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EDGECONVEX CIRCUITS AND THE TRAVELING SALESMAN PROBLEM

KENNETH KALMANSON

Introduction. This paper will continue certain investigations into the geometric nature of the well-known traveling salesman problem: that of determining the extreme Hamiltonian circuits (H-circuits) of a graph.

Let G be a graph with distinct vertices $P_{i_1}, P_{i_2}, \ldots, P_{i_n}$ such that to each edge of G, PQ, there corresponds a real number $\overline{PQ} = \overline{QP}$, called the length of the edge. Let $p = (P_1, P_2, \ldots, P_n)$ be cyclic and symmetric notation for an H-circuit, and let

$$L(p) = \sum_{i=1}^{n-1} \overline{P_iP_{i+1}} + \overline{P_nP_i}$$

denote the length of p. In this paper inequalities are given which, when satisfied, give a procedure for ordering the vertices of G so as to yield the maxima and minima of $L(p)$ over all H-circuits p of G. Geometric realizations are then provided which considerably extend the solution of the so-called convex case in the Euclidean plane.

1. Summary of results. Let G be a complete undirected graph with n vertices $P_{i_1}, P_{i_2}, \ldots, P_{i_n}$. We define an H-circuit $C = (P_1P_2 \ldots P_n)$ to be *edgeconvex* if $1 \leqq i < j < k < r \leqq n$ always implies $\overline{P_iP_j} + \overline{P_kP_r} \leqq \overline{P_iP_k} + \overline{P_jP_r}$ and $\overline{P_rP_i} + \overline{P_jP_k} \leqq \overline{P_rP_j} + \overline{P_iP_k}$.

THEOREM 1. *Let $C = (P_1P_2 \ldots P_n)$ be an edgeconvex circuit on G. Then $L(C)$ is minimal over all H-circuits of G.*

THEOREM 2(a). *Let $C = (P_1P_3P_5 \ldots P_{2n-1}P_2P_4P_6 \ldots P_{2n-2})$ be an edgeconvex circuit on a graph G having $2n - 1$ vertices. Then $K = (P_1P_2 \ldots P_{2n-1})$ has maximum length $L(C)$ among the H-circuits of G.*

THEOREM 2(b). *Let $C = (P_1{}^1P_1{}^2 \ldots P_1{}^nP_1{}^{n+1}P_1{}^{n+2} \ldots P_1{}^{2n})$ denote an edgeconvex H-circuit on a graph G having $2n$ vertices. Then one of the n circuits*

$$C^{(i)} = (\ldots P_{2n-5}{}^iP_5{}^iP_{2n-3}{}^iP_3{}^iP_{2n-1}{}^iP_1{}^iP_{2n}{}^iP_2{}^iP_{2n-2}{}^i \ldots)$$

has maximal length $L(C)$ among all the H-circuits of G, where for each $i = 1, 2, \ldots, n$, starting with $P_1{}^i$ and traversing the circuit C in the given order, the con-

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secutive points of the graph are relabelled

$$P_1{}^i, P_2{}^i, P_3{}^i, \ldots, P_n{}^i, P_{2n}{}^i, P_{2n-1}{}^i, \ldots, P_{n+1}{}^i.$$

Note. The above theorem can be pictured by imagining n copies of a circular circuit C in which the subscript labellings have been "shifted clockwise" by one each time. In Section 3 we will consider several geometric realizations of edgeconvex circuits. Among these we have the following:

PROPOSITION 1. *If the points $P_1, P_2, \ldots, P_n$ correspond to points of the Euclidean plane E_2 (or, alternatively, the Euclidean 2-sphere, S_2) which fall on the boundary of their convex hull in the given order, and $\overline{P_iP_j}$ is the distance measured by a planar norm (or the usual spherical distance) then the circuit $C = (P_1P_2\ldots P_n)$ is edgeconvex.*

COROLLARY 1. *If the points $P_1, P_2, \ldots P_n$ all lie on a great circle of S_2, then the circuit $(P_1, P_2, \ldots, P_n)$ is edgeconvex, where the p_i are in cyclic order.*

COROLLARY 2. *If the points $P_1, P_2, \ldots, P_n$ are in cyclic order on a rectifiable Jordan curve K and P_iP_j is the length of a minor arc on K, then the circuit $(P_1, P_2, \ldots, P_n)$ is edgeconvex.*

In Section 4 we shall continue an investigation (Kalmanson [**1**]) of polygons in Minkowski planes whose lengths equal that of the convex hull of their vertices as measured in the Minkowski plane metric. If, of these vertices, those taken on the boundary of their convex hull are in cyclic order then the polygon will be called a chain polygon, since it has been shown that the interior points will then form geodesic chains between adjacent boundary points.

PROPOSITION 2. *Chain polygons in any Minkowski plane correspond to edgeconvex circuits.*

While Proposition 2 extends what is known about chain polygons in the maximal case (cf. [**7**, **4**]), if there is an even number of vertices, $2n$, then Theorem 2(b) only locates the maximum circuit in a class of n circuits. The following theorem of Fred Supnick bridges this gap in certain cases:

THEOREM (*Supnick's Four Point Condition*). *Let $P_1, P_2, \ldots, P_n$ be distinct vertices such that for all i, j, k, and r, $1 \leqq i < j < k < r \leqq n$ we have*

$$\overline{P_iP_j} + \overline{P_kP_r} \leqq \overline{P_iP_k} + \overline{P_jP_r} \leqq \overline{P_iP_r} + \overline{P_jP_k}$$

then $(\ldots P_7P_5P_3P_1P_2P_4P_6\ldots)$ is minimal and $(\ldots P_5P_{n-3}P_3P_{n-1}P_1P_{n-2}P_2P_{n-4}\ldots)$ is maximal in length $L(C)$ in the class of H-circuits on the vertex set. (See Supnick [**8**]).

The condition (FPC) of the above theorem is neither implied by nor implies the "edgeconvexity" condition. In fact, while an interesting class of realizations of the FPC has been given in Lorentz space (see [**5**]), it is known that no non-collinear k points, for $k \geqq 8$, satisfies the FPC in the

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Euclidean plane. We can give a broad class of realizations in Minkowski planes, however. These will be called tracklike distributions.

2. Proof of Theorem 1. Identify each vertex P_i of graph G with a unique point Q_i on a Euclidean circle O so that the points Q_i are labelled in their natural cyclic order with respect to O. Consider the complete rectilinear graphs C^* on points Q_i, and let Q_iQ_j be the Euclidean length of segment Q_iQ_j. Define an *arc-inversion* by the symbol

$$(P_1P_2, \ldots P_{i-1}(P_i \ldots P_j)P_{j+1} \ldots P_n)$$

on a circuit

$$(P_1P_2 \ldots P_{i-1}P_i \ldots P_jP_{j+1} \ldots P_n)$$

producing

$$(P_1P_2 \ldots P_{i-1}P_jP_{j-1} \ldots P_{i+1}P_iP_{j+1} \ldots P_n).$$

Suppose that $C_0 = (P_{i_1}P_{i_2} \ldots P_{i_n}) \neq C$, the given circuit. Then the points of C_0^* are not in their natural cyclic order on circle O. Without loss of generality, we may assume either $i_1 < i_2 < i_{k+1} < i_k$ or $i_1 < i_k < i_2 < i_{k+1}$. Hence, $Q_{i_1}Q_{i_2}$ and $Q_{i_k}Q_{i_{k+1}}$ intersect at an interior point of the circle. Therefore, $C_1^* = (Q_{i_1}(Q_{i_2} \ldots Q_{i_k})Q_{i_{k+1}} \ldots Q_{i_n})$ is strictly shorter than C_0^*, where C_i^* denotes the rectilinear circuit corresponding to C_i. Since C is edgeconvex, $L(C_1) \leqq L(C_0)$. Now suppose that C_0 is of minimum length. The argument given above shows that there exists an arc-inversion, as defined in the argument, producing C_1 such that $L(C_0) \geqq L(C_1)$ and $L(C_0) > L(C_1)$. If $C_1 = C$, we are done. If not, by the above, there is a sequence of circuits C_i, and another of their rectilinear correspondents C_i^*, such that

$$L(C_0^*) > L(C_1^*) > \ldots > L(C_r^*), \text{ and}$$
$$L(C_0) \geqq L(C_1) \geqq \ldots \geqq L(C_r),$$

where $C_r = C$, the edgeconvex circuit.

The sequence of the C_i^*'s terminates in C^* because, if $C_i^* + C^*$, there is an arc-inversion which strictly decreases length, and, there are only finitely many circuits. Clearly, if $L(C_0)$ is minimal, then so is $L(C)$.

Proof of Theorem 2. It will greatly simplify the present discussion to refer to proofs of the theorems in [**7**] and [**4**] which the present theorem generalizes. Once again, let the vertices in question correspond to points Q_i on a Euclidean circle O, which are in their natural cyclic order on O. It is shown in [**7**] and [**4**] that if a rectilinear circuit C_0^* is not in the given class (of maxima), then there exists an arc-inversion producing a strictly longer circuit C_1^*. The arc-inversion merely exchanges two non-intersecting edges for two other edges which do intersect. This, in turn, re-orders the four points (that is, the endpoints) of these edges so that they *no longer* conform to the ordering of the edgeconvex

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circuit C in the statement of the theorem. Hence, $L(C_0) \leqq L(C_1)$. If C_0 is maximal, we can, using arc-inversions, produce a sequence of circuits C_0, $C_1, \ldots, C_r$ where C_r is a circuit in the relevant class, and $L(C_0) \leqq L(C_1) \leqq \ldots \leqq L(C_r)$.

COROLLARY 3. *Let C denote an edgeconvex H-circuit on a graph G. Then if C_1 and C_2 are any two minimal H-circuits on the vertices of G, there exists a finite sequence of arc-inversions and a corresponding sequence of minimal H-circuits, beginning with C_1 and ending with C_2. If G has an odd number of vertices, the analogous statement holds for the maximal H-circuits of G.*

Proof. There exist the relevant sequences from C_1 and C_2 to C, as in the proofs of Theorems 1 and 2 (a). Arc-inversions are clearly reversible. Join the sequence from C_1 to C with that from C to C_2.

3. Proof of Proposition 1. The proof for the ordinary Euclidean planar metric and the Euclidean two-sphere is given in Quintas & Supnick [**6**], so that we only need concern ourselves with general planar norm metrics. The proposition will follow if we can show that whenever three points in the plane, A, B, and C are such that $e(A, C) + e(C, B) = e(A, B)$, then we must have $m(A, C) + m(C, B) = m(A, B)$, where "$e$" and "$m$" denote the Euclidean metric and an arbitrary norm metric, respectively. Since A, B, and C are collinear, there exists a "t" such that $0 \leqq t \leqq 1$ and $c = tA + (1-t)B$. Letting $m(A, B) = ||A - B||$ where "$||\quad||$" is the norm corresponding to m, we may write

$$\begin{aligned} m(A, C) + m(C, B) &= ||A - ((1-t)B + tA)|| \\ &\quad + ||B - (tA - (1-t)B)|| = \ldots = |1-t|\,||A - B|| \\ &\quad + |t|\,||A - B|| = (1-t)||A - B|| + t||A - B|| \\ &= ||A - B|| = m(A, B). \end{aligned}$$

Proof of Corollary 1. For each t in $(0, 1)$, we may construct a sequence of points $P_1{}^t, P_2{}^t, \ldots, P_n{}^t$ such that (a) $C^t = (P_1{}^t, P_2{}^t, \ldots, P_n{}^t)$ is edgeconvex and, (b) as t approaches one, $\overline{P_i{}^tP_j{}^t}$ approaches $\overline{P_iP_j}$ uniformly for $i, j = 1, 2, \ldots, n$. This can be done by taking $P_i{}^t$ as the intersection of the geodesic arc containing P_i and a fixed north pole N of C^*, and a circle of lattitude t units north of C^*. Since each of the sets $P_1{}^t, P_2{}^t, \ldots, P_n{}^t$ are on the boundary of their convex hull and in cyclic order, by Proposition 1, C^t is edgeconvex. Taking the limits as t approaches one in

$$\overline{P_i{}^tP_j{}^t} + \overline{P_k{}^tP_r{}^t} \leqq \overline{P_i{}^tP_k{}^t} + \overline{P_j{}^tP_r{}^t}, \text{ and } \overline{P_r{}^tP_i{}^t} + \overline{P_k{}^tP_j{}^t} \leqq \overline{P_i{}^tP_k{}^t} + \overline{P_j{}^tP_r{}^t}$$

we obtain the corresponding inequalities for circuit C.

Proof of Corollary 2. If curve K has length L, associate points P_i with the points Q_i on a great circle of length L such that $\overline{P_iP_j} = \overline{Q_iQ_j}$. Apply Corollary 1.

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PROPOSITION 3. *Let A, B, C, and D be any vertices having any set of six "distances" between them. Then it is possible to relabel them P_1, P_2, P_3, P_4 so that the circuit (P_1, P_2, P_3, P_4) is edgeconvex.*

Proof. One of the following inequalities holds:

(a) $\overline{AB} + \overline{CD} \leqq \overline{AC} + \overline{BC} \leqq \overline{AD} + \overline{BC}$,

(b) $\overline{AC} + \overline{BD} \leqq \overline{AD} + \overline{BC} \leqq \overline{AC} + \overline{BD}$,

(c) $\overline{AC} + \overline{BD} \leqq \overline{AB} + \overline{CD} \leqq \overline{AD} + \overline{BC}$,

(d) $AC + BD \leqq \overline{AD} + \overline{BC} \leqq \overline{AB} + \overline{CD}$,

(e) $\overline{AD} + \overline{BC} \leqq \overline{AB} + \overline{CD} \leqq \overline{AC} + \overline{BD}$,

(f) $\overline{AD} + \overline{BC} \leqq \overline{AC} + \overline{BD} \leqq \overline{AB} + \overline{CD}$.

The edgeconvexity condition will hold for (a)–(f) if the following orderings are taken: (a) $ABDC$; (b) $ABDC$; (c) $ACDB$; (d) $ACBD$; (e) $ADCB$; (f) $ADBC$.

4. In order to prove Proposition 2, it will be necessary for us to develop some background material. Given a Minkowski plane with unit circle U, a $2n$-gon, label the n radial diameters of U: $1, 2, \ldots, n$ in some fixed cyclic order. A line through point P parallel to the rth diameter (or coincident with it) will be denoted $C_r(P)$. Let 0 designate the origin, and designate as *positive* one ray of $C_1(0)$ by $C_1^+(0)$, thereby inducing in a clockwise sense, a positive orientation the first n rays, writing $C_1^+(0), C_2^+(0), \ldots, C_n^+(0)$, and a negative orientation on the last n rays, writing $C_1^-(0), C_2^-(0), \ldots, C_n^-(0)$. Further, note that for any point P, $C_r^\pm(P)$ will denote the appropriate ray through P parallel to $C_r^\pm(0)$. Next, the positive angular region bounded by $C_r^+(P)$ and $C_{r+1}^+(P)$ will be written $C_{r,r+1}^+(P)$ (where $r < n$); similarly for $C_{r,r+1}^-(P)$. We will write

$$C_{r,r+1}(P) = C_{r,r+1}^+(P) \cup C_{r,r+1}^-(P).$$

We will say that the points $P_1, P_2, \ldots, P_k$ are cogeodesic in the Minkowski plane M if

$$\overline{P_1P_k} = \sum_{i=1}^{k-1} \overline{P_iP_{i+1}}.$$

It has been shown (Kalmanson [**2**]) that $P_1, P_2, \ldots, P_k$ are cogeodesic in M if and only if there exists an $r = 1, 2, \ldots, n$ such that

$$C_{r,r+1}^\pm(P_1) \supseteq C_{r,r+1}^\pm(P_2) \supseteq \ldots \supseteq C_{r,r+1}^\pm(P_k).$$

In this case we say that we have an *r-chain in M*, and that the line segments are *positively* (or with reverse inclusions, negatively) *oriented r-like edges*. It has likewise been shown [**2**] that a finite set of points, S, of M has a minimal circuit whose length equals that of the perimeter (as measured in M) of the convex hull of S if and only if it has such an H-circuit in which the boundary points appear in their natural cyclic order.

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LEMMA 1. *Chain polygons in any Minkowski plane M^* with a $2n$-gon unit circle correspond to edgeconvex H-circuits.*

Proposition 2 follows from Lemma 1 in the following way: The unit circle U of M, being the boundary of a centrally symmetric convex body in the plane, may be approximated by a centrally symmetric $2n$-gon U^*, such that the linear segments of U coincide with certain sides of U^*. Then, U^* defines a Minkowski plane M^* whose distances approximate those of M (uniformly for any preassigned finite set of points in M). Hence, chain polygons in M are also chain polygons in M^*. The inequalities defining edgeconvexity being satisfied in M, the proposition follows by continuity.

We now transform the unit circle, U, of M so that $C_1^+(0)$ and $C_1^-(0)$ have their slopes minus and plus one, respectively. Hence, positive orientations are "up" and "to the right" as usual. This procedure puts a chain polygon into a convenient form of four sections of chains (of the vertices in the order given by the polygon) having their endpoints on the boundary of the convex hull of the vertices. We may designate these as a leftmost and a rightmost chain $\bar{L}$ and $\bar{R}$ together with an upper and lower section "$\bar{U}$" and "$\bar{D}$" of r-like chains, $r = 2, 3, \ldots, n$ occuring sequentially. (Some r_i may be skipped, but not repeated in any section.) If we further assume that no two vertices of the polygon lie on the same $C_r(P)$ line, then we may assert that this representation is unambiguous, and in this way we will avoid certain complications of proof. This will result in no real loss of generality, since we can give a proof in this restricted case and appeal to continuity in the general case.

Let us now observe that $(P_1, P_2, \ldots, P_n)$ is edgeconvex if and only if every circuit $(P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4})$ where $i_1 \leqq i_2 \leqq i_3 \leqq i_4$, is edgeconvex and $i_j \in \{1, 2, \ldots, n\}$. In order to prove Lemma 1, we will show that in every case of four points in natural order on the given chain polygon, one of the following three lemmas apply (we will interchange the terms "polygon" and "circuit" where there is no danger of confusion):

LEMMA 2. *If $(PQRS)$ is a convex polygon, then it is edgeconvex.*

LEMMA 3. *If $(PQRS)$ has a pair of opposite sides r-like with opposite r-orientation, then it is edgeconvex.*

Proof. Lemma 3 reduces to Lemma 2 by an arc-inversion interchanging two diagonals for opposite sides whose sum has equal length.

LEMMA 4. *If $(PQRS)$ is such that P, Q, and R, or, Q, R, and S, or, R, S, and P, or S, P, and Q form an r-chain, then it is edgeconvex.*

Proof. Suppose that P, Q, and R form an r-chain (we will write $P - Q - R$ in such cases). Then $\overline{PQ} + \overline{QR} = \overline{PR}$. But $\overline{PR} + \overline{QR} = \overline{PQ} + \overline{QS} + \overline{QS} \geqq \overline{PQ} + \overline{RS}$, and we also have $\overline{PQ} + \overline{QR} + \overline{QS} \geqq \overline{QR} + \overline{PS}$, by the triangle inequality. The other cases are similar.

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In analyzing the various positions that four points in natural clockwise order ("INO") on a chain polygon can assume the following lemmas are useful. We leave their proofs for the reader. (They can be found in [**3**].)

LEMMA 5. *Suppose that PS and QR are r-like and s-like, respectively, $r, s > 1$, and both are positively (or both negatively) oriented. Then,* slope $PS >$ slope QR *implies that $r \geqq s$.*

LEMMA 6. *Every vertex Q of a chain polygon K is either within or on the same side of $C_{r,r+1}(P)$, where P is a vertex in an r-chain of K, except for possibly those Q in a different r-chain.*

LEMMA 7. *If P, Q, and R are three points in natural order in a chain polygon such that P belongs to $\bar{U}$ or $\bar{L}$, Q belongs to $\bar{U}$, and R belongs to $\bar{U}$ or $\bar{R}$, while PQ and QR are r-like and s-like, respectively, then $s = 1$ or $s \geqq r$.*

LEMMA 8. *If P, Q, and R are as in Lemma 7, and PQR is a convex arc, then $P - Q - R$.*

LEMMA 9. *If P, Q, R, and S are INO such that P is in $\bar{U}$ or $\bar{R}$, S is in $\bar{U}$ or $\bar{L}$ R and Q are in $\bar{U}$, PQ and RS are not 1-like, then either $(PQRS)$ is convex, or $P - Q - R$, or $Q - R - S$.*

LEMMA 10. (a) *If P is the "highest" point in $\bar{L}$, then $C_{1,2}(P) \cap UPr = \emptyset$.*

(b) *For all Q in $\bar{U}$, for all P^* in $\bar{L}$, Q is not in $C_{1,2}^-(P^*)$.*

(c) *If P^*Q is not 1-like, then the intersection of $C_{1,2}(P^*)$ and the points of $\bar{U}$ to the right of Q is the empty set.*

(d) *If P^*Q is 1-like, then the original chain polygon minus all points INO from P^* to Q is another chain polygon, where P^*Q is a 1-like edge in $\bar{L}$.*

A proof of Lemma 2 can now be given by letting P, Q, R, and S be vertices INO on a chain polygon K, and considering the various cases as follows:

I. If any three of these points is in the same chain or $(PQRS)$ is convex, then Lemmas 2 and 4 apply.

II. We suppose that exactly two vertices P and Q are in the same chain $(\bar{L})$ and that $(PQRS)$ is not convex.

(A) If the other two vertices are in $\bar{R}$, use Lemma 3.

(B) Suppose at least one of the remaining two is in $\bar{U}$ or $\bar{L}$ (assume $R \in \bar{U} - \bar{R}$).

(1) Suppose further that S belongs to $\bar{U}$. Then $(PQRS)$ not convex implies that QRS is a convex arc. Hence, Lemmas 9 and 4 apply.

(2) Suppose that, instead, S belongs to $\bar{R}$.

(a) If R is in $C_{1,2}^+(S)$, then by the fact that PQ and RS are r-like with opposite r-orientations, we can apply Lemma 6.

(b) If R is not in $C_{1,2}^+(S)$, then by Lemma 10, R is not in $C_{1,2}^-(S)$. So, QRS is a convex arc, as in (1), above.

(3) Suppose that R is in $\bar{D}$ but neither in $\bar{R}$ nor in $\bar{L}$.

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(a) If R is in $C_{1,2}^+(Q)$ or S is in $C_{1,2}^-(p)$ then we have either $P - Q - R$ or $Q - R - S$, respectively. Therefore, Lemma 4, applies.

(b) Suppose that (a) is false, and, without loss of generality, that R is not to the right of S. Also, let QR be r-like and PS be s-like respectively.

(i) If S is in $C_{r,r+1}^+(R)$ we have $Q - R - S$ and apply Lemma 4.

(ii) If S were below $C_{r,r+1}^+(R)$ then arc $PQRS$ would be concave and $(PQRS)$ would be a convex polygon—a contradiction.

(iii) Finally, assume S is above $C_{r,r+1}^+(R)$. If we had $s < r$, then we note that the highest point of $\bar{U}$ is below the lowest point of the same set—a contradiction. But since we have P below Q, then $a \geqq r$. Hence, $s = r$. Since PS and QR have the same r-orientation, Lemma 3 applies.

III. If R and S are in $\bar{R}$, then Lemma 3 again applies.

IV. Suppose now that no two of P, Q, R, and S are in the same chain, and that $(PQRS)$ is not a convex polygon. Without loss of generality, assume that P is in $\bar{L}$ and Q is in $\bar{U}$. Then one can show that the various subcases either reduce to one of those above (using Lemma 10), or we distinguish three cogeodesic points, or Lemma 5 applies. (for the details, see Kalmanson [**3**]).

It is well known that any minimal H-circuit on a finite set of points in the Euclidean plane must have the points on the boundary of the convex hull of the set in cyclic order. If a particular class of point sets is known to have an edgeconvex H-circuit in the Euclidean plane, then this circuit must belong to the former class. Hence, in order to find a *maximal* H-circuit on this point set, one could try to find an edgeconvex polygon in this class, and reorder the vertices according to Theorem 2. The problem with this is that *if there are a total of r points in the set, k of which are on the boundary of their convex hull, then there are precisely* $(r - 1)!/(k - 1)!$ *H-circuits on the points of the set which have the boundary vertices in cyclic order* (Supnick [**9**]).

5. Tracklike distributions and Supnick's *FPC*. Let us begin by observing that edgeconvexity does not subsume the *FPC*. We have already shown the converse. An appropriate counterexample is given by the five symbols $P_1, P_2, \ldots, P_5$ as vertices with distances $\overline{P_5P_1} = 2$, and $\overline{P_iP_j} = 1$ for all other pairs. Then $(P_1P_2P_4P_5P_3)$ satisfies the four point ordering of *FPC*. But this circuit is not edgeconvex, since

$$\overline{P_1P_4} + \overline{P_2P_5} < \overline{P_1P_5} + \overline{P_2P_4}.$$

Let $S^n = (s, E^n)$ be the metric space defined by the metric

$$s(P, Q) = \max_{i=1,2,\ldots,n} |p_i - q_i|$$

for all P, Q in E^n. A line segment PQ in E^n will be called i-like if $s(P, Q) = |p_i - q_i|$. A distribution D of points in E^n will be called tracklike if there exists $i_0, j_0 = 1, 2, \ldots, n$ such that for each point P in D we have PQ i_0-like

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for all Q in D, except possibly one point Q_p depending upon P. In that case, PQ_p must be j_0-like.

PROPOSITION 3. *Tracklike distributions in S^n satisfy the FPC if labelled in their i, j lexicographic order.*

Let M be a Minkowski plane with a $2n$-gon unit circle. Using the ideas of Section 4 we will say that a distribution of points is tracklike in M if there is an $i = 1, 2, \ldots, n$ such that for each P in D, PQ is i-like in M with the possible exception of a single point Q_p in D where Q_p depends on P.

PROPOSITION 4. *Tracklike distributions in Minkowski planes satisfy the FPC.*

A feasible labelling for the distributions in Proposition 4 can be found by first mapping the Euclidean plane onto itself using an affinity such that the side i of the unit circle U of M maps onto any side PQ of the unit circle of S^2. Then use a lexicographic ordering of the transformed points, as one would for points in S^2, where one proceeds by considering various cases. The following useful facts are not difficult to prove:

LEMMA 11(a). *Let D be a set of points $P_1, P_2, \ldots, P_m$ in S^n, $n \geqq 2$. Suppose that there exist i and j, $i \neq j$, such that for all P and Q in D, PQ is either i-like or j-like. Then the perpendicular projection of D onto the $i - j$ coordinate plane is one-to-one and S^n distance preserving.*

(b) *If $i = j$ in* (a), *then the points of D lie on an S^n geodesic.*

(c) *If the segments P_kP_{k+1}, $k = 1, 2, \ldots, m - 1$ are i-like and in their increasing* (*or decreasing*) *order, then the points of D are cogeodesic and*

$$s(P_1, P_m) = \sum_{k=1}^{m=1} s(P_k, P_{k+1}).$$

(d) *If P_1P_2 and P_2P_3 are i-like and the i-th coordinates of P_1, P_2, P_3 are monotone, then P_1P_3 is i-like.*

We refer the interested reader to Kalmanson [**2**] for the proof of this lemma.

Proposition 3 is now proved for S^2 as follows: Let $i = 1$ and let $P_{i_1}, P_{i_2}, P_{i_3}$ and P_{i_4} denote four distinct points of D such that $i_1 < i_2 < i_3 < i_4$; without loss of generality let $i_1 = 1$, $i_2 = 2$, etc. Consider all possible segments determined by these four points, P_iP_j. Our first assertion is that only the segment P_1P_2 may not be 1-like; that is, all other segments must be 1-like. For example, if P_1P_3 is not 1-like, then both P_1P_2 and P_2P_3 are 1-like. By our labelling procedure, this contradicts Lemma 11(d), above. Hence, we need only consider the following cases:

Case 1: Each of P_1P_2, P_3P_4, and P_2P_3 are 1-like. By our labelling procedure and Lemma 11(c), P_1, P_2, P_3, P_4 lie on an s-geodesic and are labelled in their geodesic order.

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Case 2: Exactly one of P_1P_2, P_2P_3, P_3P_4 are not 1-like. As all the cases are similar, we will consider only the cases where just P_1P_2 is not 1-like. By Lemma 11(c),

$$\overline{P_1P_3} + \overline{P_3P_4} = \overline{P_1P_4}, \text{ and}$$
$$\overline{P_2P_3} + \overline{P_3P_4} = \overline{P_2P_4}.$$

The triangle inequality yields

$$\overline{P_1P_2} \leqq \overline{P_1P_3} + \overline{P_3P_2}.$$

Hence,

$$\begin{aligned}\overline{P_1P_2} + \overline{P_3P_4} &= \overline{P_1P_2} + (\overline{P_2P_4} - \overline{P_2P_3}) = (\overline{P_1P_2} - \overline{P_2P_3}) + \overline{P_2P_4} \\ &\leqq \overline{P_1P_3} + \overline{P_2P_4} = \ldots (\overline{P_2P_4} - \overline{P_3P_4}) + (\overline{P_2P_3} + \overline{P_3P_4}) \\ &= \overline{P_1P_4} + \overline{P_3P_4},\end{aligned}$$

as required.

Case 3: Suppose that both P_1P_2 and P_3P_4 are not 1-like. (Note that by the definition of a tracklike distribution, this is the only remaining case.) Then, P_1, P_2, P_3, and P_4 are the vertices of a convex quadrilateral in E_2 with P_1P_2 and P_3P_4 as opposite sides. Moreover,

$$\overline{P_1P_3} + \overline{P_2P_4} = \overline{P_1P_4} + \overline{P_2P_3},$$

since all of the segments in question are 1-like, as well as the segments P_iQ, $i = 1, \ldots, 4$ where Q is the point of intersection of the diagonals of the quadrilateral. Since the diagonals give the greatest sum with any norm, we are done.

The proofs of Propositions 3 and 4 are completed as follows: For a distribution D in S^n, map D into points of the $i - j$ S^2 plane via a projection. If the distribution D is taken in a Minkowski plane with polygonal unit circle U, map the points of D into the plane via an affine transformation taking the ith sides of U onto a pair of parallel sides of the unit square, that is, the unit circle of S^2. Both of these mappings preserve i-like chains of points and, hence, preserve the relevant metric inequalities and equations of the preceeding proof.

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Montclair State College,
Upper Montclair, New Jersey

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