminus F)^{-1}x_0} |\varphi_\alpha(s)|\sigma(s)|\psi(t)|\sigma(t)
\end{aligned}$$

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$$\leq \sigma(x_0)^{-1}\|\psi\|_{1,\sigma} \sum_{s\in F^{-1}x_0} |\varphi_\alpha(s)|\sigma(s) + \sigma(x_0)^{-1} M \sum_{t\in S\setminus F} |\psi(t)|\sigma(t) \leq \varepsilon.$$

Hence, $(\varphi_\alpha * \psi)(x_0) \longrightarrow 0$ and $\varphi_\alpha * \psi \longrightarrow 0$ in the $\beta^1(S, \sigma)$-topology. □

The following example shows that Proposition 5.1 does not hold in general.

EXAMPLE 5.2. Let $S = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Then S with the operation $st = \max\{s, t\}$ is a countable semigroup with identity. It is easy to see that $\varphi \mapsto \varphi * \delta_1$ is not $\beta^1(S, \sigma)$-continuous on $\ell^1(S, 1)$; in particular, S is not finitely cancellative.

In conclusion, we give a special case of Proposition 5.1 which partially answers a question raised by Singh in [12].

COROLLARY 5.3. *If G is a countable group, then $(\ell^1(G), \beta^1(G))$ with convolution as multiplication is a complete semi-topological algebra.*

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