On certain forms of Vibration.

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Certain cases of motion in a gas contained in a spherical or in a cylindrical envelope or surrounding a sphere or a cylinder have been considered by Professor Stokes and Lord Rayleigh, of which a clear idea may be obtained from chapters xvii. and xviii. of Rayleigh's *Sound.* Their chief object is to determine the motion in the gas when the motion in the bounding surface is given, or to determine the modification in the nature of a wave being propagated through the gas by the presence of an obstacle.

Now if the spherical or cylindrical boundary be looked on as an elastic solid, then any vibration instituted in a contiguous liquid or gas will tend to produce vibrations in the boundary, precisely as any vibration set up in the solid will tend to produce vibrations in the fluid.

The object of the present paper is to consider certain simple forms of vibration when the solid boundary and the fluid are treated as one system.

If u denote the radial displacement in a sphere or spherical shell at a distance r from the centre, ρ being the density, and m, n in Thomson and Tait's notation the elastic constants for the material, the equation for radial vibrations as given in Lamè's *Leçons sur L' Elasticité* * becomes

$$\frac{d\delta}{dr}=\frac{\rho}{m+n}\frac{d^2u}{dt^2} \qquad \dots \qquad \dots \qquad (1)$$

It must be noticed that $n\equiv\mu$, $m\equiv\lambda+\mu$ and $\delta\equiv\theta$ in Lamè's notation. Also the dilatation $\delta=\frac{du}{dr}+\frac{2u}{r} \qquad \dots \qquad \dots \qquad (2)$,

as may be seen from a previous paper by the author in the *Proceedings*, noticing that $e=\frac{du}{dr}$, $f=g=\frac{u}{r}$.

The surface condition may be put in the form

$$(m+n)\delta-4n\frac{u}{r}=\text{normal traction} \qquad \dots \qquad (3).\dagger$$

* Equations (8), p. 200.

† *Cf.* Lamè, § 80.

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In (1) assume $u \propto \cos kt$, and let $\frac{\rho}{m+n} = a^2$ ... ... (4),

then we have to determine u as a function of r from

$$\frac{d}{dr}\left(\frac{du}{dr} + \frac{2u}{r}\right) + k^2a^2u = 0.$$

This may be put in the form

$$\frac{d^2}{dr^2}\left(ur^{\frac{1}{2}}\right) + \frac{1}{r}\frac{d}{dr}\left(ur^{\frac{1}{2}}\right) + ur^{\frac{1}{2}}\left\{k^2a^2 - \left(\frac{3}{2}\right)^2\right\} = 0 \quad \dots \quad (5).$$

This is the well-known Bessel's equation,* and since $\frac{3}{2}$ is not an integer a complete solution is, A′ and B′ being arbitrary constants,

$$ur^{\frac{1}{2}} = A'J_{\frac{3}{2}}(kar) + B'J_{-\frac{3}{2}}(kar) \quad \dots \quad (6).$$

If the sphere be solid B′ must be zero, but for a shell both functions may exist. Leaving out a constant multiplier, we have†

$$\left.\begin{aligned} J_{\frac{3}{2}}(z) &= \frac{1}{\sqrt{z}}\left(\frac{\sin z}{z} - \cos z\right) \\ J_{-\frac{3}{2}}(z) &= -\frac{1}{\sqrt{z}}\left(\frac{\cos z}{z} + \sin z\right) \end{aligned}\right\} \quad \dots \quad (7).$$

For convenience let $A'(ka)^2 = A$, and $B'(ka)^2 = B$, then for a solid sphere we have

$$u = \frac{A}{(ka)^2}\cos kt \frac{1}{r}\left\{\frac{\sin kar}{kar} - \cos kar\right\} \quad \dots \quad (8);$$

whence

$$\delta = A\cos kt \frac{\sin kar}{kar} \quad \dots \quad \dots \quad (9).$$

From (3) the periods $\frac{2\pi}{k}$ of the free vibrations of the sphere are given by $(m+n)\delta - 4n\frac{u}{r} = 0$ when $r = a$, if a denote the radius of the sphere.

Hence we find

$$kaa\cot kaa = 1 - \frac{m+n}{4n}(kaa)^2 \quad \dots \quad (10).$$

One root is $kaa \doteq 0$, none of the others are very small.

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* See the *Bessel'schen Functionen* of Neumann or Lommel.

† Lommel, p. 118. Also Lord Rayleigh's *Sound*, vol. i., p. 278, Eqn. (3).

Now suppose an incompressible fluid to surround the sphere and to be otherwise unlimited. Then the vibrations existing in the sphere will cause vibrations of the same pitch in the fluid of a purely radial character also. Thus the velocity potential ϕ for the fluid motion is a function only of r and t, therefore $\nabla^2\phi = 0$* reduces to

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = 0 \qquad \ldots \qquad \ldots \qquad (11).$$

Obviously a solution is $\phi = \left(\frac{C}{r} + D\right)\sin kt \qquad \ldots \qquad \ldots \qquad (12),$

where C and D are arbitrary constants. This solution gives the velocity $\frac{d\phi}{dr}$ zero at infinity, and from the equation at the surface of the sphere we can determine C and D in terms of A and so satisfy all necessary conditions.

It is obvious that the fluid in contact with the sphere must have the same normal velocity as the sphere; thus when $r = a$,

$$\frac{d\phi}{dr} = \frac{du}{dt} \qquad \ldots \qquad \ldots \qquad (13)$$

Again the pressure on the surface of the sphere is the fluid pressure, and thus the variable parts of these must be identical. Now, if σ be the density and p the variable part of the fluid pressure, and if the action of gravity be disregarded, and the square of the velocity neglected,

$$\frac{d\phi}{dt} + \frac{p}{\sigma} = 0 \qquad \ldots \qquad \ldots \qquad (14)\dagger;$$

therefore from (3), when $r = a$

$$\sigma\frac{d\phi}{dt} = (m+n)\delta - 4n\frac{u}{r} \qquad \ldots \qquad \ldots \qquad (15).$$

Determining C and D in terms of A from (13) and (15), and substituting in (12) we obtain

$$\phi = \frac{A}{k}\sin kt\left[-\left(1 - \frac{a}{r}\right)\frac{1}{a^2}\left\{\frac{\sin kaa}{kaa} - \cos kaa\right\} + \frac{1}{\sigma}\left\{(m+n)\frac{\sin kaa}{kaa} - \frac{4n}{(kaa)^2}\left(\frac{\sin kaa}{kaa} - \cos kaa\right)\right\}\right] \qquad \ldots \qquad (16)$$

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* See Lamb's *Motion of Fluids*, § 43.

† See Lamb's *Motion of Fluids*, Eqn. (4), p. 20, putting F $(t) = 0$.

This gives $$\frac{d\phi}{dr} = -\frac{\mathrm{A}a\sin kt}{ka^2r^2}\left\{\frac{\sin ka\alpha}{ka\alpha} - \cos ka\alpha\right\} \quad \ldots \quad (17).$$

Thus the velocity in the fluid is independent of σ, and of course varies inversely as the square of the distance from the centre.

If the vibration be one of those natural to the sphere, we see from (10) that (16) becomes, the co-efficient of $\frac{1}{\sigma}$ vanishing,

$$\phi = -\frac{\mathrm{A}}{ka^2}\sin kt\left(1 - \frac{a}{r}\right)\left\{\frac{\sin ka\alpha}{ka\alpha} - \cos ka\alpha\right\} \quad \ldots \quad (18).$$

In this case we see from (14) that the variable part of the fluid pressure is directly proportional to its density.

If $ka\alpha$ be small, which however excludes the vibrations natural to the sphere, an approximate solution is from (17)

$$\frac{d\phi}{dr} = -\tfrac{1}{3}\mathrm{A}k\frac{a^3}{r^2}\sin kt \quad \ldots \quad \ldots \quad (19).$$

Suppose next that the solid sphere is surrounded by a gas. Then (8) and (9) will still represent the motion in the sphere, and if σ now denote the density of the gas, whose vibrations are supposed not very large, (13) and (15)* will also give the surface conditions. But if ϕ denote the velocity potential in the gas, the vibrations in which are purely radial, the equation is now†

$$c^2\left(\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr}\right) = \frac{d^2\phi}{dt^2} \quad \ldots \quad \ldots \quad (20),$$

where c is the velocity of sound in the gas.

Assume $\phi \propto \sin kt$, then (20) may be put in the form

$$\frac{d^2}{dr^2}\left(r^{\frac{1}{2}}\phi\right) + \frac{1}{r}\frac{d}{dr}\left(r^{\frac{1}{2}}\phi\right) + \left\{\frac{k^2}{c^2} - \frac{(\frac{1}{2})^2}{r^2}\right\}(r^{\frac{1}{2}}\phi) = 0 \qquad (21).$$

The two solutions of this equation are $J_{\frac{1}{2}}\left(k\frac{r}{c}\right)$ and $J_{-\frac{1}{2}}\left(k\frac{r}{c}\right)$, both of which will apply since the gas is excluded from the origin.

From Lommel, or by direct trial, $J_{\frac{1}{2}}(z) \propto \frac{1}{\sqrt{z}}\sin z$, and $J_{-\frac{1}{2}}(z) \propto \frac{1}{\sqrt{z}}\cos z$; therefore a complete solution of (21) is

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* See Lord Rayleigh's *Sound*, Eqn. (6) of § 244.

† Lamb's *Motion of Fluids*, Eqn. (12) of § 167.

$$\phi = \sin kt \frac{\mathrm{E}\sin\left(\frac{kr}{c}\right) + \mathrm{F}\cos\frac{kr}{c}}{\frac{kr}{c}} \qquad \ldots \qquad (22),$$

where E and F are arbitrary constants.

If for shortness

$$\mathrm{H} \equiv \frac{1}{ka^2}\left\{\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right\} \qquad \ldots \qquad \ldots \qquad (23),$$

$$\text{and } \mathrm{K} \equiv \frac{1}{k\sigma}\left[(m+n)\frac{\sin k\alpha a}{k\alpha a} - \frac{4n}{k^2\alpha^2 a^2}\left\{\frac{\sin k\alpha a}{k\alpha a} - \cos k\alpha a\right\}\right] \quad (24),$$

then it will be found from (13) and (15) that

$$\mathrm{E} = \mathrm{A}\left[-\mathrm{H}\cos\frac{ka}{c} + \mathrm{K}\left\{\cos\frac{ka}{c} + \frac{ka}{c}\sin\frac{ka}{c}\right\}\right]$$

and

$$\mathrm{F} = \mathrm{A}\left[\mathrm{H}\sin\frac{ka}{c} - \mathrm{K}\left\{\sin\frac{ka}{c} - \frac{ka}{c}\cos\frac{ka}{c}\right\}\right].$$

Whence from (22)

$$\phi = \mathrm{A}\sin kt \frac{(\mathrm{K}-\mathrm{H})\sin\left(k\frac{r-a}{c}\right) + \mathrm{K}\frac{ka}{c}\cos\left(k\frac{r-a}{c}\right)}{\frac{kr}{c}} \qquad (25).$$

If the vibration be one of those natural to the sphere, then, from (10), $\mathrm{K}=0$, and reducing H we get

$$\phi = -\mathrm{A}ka^2\sin kt\frac{m+n}{4n}\,\frac{\sin k\alpha a}{k\alpha a}\,\frac{\sin\left(k\frac{r-a}{c}\right)}{\frac{kr}{c}} \qquad (26).$$

From (14) it appears that in this case the variable part of the pressure is everywhere and at all times directly proportional to σ.

If the vibration be not one of those natural to the sphere, but be such that $k\alpha a$ is small, then approximately

$$\mathrm{H} = \tfrac{1}{3}ka^2 \quad \ldots \quad (27), \qquad \text{and } \mathrm{K} = \frac{3m-n}{3k\sigma} \quad \ldots \quad (28).$$

Introducing these values in (25) and noticing that $\frac{ka}{c}$, though for large values of r not $\frac{kr}{c}$, is also small we get a very simple result.

To give some idea of the relative magnitudes of c and $\frac{1}{a}$ we may mention that in the C. G. S. system of units c is about 33000 for air at 0°C.

It is difficult to give exact values for a as different experimenters get different results, in many cases doubtless owing to the difference between specimens nominally the same; but for metals such as brass or iron it is safe to say that to take $\frac{1}{a} = 15000$ gives at all events a correct idea of its dimensions. Thus if we assume $ka a$ small we must also take $\frac{ka}{c}$ small.

From (25) we can determine A so as to get a given amplitude of vibration over any spherical surface concentric with the solid sphere. We might thus suppose the motion to be forced by an arbitrary purely radial vibration of any elastic spherical membrane containing the gas, in whole or in part, and concentric with the solid sphere. The consequent forced vibrations in the sphere are then given by (8).

Suppose next a spherical shell of elastic solid structure, the radii of whose surfaces are $r = a$, and $r = a'$, containing a gas and itself surrounded by gas. We need not in the first place consider the surrounding gas.

From (5), (6), (7), and (8) we see that a suitable solution for a vibration of period $\frac{2\pi}{k}$ in the shell is

$$u = \frac{\cos kt}{k^2 a^2 r}\left[A\left\{\frac{\sin kar}{kar} - \cos kar\right\} + B\left\{\frac{\cos kar}{kar} + \sin kar\right\}\right] \quad (29),$$

giving
$$\delta = \frac{\cos kt}{kar}\left\{A \sin kar + B \cos kar\right\} \quad \ldots \quad (30).$$

By making the expression (3), namely, $(m+n)\delta - 4n\frac{u}{r}$ vanish over both surfaces $r = a$ and $r = a'$ we find, after eliminating A and B, for the equation giving the frequency of the free radial vibrations of the shell

$$ka(a-a')\cot ka(a-a') \cdot 4n\{(m+n)k^2a^2aa' + 4n\} =$$
$$(m+n)^2k^4a^4a^2a'^2 - 4n(m+n)k^2a^2(a^2+a'^2) + 16n^2(1+k^2a^2aa') \ldots (31).$$

If $\frac{a-a'}{a}$ be small an approximate solution will be found to be

$$kaa = \frac{2\sqrt{\{n(3m-n)\}}}{m+n} \quad \ldots \quad \ldots \quad (32)$$

For the contained gas we see from (21) and (22) that a suitable solution is

$$\phi = \mathrm{E}\sin kt \frac{\sin\frac{kr}{c}}{\frac{kr}{c}} \quad \ldots \quad \ldots \quad \ldots \quad (33),$$

as this remains finite and $\frac{d\phi}{dr}$ vanishes when $r=0$.

The equations (13) and (15) apply at the surface $r=a$. From these, if H and K have the same form as in (23) and (24), and if further

$$\mathrm{L} = \frac{1}{ka}\left\{\frac{\cos kaa}{kaa} + \sin kaa\right\} \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

and

$$\mathrm{M} = \frac{1}{k\sigma}\left[(m+n)\frac{\cos kaa}{kaa} - \frac{4n}{k^2a^2a^2}\left\{\frac{\cos kaa}{kaa} + \sin kaa\right\}\right] \ldots \quad (35),$$

we find, noticing that $\mathrm{KL} - \mathrm{HM} = \frac{m+n}{k^3a^3a\sigma}$,

$$\mathrm{A} = \frac{\mathrm{E}k^2a^3\sigma c}{m+n}\left[\mathrm{L}\sin\frac{ka}{c} - \mathrm{M}\left\{\sin\frac{ka}{c} - \frac{ka}{c}\cos\frac{ka}{c}\right\}\right] \quad \ldots \quad (36)$$

and

$$\mathrm{B} = \frac{\mathrm{E}k^2a^3\sigma c}{m+n}\left[\mathrm{K}\left\{\sin\frac{ka}{c} - \frac{ka}{c}\cos\frac{ka}{c}\right\} - \mathrm{H}\sin\frac{ka}{c}\right] \quad \ldots \quad (37).$$

Substituting these values in (29) we have the solution completely expressed in terms of the one arbitrary constant E, which may be determined so that the vibration at any distance from the centre, whether in the gas or in the shell, may have a given amplitude.

If the vibration be one of those natural to the shell, on the usual theory, we have the right hand side of (15) vanishing, and thus from (33) we should also have $\sigma\mathrm{E}\frac{\sin\frac{ka}{c}}{\frac{ka}{c}} = 0$.

If we regard terms in σ as negligible this is satisfied, but otherwise the result is in general impossible. We cannot suppose E to vanish, as then no vibration would exist either in the gas or in the shell. The solution $\frac{ka}{c} = i\pi$ could happen only through accident, as k is one of a series of numbers already determined. If for instance

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we suppose the shell very thin we see from (32) that $ka = i\pi$ could happen only if $ac = \frac{2\sqrt{\{n(3m-n)\}}}{i\pi(m+n)}$ ... ... (38).

The ratio $m : n$ is not known with much certainty for any solids, in fact it forms one of the most disputed points in the range of Physical Science. We will at least approach to an average result if we take Poisson's value 2 for this ratio. This would require $iac = \frac{21}{44}$ roughly, which, even taking $i = 1$, is considerably too small for most combinations of a gas and a solid. The smaller c is, the more nearly would (38) be true; thus carbonic acid gas for which c is about 26000 would more nearly suit the conditions than air, which in turn would suit very much better than hydrogen.

The real solution of the difficulty is that if σ be not negligible a vibration natural to the shell, as usually defined, cannot really exist, while if the period approach closely to one of those natural to the shell E must be small if σ be big. Thus if the gas be at a high pressure it will very seriously muffle those vibrations of the shell which at small pressures would be far the most important. It is well known that the amplitude of vibration of any elastic body under the influence of a periodic force is comparatively small, unless the period of the force approximate to one of the natural periods of vibration of the elastic body. Thus great importance may attach to the values of σ and of ac as modifying the amplitude and period of vibration under certain forms of excitation.

For the gas surrounding the shell we may take the solution (22), writing E′ for E and F′ for F. Then to determine E′ and F′ in terms of A and B, and so of E, we have the equations (13) and (15) applied to the surface $r = a'$. If we denote the value of H &c., when a' is written for a by H′ &c., we find

$$E'\sin\frac{ka'}{c} + F'\cos\frac{ka'}{c} = \frac{ka'}{c}(K'A + M'B) \quad \ldots \quad (39)$$

and

$$E'\cos\frac{ka'}{c} - F'\sin\frac{ka'}{c} = (K' - H')A + (M' - L')B \quad (40).$$

These give E′ and F′ simply in terms of A and B, and so of E by means of (36) and (37). Thus the vibrations in the shell as well as in the enclosed and surrounding gases are all expressed in terms of the one constant E. Thus the motion might be produced by forced displacement of the surface of the shell, or of any concentric spherical membrane. We can obviously extend the method to any series of concentric elastic solid spherical shells separated by layers of gas. Exactly the same method also applies to concentric contiguous layers of different elastic solids or of different gases.

If the vibration be one natural to the shell we must have $K'A + M'B = 0$, and therefore from (39) and (40)

$$F' \operatorname{cosec} \frac{ka'}{c} = -E' \sec \frac{ka'}{c} = (H'A + L'B) \quad \ldots \quad (41).$$

There is no difficulty in satisfying these equations for all values of σ. It should however be noticed that since E' and F' are determined, a difficulty would arise if the gas surrounding the shell were in turn bounded by a surface such as to restrict the motion in the gas. If for instance this new boundary were a rigid spherical shell this would require $\frac{d\phi}{dr}$ to vanish for a given value of r, which of course could happen only accidentally; though if the radius of this boundary were large this defect might be neglected. Further, in the case of the natural vibration, the above equation (41) shews from the values of H' and L', to be obtained from (23) and (36) by writing a' for a, that E' and F' do not contain terms in $\frac{1}{\sigma}$. Thus from (22) the value of ϕ for the surrounding gas contains no term in $\frac{1}{\sigma}$, and therefore from (14) the variable part of the pressure in this gas is directly proportional to σ.

Thus by diminishing the density of a gas surrounding a sphere performing free radial vibrations the sound should appear weakened to an observer surrounded by this rarefied atmosphere. If the rarefied gas be surrounded by an elastic solid envelope, the variable gaseous pressure $\sigma\frac{d\phi}{dt}$ on the inside of this envelope being diminished will in turn diminish the variation in the pressure transmitted through the envelope to the gas surrounding it. Thus a lowering in the sound would also be perceived by an observer though not actually inside the space containing the rarefied gas.

Similar methods are applicable to the case of the vibrations in two dimensions of a right circular cylinder containing or surrounded by a fluid. We shall suppose no displacement parallel to the length of the cylinder which we assume to be so long that we may neglect the conditions at its ends. The following solution would thus strictly assume the ends of the cylinder maintained at a constant distance apart by suitably varying normal forces, which suitable supports would give. The results should also be limited to points in the fluid the distances of which from the central cross section of the cylinder are considerably less than half the length of the cylinder. If, however, the cross section be small, these limitations may be disregarded, excluding points in the axis of the cylinder produced.

If u denote the displacement for purely radial vibrations in the cross section, the equation of motion for the cylinder is

$$\frac{d\delta}{dr} = \frac{\rho}{m+n}\frac{d^2u}{dt^2} \quad \ldots \quad \ldots \quad (42),*$$

where now, however

$$\delta = \frac{du}{dr} + \frac{u}{r} \quad \ldots \quad \ldots \quad (43).*$$

Also, the surface condition is now

$$(m+n)\delta - 2n\frac{u}{r} = \text{normal traction} \ldots \quad (44).*$$

Assuming a vibration of period $\frac{2\pi}{k}$ we have to determine u as a function of r from

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + u(k^2a^2 - \frac{1}{r^2}) \quad \ldots \quad \ldots \quad (45),$$

where a has the same meaning as before.

This is a Bessel's equation, the two solutions being denoted by $J_1(kar)$ and $Y_1(kar)$, the latter occurring only for a hollow cylinder.

Thus, A being an arbitrary constant, we have for a solid cylinder

$$u = A\cos kt J_1(kar) \quad \ldots \quad \ldots \quad (46).$$

If b denote the radius of the cylinder, it will be found from (44) that the periods of free vibration, as usually defined, are given by

$$(m+n)J_1'(kab) + (m-n)\frac{J_1(kab)}{kab} = 0 \quad \ldots \quad (47)$$

where $J_1'(kab) \equiv \frac{d}{kadr}J_1(kar)$, putting $r=b$ after differentiation.

* See a paper by the author in the *Philosophical Magazine*, February 1886, pp. 81, 82.

Suppose now the cylinder surrounded by an incompressible fluid of density σ. The velocity potential will be given by

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}=0 \qquad \ldots \qquad \ldots \qquad (48).$$

A suitable solution is

$$\phi=\sin kt(C\log r+D) \qquad \ldots \qquad \ldots \qquad (49),$$

where C and D are arbitrary constants.

The surface conditions are obviously

$$\frac{du}{dt}=\frac{d\phi}{dr} \qquad \ldots \qquad \ldots \qquad (50)$$

and

$$(m+n)\delta-2n\frac{u}{r}=\sigma\frac{d\phi}{dt} \qquad \ldots \qquad \ldots \qquad (51)$$

when $r=b$.

From these two equations C and D are given in terms of A, and it will be found that (49) becomes

$$\phi=A\sin kt\Big[-kbJ_1(kab)\log\frac{r}{b}+\frac{a}{\sigma}\Big\{(m+n)J_1'(kab)$$
$$+(m-n)\frac{J_1(kab)}{kab}\Big\}\Big] \qquad \ldots \qquad \ldots \qquad (52)$$

Thus the velocity in the fluid is given by

$$\frac{d\phi}{dr}=-\frac{Akb}{r}J_1(kab)\sin kt \qquad \ldots \qquad \ldots \qquad (53),$$

which is independent of σ, and of course inversely proportional to the distance from the axis.

If the vibration be one of those natural to the cylinder the coefficient of $\frac{a}{\sigma}$ in (52) vanishes. Thus $\sigma\frac{d\phi}{dt}$, that is, the variable pressure in the fluid is directly proportional to σ.

For a gas surrounding a solid cylinder the velocity potential is given by

$$c^2\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=\frac{d^2\phi}{dt^2} \qquad \ldots \qquad \ldots \qquad (54),$$

where c denotes the velocity of sound in the gas.

Assuming a vibration of period $\frac{2\pi}{k}$, we get to determine ϕ as a function of r the Bessel's equation of zero order, viz.,

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}+\frac{k^2}{c^2}\phi=0 \quad \ldots \quad \ldots \quad (55).$$

The two solutions are denoted by $J_0\left(\frac{kr}{c}\right)$ and $Y_0\left(\frac{kr}{c}\right)$, the latter occurring only when the gas is excluded from the axis $r=0$. Supposing this the case, we may take

$$\phi=\sin kt\left[CJ_0\left(\frac{kr}{c}\right)+DY_0\left(\frac{kr}{c}\right)\right] \quad \ldots \quad (56),$$

where C and D are arbitrary constants. Determining them from (50) and (51), we get

$$\phi=\frac{A}{\Delta}\sin kt\left[cJ_1(kab)\left\{Y_0\left(\frac{kb}{c}\right)J_0\left(\frac{kr}{c}\right)-J_0\left(\frac{kb}{c}\right)Y_0\left(\frac{kr}{c}\right)\right\}\right.$$
$$+\frac{a}{\sigma}\left\{(m+n)J_1'(kab)+(m-n)\frac{J_1(kab)}{kab}\right\}\left\{J_0\left(\frac{kr}{c}\right)Y_0'\left(\frac{kb}{c}\right)\right.$$
$$\left.\left.-Y_0\left(\frac{kr}{c}\right)J_0'\left(\frac{kb}{c}\right)\right\}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (57),$$

Where for shortness $\Delta\equiv J_0\left(\frac{kb}{c}\right)Y_0'\left(\frac{kb}{c}\right)-Y_0\left(\frac{kb}{c}\right)J_0'\left(\frac{kb}{c}\right)$.

It is easy to prove* that $\Delta=\frac{kb}{c}$ (numerical factor), and so Δ can never vanish, and no forced vibration could become infinite.

If the vibration be one of those natural to the cylinder, the coefficient of $\frac{a}{\sigma}$ in (57) vanishes, and the variable part of the pressure in the gas is directly proportional to σ while the velocity is independent of σ.

Considering next a hollow cylinder, we get for the radial vibrations

$$u=\cos kt\{AJ_1(kar)+BY_1(kar)\} \quad \ldots \quad \ldots \quad (58).$$

The free vibrations, as usually defined, are given by the equation in k which results from the elimination of A and B by means of the conditions $(m+n)\delta-2n\frac{u}{r}=0$ at both surfaces $r=b$ and $r=b'$. If $\frac{b'-b}{b}$

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* See a paper by the author in the *Messenger of Mathematics*, June 1885, p. 20.

be very small an approximate solution will be found to be

$$kab = \frac{2\sqrt{(mn)}}{m+n} \qquad \ldots \qquad \ldots \qquad (59).$$

For a gas inside this cylinder the velocity potential will be given by the single term

$$\phi = \mathrm{C}\sin kt \mathrm{J}_0\left(\frac{kr}{c}\right) \qquad \ldots \qquad \ldots \qquad (60).$$

The equations (50) and (51) at the surface $r=b$ give

$$\frac{\mathrm{C}}{c}\mathrm{J}_0'\left(\frac{kb}{c}\right) = -\{\mathrm{AJ}_1(kab) + \mathrm{BY}_1(kab)\} \qquad \ldots \qquad (61),$$

and $$\sigma \mathrm{CJ}_0\left(\frac{kb}{c}\right) = \mathrm{A}a\left\{(m+n)\mathrm{J}_1'(kab) + (m-n)\frac{\mathrm{J}_1(kab)}{kab}\right\}$$

$$+ \mathrm{B}a\left\{(m+n)\mathrm{Y}_1'(kab) + (m-n)\frac{\mathrm{Y}_1(kab)}{kab}\right\} \quad \ldots \quad (62).$$

These determine A and B in terms of C.

If the vibration be one of those natural to the shell the second side of (62) vanishes. If σ be not negligible this could happen only if $\mathrm{C}=0$, or if $\mathrm{J}_0\left(\frac{kb}{c}\right)=0$. But the ratio of A : B and the value of k are determined from the conditions that the normal pressures vanish over the surfaces $r=b$ and $r=b'$; therefore $\mathrm{C}=0$ would from (61) require both A and B to vanish, and so no vibration would exist. So again, $\mathrm{J}_0\left(\frac{kb}{c}\right)$ could vanish only through accident. For instance, in a very thin shell (59) shows that this could happen only if $\mathrm{J}_0\left\{\frac{2\sqrt{(mn)}}{ac(m+n)}\right\}=0$. Since the least root of $\mathrm{J}_0(z)$ is about 2.4 there are certainly very few cases in which this could occur. The explanation of the difficulty is exactly the same as for the corresponding case of the spherical shell, and so need not be repeated.

If the shell be surrounded by the same or any other gas, the equations (50) and (51) at the outer surface $r=b'$ suffice to determine the two arbitrary constants which occur in the velocity potential for the gas in terms of C. Thus all the details of the motion are known provided the amplitude of the vibration at any one distance from the axis of the shell be given. The results are so exactly similar to those already obtained for the sphere that it is unnecessary to point out how they may be extended to any number of coaxial cylindrical layers of different substances, solid and fluid. We would only remark that a diminution of density in the gas surrounding the cylinder has exactly the same effect in diminishing the variable part of the pressure, and so lowering the sound, as it had in the case of the spherical shell.

Eighth Meeting, June 11*th*, 1886.

Dr Ferguson, F.R.S.E., President, in the Chair.

A Problem in Combinations.

By Alexander Robertson, M.A.

I.

Given sets of balls of different colours, in how many ways may they be arranged in line so that no two balls of the same colour shall come together.

If we have two colours only, and the same number m of each colour, there are evidently two arrangements possible; if we have m, $m-1$ respectively, only one arrangement is possible; if we have m, $m-2$; m, $m-3$, &c., no arrangement is possible. We may write these results

$$(m, m) = 2,\ (m, m-1) = 1,\ (m, m-2) = 0,\ (m, m-3) = 0, \text{ \&c.}$$

In the sequel we shall consider three colours only, say m white balls, n black balls, and p red balls. The number of arrangements is denoted by (mnp), and in general we shall take m greater than n and n greater than p, but not always.

Keeping m and n constant, the smallest value of p is $m-n-1$. There is also a superior limit to p; taking n as then the smallest we should have, smallest value of $n = p - m - 1$, therefore $p = m + n + 1$, and therefore the total number of values of p is $2n+3$. For example, let $m = 5$ and $n = 2$, the smallest value of p is 2 and the greatest 8; or p may have seven values, so that arrangements are possible with 522, 523, 524, 525, 526, 527, 528. If, however, m and n are equal, the case is slightly different; then the smallest value of p is 0, and the greatest is $2m+1$, or p has $2m+2$ values.

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