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NILPOTENT IDEALS IN ALTERNATIVE RINGS

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1. Introduction. It is well known and immediate that in an associative ring a nilpotent one-sided ideal generates a nilpotent two-sided ideal. The corresponding open question for alternative rings was raised by M. Slater [6, p. 476]. Hitherto the question has been answered only in the case of a trivial one-sided ideal J (i.e., in case $J^2=0$) [5]. In this note we solve the question in its entirety by showing that a nilpotent one-sided ideal K of an alternative ring generates a nilpotent two-sided ideal. In the process we find an upper bound for the index of nilpotency of the ideal generated. The main theorem provides another proof of the fact that a semiprime alternative ring contains no nilpotent one-sided ideals. Finally we note the analogous result for locally nilpotent one-sided ideals.

Recall that an alternative ring A is defined by the property $(x, x, y)=(y, x, x)=0$ for all $x, y \in A$ where the associator (x, y, z) denotes $(xy)z-x(yz)$. The fundamental property that we shall use repeatedly is that $(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})=(\operatorname{sgn}\sigma)(x_1, x_2, x_3)$ for all x_i in A, $i=1,2,3$, and $\sigma \in S_3$ [4]. The nucleus, $N(A)$, of A and the center, $Z(A)$, of A are defined by $N(A)=\{n \in A \mid (n, a_1, a_2)=0 \ \forall a_1, a_2, \in A\}$ and $Z(A)=\{z \in N(A) \mid za=az \ \forall a \in A\}$. For $a \in A$ the right multiplication map determined by a is given by $R_a : x \mapsto xa$. Similarly one defines $L_a : x \mapsto ax$. Let $A_\ell=\{L_a \mid a \in A\}$, $A_r=\{R_a \mid a \in A\}$, and $M(A)$ be the subring of End A generated by A_ℓ and A_r. We also denote by A' the ring obtained after adjoining an identity element to A in the usual way.

In any non-associative ring R, R^s denotes the ring spanned by all monomials of R of degree s (no matter how associated) and R is nilpotent if $R^s=0$ for some positive integer s. Finally, we define right powers of R inductively by $R^{[1)}=R$, and $R^{[n+1)}=R^{[n)}R$. We say that R is right nilpotent if $R^{[n)}=0$ for some positive integer n.

Throughout we shall assume that K denotes a left ideal. Similar results and proofs apply to right ideals.

2. Main results. It is well known that if K is a left ideal of an alternative ring A then the two-sided ideal generated by K is $KA'=K+KA$. Thus, we shall be interested in the effect on KA' of the nilpotence of K. It should be noted that K^s is not in general a left ideal of A for a positive integer s.

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LEMMA 1. *$(M(A')(K^s))(KA')\subseteq M(A')(K^{s+1})$ for any positive integer s.*

Proof. We shall show that $(T_{x_t}\cdots T_{x_2}T_{x_1}(k_s))(ky)\in M(A')(K^{s+1})$ for $T=R$ or L, $k_s\in K^s$, $k\in K$, t any non-negative integer and y, x_i arbitrary elements of A' for $i=1,2,\ldots,t$. The proof is by induction on t. Suppose $t=0$. Then, since $(k_s,k,y)=-(y,k,k_s)$ we have:

$$
\begin{aligned}
k_s(ky)&=(k_sk)y-(k_s,k,y)\\
&=(k_sk)y+(y,k,k_s)\qquad(1)\\
&=(k_sk)y+(yk)k_s-y(kk_s).
\end{aligned}
$$

It is easy to see that the right hand side of (1) is in $M(A')(k^{s+1})$. Thus, if $t=0$ we have our result.

Assume now that the result holds for $t<n$ and consider an element of the form $u=(T_{x_n}\cdots T_{x_2}T_{x_1}(k_s))(ky)$ using the previous notation. Let $k_1=T_{x_{n-1}}\cdots T_{x_2}T_{x_1}(k_s)$. Then by the induction hypothesis we have $k_1(KA')\subseteq M(A')(K^{s+1})$. Now if $T_{x_n}=R_{x_n}$ then

$$
\begin{aligned}
u=(k_1x_n)(ky)=k_1[x_n(ky)]+(k_1,x_n,ky)&=k_1[x_n(ky)]-(k_1,ky,x_n)\\
&=k_1[x_n(ky)]-[k_1(ky)]x_n+k_1[(ky)x_n].
\end{aligned}
$$

Since $k_1(KA')\subseteq M(A')(K^{s+1})$ the second term on the right is in $M(A')(K^{s+1})$. Since

$$x_n(ky)=(x_nk)y-(x_n,k,y)=(x_nk)y+(x_n,y,k)=(x_nk)y+(x_ny)k-x_n(yk)$$

it follows that $k_1[x_n(ky)]\in k_1(KA')\subseteq M(A')(K^{s+1})$. Similarly $(ky)x_n=k(yx_n)-(x_ny)k+x_n(yk)$. Therefore

$$k_1[(ky)x_n]\in k_1(KA')\subseteq M(A')(K^{s+1}).$$

Thus, if $T_{x_n}=R_{x_n}$ we have $u\in M(A')(K^{s+1})$. On the other hand, if $T_{x_n}=L_{x_n}$ then

$$u=(x_nk_1)(ky)=x_n[k_1(ky)]+[k_1(ky)]x_n-k_1[(ky)x_n].$$

As before, all three terms on the right are in $M(A')(K^{s+1})$ by the induction hypothesis. Thus, in all cases we have

$$u=(T_{x_n}\cdots T_{x_2}T_{x_1}(K^s))(KA')\subseteq M(A')(K^{s+1})$$

and the result follows by mathematical induction.

THEOREM 1. *If the left ideal K of the alternative ring A is nilpotent of index n, then the ideal $KA'=K+KA$ is right nilpotent of index n.*

Proof. We prove by mathematical induction that $(KA')^{[s]}\subseteq M(A')(K^s)$ for all positive integers s. The case $s=1$ is obvious. Assume true in case $s=t$. Then $K(A')^{[t+1]}=(KA')^{[t]}(KA')\subseteq(M(A')K^t))(KA')$ by the induction hypothesis. But by Lemma 1 $(M(A')(K^t))(KA') \subseteq M(A')(K^{t+1})$ to complete the proof. Now if $s = n$ we have $(KA')^{[n)} = 0$.

This enables us to prove our main result in short order.

THEOREM 2. *If the left ideal K of the alternative ring A is nilpotent of index n, then the ideal KA' is nilpotent of index $\leq n^2$.*

Proof. Let w be a monomial of $(KA')^{n^2}$. Then w is a product of n^2 terms of the form $k_i a_i$ with $a_i \in A'$. By [2, Proposition 3] we may assume that w is a linear combination of second-order monomials of degree n^2 in the $k_i a_i$, i.e., w is a sum of terms of the form $u = R_{z_r} \cdots R_{z_2} R_{z_1}(1)$ where $z_i = R_{x_{i_{s(i)}}} \cdots R_{x_{i_2}} R_{i_1}(1)$ for $i = 1, 2, \ldots, r$ for some r, s and some choice of $x_{i_j} = k_t a_t$ where the degree of u in the x_{i_j} is n^2. It then follows that either $i_{s(i)} \geq n$ for some i or $r > n$. Note that $z_i \in (KA')^{[i_s)}$. Therefore, if $i_{s(i)} \geq n$ Theorem 1 provides that $z_i = 0$. Thus $u = 0$. Suppose, on the other hand, that $i_{s(i)} < n$ for each i, and $r > n$. Now, by Lemma 1 (since $M(A')(K) \subseteq KA'$) $z_i \in M(A')(K)$ for each i. Then by repeated use of Lemma 1 we have $u \in M(A')(K^r) \subseteq M(A')(K^n) = 0$. Thus, any product of n^2 terms of KA' reduces to zero and the proof is completed.

We thus have another way at arriving at the following result.

COROLLARY. *A semiprime alternative ring contains no non-zero nilpotent one-sided ideals.*

REMARKS. Independent of our Theorem 2, the result of the corollary can be obtained as an immediate consequence of a result of Slater [7, Prop. 11.6]. In fact, it also follows from an earlier result of Kleinfeld. For he has shown that if K is a left ideal of A then $S(K) = \{a \in K \mid aA \subseteq K\}$ is a two-sided ideal of A contained in K and $(A, K, K) \subseteq S(K)$ [1]. Therefore, if K is nilpotent and $S(K) \neq 0$ we have the result while if $S(K) = 0$ we have $(A, K, K) = 0$. In particular, K^n is a left ideal of A for each positive integer n. Therefore K^t is a trivial left ideal of A for some t and by [5, Lemma 3.3] the ideal of A generated by K^t is a trivial ideal of A. Moreover, in case A is 3-torsion free then a stronger result than that of the corollary is known. Namely, a semiprime 3-torsion free alternative ring contains no one-sided ideals which are nil of bounded index [3, 8].

We will now establish an analog of Theorem 2 with local nilpotence in place of nilpotence. Recall that the Levitzki radical, $\mathscr{L}(A)$, of A is the locally nilpotent ideal of A which contains every other locally nilpotent ideal of A and that an ideal J is locally nilpotent if every finitely generated subring of J is nilpotent. We shall also make use of the fact that $\mathscr{L}(A/\mathscr{L}(A)) = 0$. As a preliminary result (and as an analog to the previous Corollary) we prove

LEMMA 2. *A Levitzki semisimple 3-torsion free alternative ring A contains no locally nilpotent one-sided ideals.*

Proof. Let K be a left ideal of A, with $\mathscr{L}(A)=0$. Then A is semiprime. Hence, by [5, Corollary 7.7] either $3K \subseteq N(A)$ or $K \cap Z(A) \neq 0$. Suppose, then, that K is a locally nilpotent left ideal. If $3K \subseteq N(A)$ then $3K$ is a non-zero left ideal in $N(A)$ so that the ideal generated by it, $3K+3KA$, is a locally nilpotent ideal just as in the case of associative rings. If $3K \not\subseteq N(A)$ let $0 \neq z \in K \cap Z(A)$ such that $z^2=0$. Then either zA or Iz ($I=$ integers) forms a non-zero nilpotent ideal. In either case, the existence of a non-zero locally nilpotent ideal contradicts the assumption $\mathscr{L}(A)=0$ to complete the proof.

Lemma 3. *If an alternative ring A is n-torsion free then $\bar{A}=A/\mathscr{L}(A)$ is also n-torsion free.*

Proof. Suppose that $n\bar{a}=\bar{0}$ for some $a \in A$. Then $na \in \mathscr{L}(A)$. We show that $a \in \mathscr{L}(A)$. For if not then the ideal $\mathscr{L}_a$ generated by $\mathscr{L}$ and a properly contains $\mathscr{L}(A)$. Note that a typical element of $\mathscr{L}_a$ is of the form $\ell+m(a)$ for some $\ell \in \mathscr{L}(A)$ and $m \in M(A')$. But this implies that $\mathscr{L}_a$ is locally nilpotent. For if we pick any finite set $T=\{t_1, t_2, \ldots, t_s\}$ of elements of $\mathscr{L}_a$ then, since $na \in \mathscr{L}(A)$, the subring generated by $nT=\{nt_1, nt_2, \ldots, nt_s\}$ is nilpotent, say of index k. Thus, if we consider any product $t_{i_1}t_{i_2}\cdots t_{i_k}$ for $t_{i_j} \in T$ it follows that $n^k t_{i_1}t_{i_2}\cdots t_{i_k}=0$. But since A is n-torsion free this means that $t_{i_1}t_{i_2}\cdots t_{i_k}=0$. Hence, the subring generated by T is nilpotent of index k and $\mathscr{L}_a$ is locally nilpotent. Since $\mathscr{L}(A)$ contains all locally nilpotent ideals it follows that $\mathscr{L}_a=\mathscr{L}(A)$ or $a \in \mathscr{L}(A)$. Thus, $\bar{a}=\bar{0}$ and $\bar{A}$ is n-torsion free.

Theorem 3. *If K is a locally nilpotent left ideal of the alternative ring A and A is 3-torsion free, then the ideal KA′ of A generated by K is also locally nilpotent.*

Proof. Let $\bar{A}=A/\mathscr{L}(A)$. Then A is 3-torsion free by Lemma 3. Since $\bar{A}$ is Levitzki semisimple and the image $\bar{K}$ of K in $\bar{A}$ is locally nilpotent it follows from Lemma 2 that $\bar{K}=\bar{0}$. Therefore $K \subseteq \mathscr{L}(A)$. Since $\mathscr{L}(A)$ is an ideal of A we have $KA' \subseteq \mathscr{L}(A)$. Thus, KA' is locally nilpotent.

Note. The results beginning with Lemma 2 can be easily modified to apply to local finiteness instead of local nilpotence.

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