SPACINGS BETWEEN SELECTED PRIME DIVISORS
OF AN INTEGER

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Abstract

Given a large integer n, determining the relative size of each of its prime divisors as well as the spacings between these prime divisors has been the focus of several studies. Here, we examine the spacings between particular types of prime divisors of n, such as prime divisors in certain congruence classes of primes and various other subsets of the set of prime numbers.

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1. Introduction

Given a large integer n, determining the size of its distinct prime factors as well as their relative sizes has been the focus of several studies over the past 75 years. For instance, if we let $\omega(n)$ stand for the number of distinct prime divisors of an integer $n \geq 2$ and

$$p_1(n) < p_2(n) < \cdots < p_{\omega(n)}(n) \quad \text{or for short} \quad p_1 < p_2 < \cdots < p_{\omega(n)},$$

be these prime divisors, Erdoős [6] proved in 1946 that, letting $\xi(n) \to \infty$ as $n \to \infty$ and given any small number $\varepsilon > 0$, then

$$e^{e^{k(1-\varepsilon)}} < p_k(n) < e^{e^{k(1+\varepsilon)}} \quad (\xi(n) \leq k \leq \omega(n)) \quad \text{for almost all } n,$$

thereby providing bounds on the size of the kth prime factor for most integers.

Thirty years later, Galambos [7] provided important information on the relative size of the consecutive prime factors of an integer by showing that, given any small $\varepsilon > 0$ and a function $j = j(x)$ which tends to infinity with x in such a manner that $j(x) \leq (1 - \varepsilon) \log \log x$, then, for any fixed real number $z > 1$,

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \leq x : \frac{\log p_{j+1}(n)}{\log p_j(n)} < z \right\} = 1 - \frac{1}{z},$$

thereby providing bounds on the gaps between the successive prime factors of most integers. Further refinements of these results were provided by many others, such as De Koninck and Galambos [1] in 1987, and Granville [8, 9] in 2007.

More recently, in [3, 4], we further explored this topic by examining the spacings between those consecutive prime factors of an integer which are 'close' to one another, in the following sense. Given a real number $\lambda \in (0, 1]$, we introduced the arithmetic function

$$U_\lambda(n) := \#\{i \in [1, \omega(n) - 1] : p_i < p_{i+1}^\lambda\}$$

and proved (see Theorem 1 in [3]) that

$$\sum_{n \le x} U_\lambda(n) = (1 + o(1))\lambda x \log\log x \quad (x \to \infty) \tag{1.1}$$

and that, for any given $\varepsilon > 0$,

$$\lim_{x \to \infty} \frac{1}{x} \#\left\{n \le x : \left|\frac{U_\lambda(n)}{\omega(n)} - \lambda\right| \ge \varepsilon\right\} = 0. \tag{1.2}$$

Here, we examine the spacings between particular types of prime divisors of n, such as prime divisors in certain congruence classes of primes and various other subsets of the set of prime numbers.

2. Setting the table

Let $\wp$ be the set of all primes. From here on, unless indicated otherwise, the letters p, q and π will stand for primes, whereas $\pi(x)$ will stand for the number of primes not exceeding x. Also, we will be using the logarithmic integral $\mathrm{li}(x) := \int_2^x dt/\log t$. Finally, given a real number $x \ge e^{e^e}$, we set

$$Y_1 := Y_1(x) = \exp\{(\log x)^{\varepsilon(x)}\} \quad \text{and} \quad Y_2 := Y_2(x) = \exp\{(\log x)^{1-\varepsilon(x)}\}, \tag{2.1}$$

where

$$\varepsilon(x) := \frac{1}{2}\frac{\log\log\log x}{\log\log x}.$$

Let $\mathcal{B}$ be a subset of $\wp$ whose counting function $B(x) := \#\{p \le x : p \in \mathcal{B}\}$ is such that, for some real number $\beta > 0$,

$$B(x) = \beta\,\mathrm{li}(x) + O\left(\frac{x}{\log^3 x}\right), \tag{2.2}$$

so that in particular, for some constant β_2,

$$\sum_{\substack{p \le x \\ p \in \mathcal{B}}} \frac{1}{p} = \beta \log\log x + \beta_2 + O\left(\frac{x}{\log x}\right), \tag{2.3}$$

$$\sum_{\substack{u < p < v \\ p \in \mathcal{B}}} \frac{\log p}{p} = \beta \log \frac{v}{u} + O(1),$$

$$\prod_{\substack{u < p < v \\ p \in \mathcal{B}}} \left(1 - \frac{1}{p}\right) = \left(\frac{\log u}{\log v}\right)^{\beta} \left(1 + O\left(\frac{1}{\log u}\right)\right).$$

Any such set $\mathcal{B}$ satisfying (2.2) will be called a *B-set*.

Observe that given a *B*-set and its associated function $\omega_{\mathcal{B}}(n) := \sum_{p|n,\, p \in \mathcal{B}} 1$, it is immediate that

$$\sum_{n \le x} \omega_{\mathcal{B}}(n) = \beta x \log \log x + O(x)$$

and one can show that it follows from the Turán–Kubilius inequality that

$$\sum_{n \le x} (\omega_{\mathcal{B}}(n) - \beta \log \log n)^2 \ll x \log \log x,$$

implying that the average value and the normal order of the function $\omega_{\mathcal{B}}(n)$ are each $\beta \log \log n$. Moreover, one can establish that, for some positive constant β_3,

$$\lim_{x \to \infty} \frac{1}{x} \#\left\{n \le x : \frac{\omega_{\mathcal{B}}(n) - \beta \log \log n}{\beta_3 \sqrt{\log \log n}} < y\right\} = \Phi(y),$$

where $\Phi(y) := (1/\sqrt{2\pi}) \int_{-\infty}^{y} e^{-t^2/2}\, dt$.

Now, given positive real numbers $u < v$, let us set

$$Q(u, v) := \prod_{\substack{u < p < v \\ p \in \wp}} p.$$

Also, given an integer $n > 1$ and a prime divisor p of n which is smaller than $P(n)$, the largest prime divisor of n, we set

$$v_p = v_p(n) := \min\{q \mid n : q > p\},$$

so that

$$\left(\frac{n}{p v_p}, Q(p, v_p)\right) = 1.$$

Now, consider the arithmetic function

$$U_{\lambda, \mathcal{B}}(n) := \sum_{\substack{p|n \\ p \in \mathcal{B} \\ \log p / \log v_p(n) < \lambda}} 1.$$

The technique we used in [3] to prove (1.1) can also be used to determine the mean value of the function $U_{\lambda, \mathcal{B}}(n)$ and in fact the following more accurate result.

THEOREM 2.1. *Given $\lambda \in (0, 1]$ and a set of primes $\mathcal{B}$ satisfying (2.2),*

$$\sum_{n \leq x} U_{\lambda,\mathcal{B}}(n) = (1 + o(1))\lambda \beta x \log \log x \quad (x \to \infty) \tag{2.4}$$

and, for an arbitrarily small $\varepsilon > 0$,

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \leq x : \left| \frac{U_{\lambda,\mathcal{B}}(n)}{\omega(n)} - \lambda \beta \right| \geq \varepsilon \right\} = 0.$$

Estimates (1.1) and (1.2) can be modified to hold for shifted primes. Indeed, we also have the following result.

THEOREM 2.2. *Given $\lambda \in (0, 1]$, a set of primes $\mathcal{B}$ satisfying (2.2) and a fixed integer $a \neq 0$,*

$$\frac{1}{\pi(x)} \sum_{p \leq x} U_{\lambda,\mathcal{B}}(p + a) = (1 + o(1))\lambda \beta \log \log x \quad (x \to \infty)$$

and, moreover, for any arbitrarily small $\varepsilon > 0$,

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \# \left\{ p \leq x : \left| \frac{U_{\lambda,\mathcal{B}}(p + a)}{\omega(p + a)} - \lambda \beta \right| \geq \varepsilon \right\} = 0.$$

3. An extension of Theorem 2.1

Given $\lambda \in (0, 1]$ and a set of primes $\mathcal{B}$ satisfying (2.2), we introduce the arithmetic function $\widetilde{U}_{\lambda,\mathcal{B}}(n)$ which counts the number of prime divisors p of n which belong to $\mathcal{B}$ and for which the next prime divisor q of n also belonging to $\mathcal{B}$ satisfies $\log p / \log q < \lambda$. In short, setting $Q_{\mathcal{B}}(u, v) := \prod_{u < p < v, \, p \in \mathcal{B}} p$, we can write

$$\widetilde{U}_{\lambda,\mathcal{B}}(n) := \sum_{\substack{p|n \\ \log p/\log q < \lambda \\ p,q \in \mathcal{B} \\ (n/pq, Q_{\mathcal{B}}(p,q))=1}} 1.$$

In the case of the function $\widetilde{U}_{\lambda,\mathcal{B}}(n)$, we can obtain an asymptotic formula for its average value with greater accuracy than the one obtained for the function $U_{\lambda,\mathcal{B}}(n)$ (whose average value was revealed through the estimate (2.4)). Indeed, we can prove the following result.

THEOREM 3.1. *Let $\lambda \in (0, 1]$ and $\mathcal{B}$ be a set of primes satisfying (2.2). Then*

$$\sum_{n \leq x} \widetilde{U}_{\lambda,\mathcal{B}}(n) = \beta \lambda^{\beta} x \log \log x + O(x \log \log \log x). \tag{3.1}$$

Moreover, for any arbitrarily small $\varepsilon > 0$,

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \leq x : \left| \frac{\widetilde{U}_{\lambda,\mathcal{B}}(n)}{\omega(n)} - \beta \lambda^{\beta} \right| \geq \varepsilon \right\} = 0. \tag{3.2}$$

Before we begin the proof of Theorem 3.1, let us recall a lemma from our recent paper [4].

LEMMA 3.2 [4, Lemma A]. *Let $x \geq e^{e^{100}}$ and let $Y_1 = Y_1(x)$ and $Y_2 = Y_2(x)$ be the functions defined in (2.1). Let $\pi_1 < \pi_2 < \cdots < \pi_s$ be s primes located in the interval (Y_1, Y_2). Write their product as $R = \pi_1\pi_2\cdots\pi_s$. Further, set*

$$\eta := \sum_{i=1}^{s} \frac{1}{\pi_i}$$

and

$$S_R(x) := \sum_{\substack{n \leq x \\ (n,R)=1}} 1.$$

Assume that $\eta \leq K$, where K is an arbitrary number, and let h be a positive integer satisfying $h \geq 3e^2 K$. Then, letting ϕ stand for the Euler totient function,

$$\left| S_R(x) - \frac{\phi(R)}{R} x \right| \leq x\,(3e)^{-h} + 2Y_2^h,$$

so that in particular, choosing $h = \lfloor \log\log\log x \rfloor$, there exists a positive constant c such that

$$\left| S_R(x) - \frac{\phi(R)}{R} x \right| \leq \frac{c\,x}{(\log\log x)^2}.$$

REMARK 3.3. Observe that, as detailed in [3], the choice of h in Lemma 3.2 is optimal as it ensures that each of the two terms $x(3e)^{-h}$ and $2Y_2^h$ is much less than $x/(\log\log x)^2$.

We are now ready for the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Let $(p, q) \in \mathcal{B} \times \mathcal{B}$, with $p < q$, be a pair of prime divisors of an integer n which satisfy

$$\left(\frac{n}{pq}, Q_{\mathcal{B}}(p, q) \right) = 1.$$

Let us first assume that $Y_1 < p < q < Y_2$, where $Y_1 = Y_1(x)$ and $Y_2 = Y_2(x)$ are the functions defined in (2.1). Using Lemma 3.2 and estimate (2.3),

$$\sum_{n \leq x} \widetilde{U}_{\lambda,\mathcal{B}}(n) = \sum_{\substack{Y_1 < p < q < Y_2 \\ p,q \in \mathcal{B} \\ \log p/\log q < \lambda}} \frac{x}{pq} \prod_{\substack{p < \pi < q \\ \pi \in \mathcal{B}}} \left(1 - \frac{1}{\pi} \right) \left(1 + O\left(\frac{1}{\log p} \right) \right)$$

$$= \sum_{\substack{Y_1 < p < q < Y_2 \\ p,q \in \mathcal{B} \\ \log p/\log q < \lambda}} \frac{x}{pq} \left(\frac{\log p}{\log q} \right)^{\beta} \left(1 + O\left(\frac{1}{\log p} \right) \right)$$

$$= x \sum_{\substack{Y_1 < p < q < Y_2 \\ p,q \in \mathcal{B} \\ \log p / \log q < \lambda}} \frac{(\log p)^\beta}{p} \cdot \frac{1}{q(\log q)^\beta}\left(1 + O\left(\frac{1}{\log p}\right)\right). \tag{3.3}$$

First assume that $\beta < 1$. In this case, fixing q and summing over p,

$$\sum_{\substack{Y_1 < p < q^\lambda \\ p \in \mathcal{B}}} \frac{(\log p)^\beta}{p} = \int_{Y_1}^{q^\lambda} \frac{(\log u)^\beta}{u} \, dB(u)$$

$$= \beta \int_{Y_1}^{q^\lambda} \frac{(\log u)^\beta}{u \log u} \, du + \int_{Y_1}^{q^\lambda} \frac{(\log u)^\beta}{u} \, d(B(u) - \beta \mathrm{li}(u))$$

$$= I_1(x) + I_2(x), \tag{3.4}$$

say. On the one hand, recalling that $\beta > 0$ (see (2.2)),

$$I_1(x) = \beta \int_{\log Y_1}^{\lambda \log q} v^{\beta-1} \, dv = \lambda^\beta (\log q)^\beta + O((\log Y_1)^\beta). \tag{3.5}$$

On the other hand,

$$I_2(x) \ll (B(u) - \beta \mathrm{li}(u)) \frac{(\log u)^\beta}{u}\Big|_{Y_1}^{q^\lambda} - \int_{Y_1}^{q^\lambda} \frac{d}{du}\left(\frac{(\log u)^\beta}{u}\right) du$$

$$\ll (\log q)^{\beta-2} + \int_{Y_1}^{q^\lambda} \frac{u}{\log^2 u} \cdot \frac{\beta(\log u)^{\beta-1} - (\log u)^\beta}{u^2} \, du$$

$$\ll (\log q)^{\beta-2} + \int_{\log Y_1}^{\lambda \log q} v^{\beta-2} \, dv$$

$$\ll (\log q)^{\beta-2} + (\log q)^{\beta-1} \ll (\log q)^{\beta-1}. \tag{3.6}$$

Gathering estimates (3.5) and (3.6) in (3.4), we obtain, in the case $\beta < 1$,

$$\sum_{\substack{Y_1 < p < q^\lambda \\ p \in \mathcal{B}}} \frac{(\log p)^\beta}{p} = \lambda^\beta (\log q)^\beta + O((\log q)^{\beta-1}) + O((\log Y_1)^\beta). \tag{3.7}$$

Finally, it is immediate that (3.7) also holds in the case $\beta = 1$.

In light of estimate (3.7), the sum on the right-hand side of (3.3) can therefore be replaced by

$$\lambda^\beta \sum_{\substack{Y_1 < q < Y_2 \\ q \in \mathcal{B}}} \frac{(\log q)^\beta}{q(\log q)^\beta} = \beta \lambda^\beta \log \log x + O(\log \log \log x), \tag{3.8}$$

where we could ignore the contribution of the last error term on the right-hand side of (3.7) because

$$\sum_{Y_1 < q < Y_2} \frac{1}{q(\log q)^\beta} \ll \int_{Y_1}^{\infty} \frac{1}{t(\log t)^{\beta+1}} \, dt \ll \frac{1}{(\log Y_1)^\beta}.$$

Then, in light of (3.8) and with the help of (2.3), it is clear that the estimate (3.3) can be replaced by

$$\sum_{n \le x} \widetilde{U}_{\lambda, \mathcal{B}}(n) = \beta \lambda^\beta x \log \log x + O(x \log \log \log x),$$

thus completing the proof of (3.1).

The proof of (3.2) rests on the evaluation of $\widetilde{U}_{\lambda, \mathcal{B}}(n)^2$, which can be handled using an approach similar to that used to obtain (3.1). We will therefore omit this proof. □

4. Two *B*-sets involving congruence classes

Given an integer $k \ge 3$, let $\ell_1, \ldots, \ell_r$ be the reduced residue system modulo k, with $r = \phi(k)$. Then it is clear that each of the residue classes

$$\mathcal{B}_{\ell_j} := \{p \in \wp : p \equiv \ell_j \pmod{k}\} \quad (j = 1, \ldots, r)$$

satisfies condition (2.2) and is therefore a *B*-set.

A second interesting *B*-set involves the sum-of-digits function $s_q(n)$ which stands for the sum of the base q digits of an integer n (here, $q \ge 2$). First note that it is known (see Mauduit and Rivat [11] as well as Drmota *et al.* [5]) that if $(k, q - 1) = 1$, then there exists a real number $\sigma = \sigma_{q,k} > 0$ such that

$$\frac{1}{\pi(x)} \#\{p \le x : s_q(p) \equiv \ell \pmod{k}\} = \frac{1}{k} + O_{q,k}(x^{-\sigma} \log x). \tag{4.1}$$

Clearly, (4.1) guarantees that if, for a given integer $k \ge 3$, we set

$$\mathcal{B}_\ell := \{p \in \wp : s_q(p) \equiv \ell \pmod{k}\} \quad (\ell = 0, 1, \ldots, k - 1),$$

then $\mathcal{B}_\ell$ is indeed a *B*-set for each $\ell = 0, 1, \ldots, k - 1$.

5. Two *B*-sets involving fractional parts

We first identify a *B*-set involving fractional parts of prime powers. Let $\alpha, \sigma \in (0, 1)$. In what follows, $\{y\}$ stands for the fractional part of y. Using the method of exponential sums initiated in 1940 by Vinogradov [13], one can prove that

$$\sum_{\substack{p \le x \\ \{p^\alpha\} < \sigma}} 1 = \sigma \pi(x) + R(x) \tag{5.1}$$

(see the recent paper of Shubin [12] for a thorough discussion), where $R(x) \ll x^{\theta(\alpha)+\varepsilon}$ with

$$\theta(\alpha) = \begin{cases} 1 - 2\alpha/15 & \text{if } 0 < \alpha \leq 3/5, \\ (4+\alpha)/5 & \text{if } 3/5 < \alpha < 1. \end{cases}$$

In light of (5.1), it is clear that indeed, given $\alpha, \sigma \in (0, 1)$, the set

$$\mathcal{B} = \{p \in \wp : \{p^\alpha\} < \sigma\}$$

is a B-set.

Another B-set involving fractional parts is as follows. Let α be an irrational number. As was shown in 1940 by Vinogradov [13],

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} e^{2\pi i \alpha p} = 0.$$

Consequently, the sequence $(\alpha p)_{p \in \wp}$ is uniformly distributed modulo 1. Thus,

$$E_\alpha(x) := \sup_{0 \leq u < v < 1} \left| \frac{1}{\pi(x)} \sum_{\substack{p \leq x \\ \{\alpha p\} \in [u, v)}} 1 - (v - u) \right| \to 0 \quad \text{as } x \to \infty.$$

Our next theorem will depend on the following result of Harman.

THEOREM 5.1 [10, Theorem 2.2]. *Let $\alpha \in \mathbb{R}$ and suppose that, for integers a, q with $(a, q) = 1$, the inequality*

$$|q\alpha - a| < \frac{1}{q}$$

holds. Let $\delta \in (0, 1]$ be given. Write

$$\chi(\theta) = \begin{cases} 1 & \text{if } \|\theta\| < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any real β and positive integers N, H such that $N^\varepsilon \ll q \ll N^{1-\varepsilon}$ for some $\varepsilon > 0$, we have

$$\sum_{n \leq N} \Lambda(n) \chi(n\alpha + \beta) = 2\delta \sum_{n \leq N} \Lambda(n) + O\left(\frac{N\delta}{H}\right)$$
$$+ O\left(N(\log H)(\log N)^7 \left(\frac{q\delta}{N} + \frac{1}{q} + \frac{1}{N^{1/2}} + \frac{\delta}{N^{1/3}}\right)^{1/2}\right).$$

(Here $\Lambda(n)$ stands for the von Mangoldt function. Also $\|\theta\| := \min_{m \in \mathbb{Z}} |\theta - m|$.)

From Theorem 5.1, one can easily deduce the following result.

THEOREM 5.2. *Let $\varepsilon > 0$ be an arbitrarily small number. Let $\alpha \in (0, 1)$ be an irrational number such that, for every real $x > x_0(\alpha)$, there exists a positive integer*

$q \in [x^\varepsilon, x^{1-\varepsilon}]$ *for which* $\|q\alpha\| < 1/q$. *Then*

$$E_\alpha(x) \ll \exp\left\{ -\frac{\varepsilon}{2}\log x \right\}.$$

REMARK 5.3. Observe that the condition imposed on α in Theorem 5.2 is known to hold for almost all real numbers α and for every irrational algebraic number α.

Given $0 \le u < v < 1$, let α be an irrational number satisfying the conditions of Theorem 5.2 and set $\mathcal{B} := \{p \in \wp : \{\alpha p\} \in [u, v)\}$. Then it follows from Theorem 5.2 that $\mathcal{B}$ is a B-set.

6. Disjoint classification of primes

Given an integer $k \ge 2$, we will now be interested in sets of prime numbers $\wp_1, \wp_2, \ldots, \wp_k$ such that

$$\wp = \wp_1 \cup \wp_2 \cup \cdots \cup \wp_k \cup \mathcal{D} \quad \text{and} \quad \wp_i \cap \wp_j = \emptyset \quad \text{if } i \ne j,$$

where $\mathcal{D}$ is a finite (perhaps empty) set of primes. We assume that, for each $j \in \{1, 2, \ldots, k\}$,

$$\pi_j(x) := \sum_{\substack{p \le x \\ p \in \wp_j}} 1 = \beta_j \operatorname{li}(x) + O\left(\frac{x}{\log^3 x}\right),$$

where each $\beta_j \in (0, 1]$ and $\sum_{j=1}^k \beta_j = 1$. Such a collection of subsets of primes is called a *disjoint classification of primes*, a notion we introduced in [2] a decade ago in order to create large families of normal numbers. Here, we use this concept for a different purpose.

For each $j = 1, 2, \ldots, k$, let E_j be a subinterval of $[0, 1]$. Given a large integer n, we will now count those k-tuples of primes $(q_1, q_2, \ldots, q_k)$ which satisfy the conditions

$$q_1 q_2 \cdots q_k \mid n, \quad \left(\frac{n}{q_1 q_2 \cdots q_k}, Q(q_1, q_k) \right) = 1,$$

$$q_j \in \wp_{\ell_j}, \ q_{j+1} \in \wp_{\ell_{j+1}}, \ \log q_j / \log q_{j+1} \in E_j \quad (j = 1, 2, \ldots, k-1). \tag{6.1}$$

In fact, let $E(n)$ be the number of these primes q_1, that is, primes q_1 for which there exist primes $q_2, \ldots, q_k$ satisfying the conditions listed in (6.1).

By using our usual technique and writing $\lambda(I)$ for the length of the interval I, we can prove that, if we set

$$T(\ell_1, \ell_2, \ldots, \ell_k) := \beta_{\ell_1} \beta_{\ell_2} \cdots \beta_{\ell_k} \lambda(E_1) \lambda(E_2) \cdots \lambda(E_{k-1}),$$

then,

$$\frac{1}{x} \sum_{n \le x} E(n) = T(\ell_1, \ell_2, \ldots, \ell_k) \log\log x + O(\log\log\log x),$$

and moreover that, given any small number $\varepsilon > 0$,

$$\lim_{x \to \infty} \frac{1}{x} \#\left\{ n \le x : \left| \frac{E(n)}{\omega(n)} - T(\ell_1, \ell_2, \ldots, \ell_k) \right| \ge \varepsilon \right\} = 0.$$

An analogous result for shifted primes can also be proved, namely that, for any fixed integer $a \ne 0$,

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \#\left\{ p \le x : \left| \frac{E(p+a)}{\omega(p+a)} - T(\ell_1, \ell_2, \ldots, \ell_k) \right| \ge \varepsilon \right\} = 0.$$

7. The sequence $n^2 + 1$

It is known that if $p \mid n^2 + 1$, then either $p = 2$ or $p \equiv 1 \pmod 4$ (since $(-1/p) = 1$ if and only if $p \equiv 1 \pmod 4$). On the one hand,

$$\sum_{n \le x} \omega(n^2 + 1) = x \log \log x + O(x). \tag{7.1}$$

For the sake of completeness, this can be proved by first observing that

$$\sum_{n \le x} \omega(n^2 + 1) = \left\lfloor \frac{x+1}{2} \right\rfloor + \sum_{\substack{p \le x \\ p \equiv 1 \ (\text{mod } 4)}} \sum_{\substack{n \le x \\ n^2 + 1 \equiv 0 \ (\text{mod } p)}} 1 + O(x), \tag{7.2}$$

where the last term accounts for the contribution of those primes $p \in (x, x^2 + 1]$. Since for each prime $p \equiv 1 \pmod 4$, the congruence $n^2 + 1 \equiv 0 \pmod p$ has exactly two solutions, the number of positive integers $n \le x$ for which $n^2 + 1 \equiv 0 \pmod p$ is equal to $2x/p + O(1)$. Hence,

$$\sum_{\substack{p \le x \\ p \equiv 1 \ (\text{mod } 4)}} \sum_{\substack{n \le x \\ n^2 + 1 \equiv 0 \ (\text{mod } p)}} 1 = \sum_{\substack{p \le x \\ p \equiv 1 \ (\text{mod } 4)}} \left(\frac{2x}{p} + O(1) \right) = 2x \sum_{\substack{p \le x \\ p \equiv 1 \ (\text{mod } 4)}} \frac{1}{p} + O\left(\frac{x}{\log x} \right)$$

$$= 2x \left(\frac{1}{2} \log \log x + O(1) \right) + O\left(\frac{x}{\log x} \right) = x \log \log x + O(x),$$

which, combined with (7.2), proves (7.1).

On the other hand, using estimate (7.1), it follows from the Turán–Kubilius inequality that

$$\sum_{n \le x} \left(\omega(n^2 + 1) - \log \log n \right)^2 \ll x \log \log x. \tag{7.3}$$

In light of (7.1) and (7.3), we may conclude that both the average value and the normal order of the function $\omega(n^2 + 1)$ are $\log \log n$.

For convenience, we will now be using the notation

$$\Lambda(p, q) := \frac{\log p}{\log q} \quad \text{and} \quad Q^{(1)}(p, q) = \prod_{\substack{p < \pi < q \\ \pi \equiv 1 \ (\text{mod } 4)}} \pi.$$

Given a small number $\varepsilon > 0$ and $\lambda \in (\varepsilon, 1)$, let us count those pairs of primes $p < q$ such that

$$pq \mid n^2 + 1, \quad \varepsilon \le \Lambda(p, q) < \lambda, \quad p \equiv q \equiv 1 \pmod 4, \quad \left(\frac{n^2 + 1}{pq}, Q^{(1)}(p, q)\right) = 1.$$
(7.4)

For each such pair of primes p, q, let $E_{p,q}$ be the number of those positive integers $n \le x$ for which conditions (7.4) hold. Using Lemma 3.2, we may write

$$\sideset{}{'}\sum_{Y_1 < p < q < Y_2} E_{p,q} = \sideset{}{'}\sum_{Y_1 < p < q < Y_2} \frac{x}{pq} \prod_{\substack{p < \pi < q \\ \pi \equiv 1 \,(\mathrm{mod}\ 4)}} \left(1 - \frac{1}{\pi}\right) + O(x)$$

$$= \sum_{\substack{Y_1 < q < Y_2 \\ q \equiv 1 \,(\mathrm{mod}\ 4)}} \frac{1}{q \log q} \sum_{\substack{q^\varepsilon < p < q^\lambda \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \frac{\log p}{p} + O(x), \tag{7.5}$$

where the prime ($'$) on the above sums indicates that the summation runs over those pairs of primes p, q which satisfy the conditions listed in (7.4).

It is easily established that

$$\sum_{\substack{q^\varepsilon < p < q^\lambda \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \frac{\log p}{p} = \frac{\lambda - \varepsilon}{2} \log q + O\left(\frac{1}{\log q}\right). \tag{7.6}$$

Using (7.6) in (7.5), we obtain

$$\frac{1}{x} \sideset{}{'}\sum_{Y_1 < p < q < Y_2} E_{p,q} = \frac{\lambda - \varepsilon}{2} \sum_{\substack{Y_1 < q < Y_2 \\ q \equiv 1 \,(\mathrm{mod}\ 4)}} \frac{1}{q} = \frac{\lambda - \varepsilon}{4} \log\log x + O(\log\log\log x). \tag{7.7}$$

Denote by $K_{\varepsilon,\lambda}(n)$ the number of those primes p for which there exists a prime $q > p$ satisfying the list of conditions (7.4). Since the contribution of those primes $p < Y_1$ and $q > Y_2$ to the above sums is relatively small, it follows from (7.7) that

$$\frac{1}{x \log\log x} \sum_{n \le x} K_{\varepsilon,\lambda}(n) = \frac{\lambda - \varepsilon}{4} + O\left(\frac{\log\log\log x}{\log\log x}\right). \tag{7.8}$$

Since ε can be chosen arbitrarily small, it follows from (7.8) that

$$\lim_{x \to \infty} \frac{1}{x \log\log x} \sum_{n \le x} K_{0,\lambda}(n) = \frac{\lambda}{4}.$$

Moreover, one can show that, given any arbitrarily small number $\delta > 0$,

$$\lim_{x \to \infty} \frac{1}{x} \#\left\{n \le x : \left|\frac{K_{0,\lambda}(n)}{\log\log n} - \frac{\lambda}{4}\right| \ge \delta\right\} = 0.$$

We can also prove an analogue of [3, Theorem 5]. Indeed, let $\delta_0, \delta_1, \ldots, \delta_{k-1} \in (0, 1)$ and set $H := \delta_0 \delta_1 \cdots \delta_{k-1}$. Further, let $\mathcal{F}$ be the set of all $(k + 1)$-tuples of primes

$(p_0, p_1, \ldots, p_k)$ where $p_0 < p_1 < \cdots < p_k$ satisfy

$$1 - \delta_j < \Lambda(p_j, p_{j+1}) < 1 \quad (j = 0, 1, \ldots, k - 1).$$

Also let $V_{\delta_0, \delta_1, \ldots, \delta_{k-1}}(n)$ stand for the number of those prime divisors p_0 of $n^2 + 1$ for which

$$p_1 p_2 \cdots p_k \mid n^2 + 1 \quad \text{and} \quad \left(\frac{n^2 + 1}{p_0 p_1 \cdots p_k}, Q(p_0, p_k) \right) = 1 \quad \text{for } (p_0, p_1, \ldots, p_k) \in \mathcal{F}.$$

Then we have the following result.

THEOREM 7.1. *Let $\delta_0, \delta_1, \ldots, \delta_{k-1}$, H and $\mathcal{F}$ be as above. Then, as $x \to \infty$,*

$$\sum_{n \leq x} V_{\delta_0, \delta_1, \ldots, \delta_{k-1}}(n) = (H + o(1))x \log \log x,$$

$$\sum_{n \leq x} V^2_{\delta_0, \delta_1, \ldots, \delta_{k-1}}(n) = (H^2 + o(1))x (\log \log x)^2.$$

REMARK 7.2. It follows from Theorem 7.1 that

$$V_{\delta_0, \delta_1, \ldots, \delta_{k-1}}(n) = (1 + o(1))H \log \log n \quad \text{for almost all } n.$$

On the other hand, most likely, for some positive constant c, the function

$$\frac{V_{\delta_0, \delta_1, \ldots, \delta_{k-1}}(n) - H \log \log n}{c \sqrt{\log \log n}}$$

is distributed according to the normal law. However, in order to prove this, one would need to compute the asymptotic value of the sum $\sum_{n \leq x} V_{\delta_0, \delta_1, \ldots, \delta_{k-1}}(n)^k$ with a good remainder term, for every k. We were not successful in doing so.

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