GRAVITY EFFECTS IN THE INITIAL VALUE PROBLEMS OF WATER WAVE THEORY

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1. Introduction

The theory described in this paper is directed towards obtaining a general expression for the development of the free surface of a fluid, subsequent to a given initial state and prescribed boundary conditions, as a power series in g, the gravitational acceleration. In an earlier paper [4], a result, applicable to the particular case of the entry of a thin wedge into an incompressible fluid, was obtained and gave the shape of the free surface as such a power series. This series was valid for values of the ratio $ut/x < 1$, where u was the (constant) velocity of entry of the wedge; x the horizontal distance from the vertex of the wedge; and t the time elapsed after entry. This particular problem was first investigated by Mackie [6] who derived an asymptotic solution.

Here, we consider the more general problem of the relationship between the behaviour of the free surface, due to the development of two-dimensional gravity waves on a semi-infinite body of liquid, and the horizontal velocity $U(y, t)$, on $x = 0$, of a vertical wave-making agency. A solution of this problem, for a general velocity distribution $U(y, t)$ was first obtained by Mackie [5]. The method of solution used by Mackie is the basis of the theory in § 2 and his fundamental result is contained in that section. However, our prime concern is with the derivation of a power series, in g, for the shape of the free surface resulting from the onset of a velocity distribution $U(y, t)$ on $x = 0$.

2. Statement and formal solution of the problem

We consider only the motion of the fluid to the right of the wave-maker, i.e. $x > 0$. Taking the y-axis vertically downwards and letting $\varphi(x, y, t)$ denote the velocity potential, $\eta(x, t)$ the free surface displacement, and ρ the density of the liquid, we then have the following initial value problem for the determination of η:

$$\varphi_{xx}+\varphi_{yy} = 0, \tag{1}$$

with

$$\varphi_x = U(y, t) \text{ on } x = 0, \tag{2}$$

$$\eta_t = \varphi_y \quad \text{ on } y = 0, \tag{3}$$

together with

$$\varphi = 0 \quad \text{ at } t = 0, \tag{4}$$

and

$$\eta = 0 \quad \text{ at } t = 0. \tag{5}$$

Let us take an even Fourier transform in x and a Laplace transform in t and write

$$\bar{\tilde{f}}(k, y, p) = \int_0^\infty dx \cos(kx) \int_0^\infty dt\, e^{-pt} f(x, y, t). \tag{6}$$

Where applicable, we will use the notation

$$\bar{f}(k, y, t) = \int_0^\infty f(x, y, t) \cos(kx)\, dx \tag{7}$$

and

$$\tilde{f}(x, y, p) = \int_0^\infty f(x, y, t)\, e^{-pt}\, dt. \tag{8}$$

Transforming equations (1), (2) and (4) by means of (6) we find

$$\frac{d^2\bar{\tilde{\varphi}}}{dy^2} - k^2\bar{\tilde{\varphi}} = \tilde{U}(y, p). \tag{9}$$

Following Mackie [5], (9) can be solved in terms of $\bar{\tilde{\varphi}}(k, o, p)$ when the condition of boundedness at $y = \infty$ is used. The solution is

$$\bar{\tilde{\varphi}}(k, y, p) = \bar{\tilde{\varphi}}(k, o, p)e^{-ky} + \int_0^\infty \tilde{U}(\alpha, p)G(k, \alpha, y)d\alpha$$

where $G(k, \alpha, y)$, a Green's function, is given by

$$\begin{aligned} G(k, \alpha, y) &= -e^{-ky} \sinh k\alpha/k && (\alpha < y), \\ &= -e^{-k\alpha} \sinh ky/k && (\alpha > y). \end{aligned}$$

Thus,

$$\left(\frac{d\bar{\tilde{\varphi}}}{dy}\right)_{y=0} = -k\bar{\tilde{\varphi}}\,(k, o, p) - \int_0^\infty \tilde{U}(\alpha, p)\, e^{-k\alpha}\, d\alpha. \tag{10}$$

The transformation of equation (3) by (6) yields

$$p\bar{\tilde{\eta}} = \left(\frac{d\bar{\tilde{\varphi}}}{dy}\right)_{y=0}.$$

This last equation, together with (10), gives

$$\tilde{\bar{\eta}}(k, p) = -\frac{p}{p^2+kg}\int_0^\infty \bar{U}(\alpha, p)e^{-k\alpha}d\alpha. \tag{11}$$

Equation (11) is the fundamental result of this paper in that the equation of the free surface $\eta(x, t)$ can be recovered by inversion. The order, in which the two inversions involved are performed, determines the form of expression of the free surface profile.

It is easily seen that effecting the Laplace inversion first yields Mackie's [5] basic result

$$\tilde{\eta}(k, t) = -\int_0^t \tilde{U}(k, \tau)\cos\sqrt{gk}\,(t-\tau)d\tau \tag{12}$$

where

$$\tilde{U}(k, \tau) = \int_0^\infty U(\alpha, \tau)\, e^{-k\alpha}d\alpha. \tag{13}$$

3. The surface profile as a power series in g

In the previous section, equation (12) was obtained from equation (11) by Laplace inversion. In this section, however, we will consider the effect of taking, first, the inverse Fourier transform of equation (11). This can be obtained in terms of the Whittaker functions $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ (see, for example, Erdélyi [1a], pp. 264—267) and we find (Erdélyi [2], p. 137) for $|\arg p^2| < \pi$, Re $\alpha > 0$,

$$\begin{aligned}\bar{\eta}(x, p) = \frac{-p}{\pi g}\int_0^\infty \bar{U}(\alpha, p)\left\{\sqrt{\left(\frac{q}{\alpha-ix}\right)}\,\frac{e^{(p^2(\alpha-ix))/2g}}{p}\,W_{-\frac{1}{2},o}\left(\frac{p^2(\alpha-ix)}{g}\right)\right.\\ \left.+\sqrt{\left(\frac{g}{\alpha+ix}\right)}\,\frac{e^{(p^2(\alpha+ix))/2g}}{p}\,W_{-\frac{1}{2},o}\left(\frac{p^2(\alpha+ix)}{g}\right)\right\}d\alpha.\end{aligned}$$

If this last equation is written in the form

$$\bar{\eta}(x, p) = \frac{-p}{\pi g}\int_0^\infty \bar{U}(\alpha, p)\bar{F}(\alpha, p)\,d\alpha \tag{15}$$

it is seen that, on using the convolution theorem of the Laplace transform,

$$\eta(x, t) = -\frac{1}{\pi g}\int_0^\infty\int_0^t U(\alpha, t-\tau)\left\{\frac{\partial F(\alpha, \tau)}{\partial \tau}+F(\alpha, o)\delta(\tau)\right\}d\tau d\alpha, \tag{16}$$

where $\delta(\tau)$ is the Dirac delta function.

Thus, we require a function $F(\alpha, t)$ where

$$\tilde{F}(\alpha, p) = \frac{2}{\sqrt{\beta}} \frac{e^{(\beta p^2)/8}}{p} W_{-\frac{1}{2}, 0}\left(\frac{\beta p^2}{4}\right) + \frac{2}{\sqrt{\beta^*}} \frac{e^{(\beta^* p^2)/8}}{p} W_{-\frac{1}{2}, 0}\left(\frac{\beta^* p^2}{4}\right),$$
$$\beta = \frac{4}{g}(\alpha - ix), \tag{17}$$

and β^* is the conjugate complex of β. We find ([2], p. 215) that, for $\operatorname{Re} p > 0$, $\operatorname{Re} \beta > 0$,

$$F(\alpha, t) = \frac{4}{\beta^{\frac{1}{4}} t^{\frac{1}{2}}} e^{(-t^2)/2\beta} M_{\frac{1}{4}, \frac{1}{4}}\left(\frac{t^2}{\beta}\right) + \frac{4}{\beta^{*\frac{1}{4}} t^{\frac{1}{2}}} e^{(-t^2)/2\beta^*} M_{\frac{1}{4}, \frac{1}{4}}\left(\frac{t^2}{\beta^*}\right). \tag{18}$$

Now, $F(\alpha, t)$ may be written in terms of the error function, Erf (z), by means of the relation ([1a], pp. 264—266)

$$M_{\frac{1}{4}, \frac{1}{4}}(z^2) = i z^{\frac{1}{2}} e^{-z^2/2} \operatorname{Erf}(-iz) \tag{19}$$

where

$$\operatorname{Erf}(z) = \int_0^z e^{-t^2} dt.$$

Then, using the fact ([1b], p. 147) that

$$\operatorname{Erf}(z) = e^{-z^2} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(\frac{3}{2})_n} \tag{20}$$

where $(\frac{3}{2})_n = \Gamma(\frac{3}{2}+n)/\Gamma(\frac{3}{2})$, we have, from (18), (19) and (20),

$$F(\alpha, t) = 4 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(\frac{3}{2})_n} \left\{\frac{1}{\beta^{n+1}} + \frac{1}{\beta^{*n+1}}\right\} \tag{21}$$

where β is given by equation (17). Hence, from (16),

$$\eta(x, t) = -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n g^n}{(\frac{3}{2})_n 2^{2n}} \int_0^{\infty} \int_0^t (2n+1) \tau^{2n} U(\alpha, t-\tau) \left\{\frac{1}{(\alpha - ix)^{n+1}} + \frac{1}{(\alpha + ix)^{n+1}}\right\} d\tau \, d\alpha. \tag{22}$$

Equation (22) determines, for a given velocity $U(y, t)$ of the wave-maker, the shape of the free surface as a power series in g.

4. Particular series

The following particular cases of (22) are readily determined.

i) If a thin wedge, of angle 2ε, is suddenly plunged, with constant speed u along the y-axis, into the liquid at rest, then

$$U(y, t) = \varepsilon u\{1 - H(y - ut)\} \tag{23}$$

where $H(y-ut)$ is the Heaviside step function. Substitution of (23) in (22) gives

$$\eta(x, t) = -\frac{\varepsilon u^2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n g^n}{(\frac{3}{2})_n 2^{2n}}\int_0^t (t-\tau)^{2n+1}\left[\frac{1}{(ut-ix)^{n+1}}+\frac{1}{(ut+ix)^{n+1}}\right]d\tau.$$

This agrees with a result obtained earlier (Low [4]) and allows $\eta(x, t)$ to be evaluated as a series of hypergeometric functions.

ii) For an impulsive velocity at $y = 0$,

$$U(y, t) = u\delta(y)\delta(t)$$

and (22) becomes, for $x > 0$,

(24)

$$\eta(x, t) = -\frac{u}{\pi}\lim_{\alpha\to 0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)\cdots 5\cdot 3\cdot 1}\left(\frac{gt^2}{2}\right)^n\left[\frac{1}{(\alpha-ix)^{n+1}}+\frac{1}{(\alpha+ix)^{n+1}}\right]$$
$$= -\frac{2u}{\pi x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+1)\cdots 5\cdot 3\cdot 1}\left(\frac{gt^2}{2x}\right)^{2n+1}.$$

This, in fact, is Lamb's result ([3]), p. 385) for the shape of the free surface due to an initial elevation of the surface confined to the immediate neighbourhood of the origin. In comparing equation (24) with Lamb's result, it should be noted that the factor 2 appearing in (24) is explained by our adoption of the convention that

$$\int_0^{\infty}\delta(\alpha)d\alpha = 1.$$

iii) For harmonic oscillations of a wavemaker located at the origin we have

$$U(y, t) = e^{iwt}H(t)\delta(y), \tag{25}$$

where w is the angular frequency. Then, for $x > 0$, the profile of the free surface is given by

$$\eta(x, t) = -\frac{2}{\pi x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+1)\cdots 5\cdot 3\cdot 1}\left(\frac{g}{2x}\right)^{2n+1}\int_0^t \tau^{4n+2}e^{iw(t-\tau)}\,d\tau.$$

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