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ON STRONGLY RIGHT BOUNDED FINITE RINGS

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An associative ring is called strongly right (left) bounded if every nonzero right (left) ideal contains a nonzero ideal. We prove that if R is a strongly right bounded finite ring with unity and the order $|R|$ of R has no factors of the form p^5, then R is strongly left bounded. This answers a question of Birkenmeier and Tucci.

Throughout this paper, rings are associative with unity. From [**1**], a ring R is *right (left) duo* if every right (left) ideal is an ideal, and R is *strongly right (left) bounded* if every nonzero right (left) ideal contains a nonzero ideal. Birkenmeier and Tucci [**1**] have constructed a ring R with 32 elements such that R is strongly right bounded, not right duo, and not strongly left bounded. They raised the following question in [**1**, p.1110]:

QUESTION: Does there exist a ring with unity which is strongly right bounded, not right duo, and not strongly left bounded with fewer than 32 elements?

The purpose of this note is to answer the above question in the negative. In fact, we prove

THEOREM. *If R is a strongly right bounded finite ring with unity and the order $|R|$ of R has no factors of the form p^5 (in particular, if $|R| < 32$), then R is strongly left bounded.*

To establish our theorem, we need the following

PROPOSITION 1. (Eldridge [3]) *Let R be a finite ring with unity. (1) If $|R|$ has a cube free factorisation, then R is a commutative ring; and (2) if $|R| = p^3$, p a prime, and R is noncommutative, then $R \cong \begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$, where $GF(p)$ denotes the field with p elements.*

Clearly, $\begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$ is not strongly right bounded, since it has a minimal right ideal $\begin{bmatrix} 0 & 0 \\ 0 & GF(p) \end{bmatrix}$ which is not an ideal. Hence any noncommutative ring with p^3 elements is not strongly right bounded.

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PROPOSITION 2. *Let R be a strongly right bounded ring with unity and with p^4 elements, p a prime. If R is not local, then R is strongly left bounded.*

PROOF: Suppose that R is a noncommutative ring. If R has a ring decomposition, $R = R_1 \oplus R_2$, then either $|R_1| = p^3$ or $|R_2| = p^3$, since R is noncommutative and any ring of order p^2 is commutative. Let $|R_1| = p^3$; then $|R_2| = p$. It follows that R_1 is noncommutative since R_2 is commutative. So by the above discussion, R_1 is not strongly right bounded, neither is R. Hence R must be indecomposable, and then R has a non-trivial idempotent e since R is not local. Let $f = 1 - e$ and identify $R = \begin{bmatrix} eRe & eRf \\ fRe & fRf \end{bmatrix}$.

(I) Let $eRf \neq 0$ and $fRe \neq 0$. If $eRfRe \neq 0$ (similarly for $fReRf \neq 0$), R has a minimal right ideal $\begin{bmatrix} eRe & eRf \\ 0 & 0 \end{bmatrix}$ which is not an ideal. This is a contradiction. So $eRfRe = fReRf = 0$, and then R is strongly left bounded.

(II) Now let $eRf \neq 0$ and $fRe = 0$. Then $R = \begin{bmatrix} eRe & eRf \\ 0 & fRf \end{bmatrix}$. If $|fRf| = p$, R has a minimal right ideal $\begin{bmatrix} 0 & 0 \\ 0 & fRf \end{bmatrix}$ which is not an ideal. So we assume that $|fRf| = p^2$. If $fRf = R_1 \oplus R_2$ is decomposable and $eRfR_i \neq 0$ for $i = 1$ or 2, then R has a minimal right ideal $\begin{bmatrix} 0 & 0 \\ 0 & R_i \end{bmatrix}$ which is not an ideal. Hence we suppose that fRf is a local ring of length 2. (Since $|eRf| = p$, fRf is not a field.) Let $0 \neq a \in eRf$ and $0 \neq b \in \mathrm{Rad}(fRf)$. Then $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} R$ is a minimal right ideal of R that is not an ideal. ∎

PROPOSITION 3. *A finite right duo ring R is left duo.*

PROOF: By [**2**, Theorem 3.1], R is the ring direct sum of local right duo rings. So we may assume that R itself is a local right duo ring. It follows that the simple left R-module and the simple right R-module have the same number of elements. Hence the composition length of ${}_RR$ is equal to that of R_R. Therefore R is a left duo ring by [**2**, Theorem 2.2]. ∎

PROOF OF THE THEOREM: Suppose that R is strongly right bounded, but not strongly left bounded. Since R is a direct sum of rings each of which has prime power order, we may assume from Proposition 1 that $|R| = p^4$. Now R is not a right duo ring by Proposition 3. Hence by [**1**, Proposition 6], there exists a nonzero ideal T such that R/T is not strongly right bounded. It follows from Proposition 1 again that $|R/T| = p^3$ and R/T is not a local ring. Hence R is not a local ring, and then R is strongly left bounded by Proposition 2. ∎

Adapting the above proof and using Proposition 3, we have

REMARK 4. Let R be a strongly right bounded ring with p^4 elements, p a prime. If R is local, then R is both right and left duo. We note that $R = GF(p)[x, y]/\langle x^3, y^3, yx, x^2 - xy, y^2 - xy\rangle$ is a non-commutative local duo ring with p^4 elements.

In view of Proposition 2 we mention the following examples: Let $S = \begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$ and $T = \begin{bmatrix} GF(p) & 0 \\ GF(p) & GF(p) \end{bmatrix}$. Let R be the subring of $S \times T$ consisting of elements of the form

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \times \begin{bmatrix} a & 0 \\ d & b \end{bmatrix}.$$

Then R is a ring with p^4 elements that is strongly right bounded but not right duo.

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