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Transformation Formulas for Bilinear Sums of Basic Hypergeometric Series

Dedicated to Professor Richard Askey for his 80th birthday

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Abstract. A master formula of transformation formulas for bilinear sums of basic hypergeometric series is proposed. It is obtained from the author's previous results on a transformation formula for Milne's multivariate generalization of basic hypergeometric series of type A with different dimensions and it can be considered as a generalization of the Whipple–Sears transformation formula for terminating balanced ${}_4\phi_3$ series. As an application of the master formula, the one-variable cases of some transformation formulas for bilinear sums of basic hypergeometric series are given as examples. The bilinear transformation formulas seem to be new in the literature, even in the one-variable case.

1 Introduction

Classical orthogonal polynomials (or orthogonal polynomials of (basic) hypergeometric type), which notion was introduced in G. E. Andrews and R. Askey [2], have been investigated in various aspects, and important applications to other branches of mathematics and related areas were found. Askey and Wilson [3] arranged the classical orthogonal polynomials for $q = 1$ in a scheme that soon became known as the *Askey scheme*. A q-Askey scheme was also arranged. The first published version of the Askey scheme is in [14], which was presented at the Bar-Le-Duc conference. For the proofs of various properties of these polynomials, transformation and summation formulas of (basic) hypergeometric series turned out to be useful.

On the other hand, S. C. Milne [15] has introduced a class of multivariate generalizations of basic hypergeometric series that are nowadays called A_n basic hypergeometric series (or basic hypergeometric series in $SU(n+1)$). Transformation and summation formulas for A_n hypergeometric series including their elliptic generalization, and applications to other branches in mathematics and related areas have been investigated by many authors.

Among these results, the author [8] obtained a number of transformation formulas that relate (mainly basic) hypergeometric series of type A with different dimensions (see also [7, 9–11]). In this paper, we propose a master formula (see (3.1)) of transformation formulas for bilinear sums of basic hypergeometric series. We obtain it from our previous results on the Euler transformation formula [8] for basic hypergeometric series of type A with different dimensions, and it can be considered a generalization

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of the Whipple–Sears transformation formula:

$$ (1.1) \quad {}_4\phi_3\left[\begin{matrix} a, b, c, q^{-N} \\ d, e, f, \end{matrix}; q; q\right] = \frac{(e/a, de/bc)_N}{(e, de/abc)_N} {}_4\phi_3\left[\begin{matrix} a, d/b, d/c \\ d, df/bc, de/bc \end{matrix}; q; q\right], \quad (abc = defq^{N-1}) $$

for terminating balanced ${}_4\phi_3$ series. As an application of the master formula (3.1), we give one-variable cases of some transformation formulas for bilinear sums of basic hypergeometric series as examples. The bilinear transformation formulas involve a high degree of freedom of parameters and seem to be new in the literature, even in the one-variable case. What is remarkable is that transformations for strictly multivariate basic hypergeometric series of type A, especially with different dimensions, may shed light on future investigations of basic hypergeometric series and related classical orthogonal polynomials, even in the one variable case. It is expected that the bilinear transformations in this paper will have applications to the moment representations (see Ismail and Stanton [5,6]) and the Poisson kernel (see Rahman [18,19]) for classical orthogonal polynomials. Furthermore, it would be interesting if the bilinear transformations could be useful for a deeper understanding of fundamental properties of classical orthogonal polynomials (see the lecture notes by Askey [1]), such as orthogonality, addition formulas (see Noumi and Mimachi [17]), linearization of the products, connection coefficients, positivity, and, in particular, convolution structures (see Koelink and van der Jeugt [13]).

2 Notation and Terminology

In this section, we give some notation for basic hypergeometric series, and we recall a result of our previous work [8]. In this paper we basically follow the notation from the book by Gasper and Rahman [4]. Let q be a complex number under the condition $0 < |q| < 1$. Define the q-shifted factorial as

$$ (a)_\infty := (a;q)_\infty = \prod_{n\in\mathbb{N}} (1 - aq^n), \quad (a)_k := (a;q)_k = \frac{(a)_\infty}{(aq^k)_\infty} \quad \text{for } k \in \mathbb{C}. $$

Unless otherwise indicated, we omit the basis q. We will write

$$ (a_1, a_2, \dots, a_n)_k := (a_1)_k (a_2)_k \cdots (a_n)_k. $$

We denote the basic hypergeometric series ${}_{n+1}\phi_n$ as

$$ {}_{n+1}\phi_n\left[\begin{matrix} a_0, \{a_i\}_n \\ \{c_i\}_n \end{matrix}; q; u\right] := {}_{n+1}\phi_n\left[\begin{matrix} a_0, a_1, \dots, a_n \\ c_1, \dots, c_n \end{matrix}; q; u\right] = \sum_{k\in\mathbb{N}} \frac{(a_0, a_1, \dots, a_n)_k}{(q, c_1, \dots, c_n)_k} u^k. $$

A ${}_{n+1}\phi_n$ series is called *well-poised* if $a_0 q = a_1 c_1 = \cdots = a_n c_n$. In addition, if $a_1 = q\sqrt{a_0}$ and $a_2 = -q\sqrt{a_0}$, then the ${}_{r+1}\phi_r$ is called *very well-poised*. Throughout this paper, we denote the very well-poised basic hypergeometric series ${}_{r+1}\phi_r$ as ${}_{r+1}W_r$

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series defined as follows:

$$
\begin{aligned}
&{}_{n+1}\phi_n\left[\begin{matrix} a_0, q\sqrt{a_0}, -q\sqrt{a_0}, & a_3, & \dots, & a_n \\ \sqrt{a_0}, \ -\sqrt{a_0}, & a_0q/a_3, & \dots, & a_0q/a_n \end{matrix} ; q ; u\right] \\
&= \sum_{k\in\mathbb{N}} \frac{1-a_0q^2}{1-a_0} \frac{(a_0)_k(a_3)_k\cdots(a_n)_k}{(q)_k(a_0q/a_3)_k\cdots(a_0q/a_n)_k} u^k \\
&:= {}_{n+1}W_n[a_0 ; a_3, \dots, a_n ; q ; u].
\end{aligned}
$$

Our fundamental tool to derive the master formula (3.1) will be the multiple Euler transformation formula for basic hypergeometric series of type A.

Theorem 2.1 ([8, Theorem 1.1])

$$
\begin{aligned}
(2.1)\qquad &\sum_{\gamma\in\mathbb{N}^n} u^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{\substack{1\le i,\\ j\le n}} \frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{\substack{1\le i\le n,\\ 1\le k\le m}} \frac{(b_kx_iy_k)_{\gamma_i}}{(cx_iy_k)_{\gamma_i}} \\
&= \frac{(ABu/c^m)_\infty}{(u)_\infty} \sum_{\delta\in\mathbb{N}^m} \left(\frac{ABu}{c^m}\right)^{|\delta|} \frac{\Delta(yq^\delta)}{\Delta(y)} \\
&\quad \times \prod_{\substack{1\le k,\\ l\le m}} \frac{((c/b_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}} \prod_{\substack{1\le i\le n,\\ 1\le k\le m}} \frac{((c/a_i)x_iy_k)_{\delta_k}}{(cx_iy_k)_{\delta_k}}
\end{aligned}
$$

where $A := a_1a_2\cdots a_n$, $B := b_1b_2\cdots b_m$ *and*

$$
\Delta(x) := \prod_{1\le i<j\le n}(x_i - x_j) \quad \text{and} \quad \Delta(xq^\gamma) := \prod_{1\le i<j\le n}(x_iq^{\gamma_i} - x_jq^{\gamma_j})
$$

are the Vandermonde determinant for the sets of variables $x = (x_1, \dots, x_n)$ *and* $x^\gamma = (x_1q^{\gamma_1}, \dots, x_nq^{\gamma_n})$, *respectively.*

Remark 2.1 In the case $m = n = 1$ and $x_1 = y_1 = 1$, (2.1) reduces to the third Heine transformation formula for basic hypergeometric series ${}_2\phi_1$ (a basic analogue of Euler transformation formula for the Gauss hypergeometric series ${}_2F_1$):

$$
{}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix} ; q ; u\right] = \frac{(abu/c)_\infty}{(u)_\infty} {}_2\phi_1\left[\begin{matrix} c/b, c/a \\ c \end{matrix} ; q ; abu/c\right].
$$

Here we present the multiple Euler transformation formula (2.1) in a slightly different expression than in [8]. We note that (2.1) is valid for any pair of positive integers n and m, and (2.1) is a transformation formula between A_n ${}_{m+1}\phi_m$ series and A_m ${}_{n+1}\phi_n$ series. The definitions and terminology for A_n basic hypergeometric series can be found in [10].

3 Master Formula

In this section, we present the master formula (3.1).

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First, notice that (2.1) is an identity for formal power series of the variable u. Set the homogeneous part in A_n basic hypergeometric series in (2.1) as $\Phi_N^{n,m}$:

$$\Phi_N^{n,m}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_m\\ \{c y_k\}_m\end{matrix}\right) := \sum_{\substack{\gamma\in\mathbb{N}^n,\\ |\gamma|=N}} \frac{\Delta(xq^\gamma)}{\Delta(x)} \times \prod_{\substack{1\le i,\\ j\le n}} \frac{(a_j x_i/x_j)_{\gamma_i}}{(q x_i/x_j)_{\gamma_i}} \prod_{\substack{1\le i\le n,\\ 1\le k\le m}} \frac{(b_k x_i y_k/x_n y_m)_{\gamma_i}}{(c x_i y_k/x_n y_m)_{\gamma_i}}.$$

The multiple Euler transformation (2.1) can be expressed in terms of $\Phi_N^{n,m}$. Namely, it can be stated as

$$\sum_{K\in\mathbb{N}} \Phi_K^{n,m}\left(\begin{matrix}\{a_i\}_{n_1}\\ \{x_i\}_{n_1}\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_{m_1}\\ \{c y_k\}_{m_1}\end{matrix}\right) u^K = \frac{(ABu/c^m)_\infty}{(u)_\infty} \times \sum_{L\in\mathbb{N}} \Phi_L^{m,n}\left(\begin{matrix}\{c/b_k\}_{m_1}\\ \{y_k\}_{m_1}\end{matrix}\middle|\begin{matrix}\{(c/a_i)x_i\}_{n_1}\\ \{c x_i\}_{n_1}\end{matrix}\right)\left(\frac{ABu}{c^m}\right)^L.$$

Now consider the product of the multiple series

$$\left(\sum_{K\in\mathbb{N}} \Phi_K^{n_1,m_1}\left(\begin{matrix}\{a_i\}_{n_1}\\ \{x_i\}_{n_1}\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_{m_1}\\ \{c y_k\}_{m_1}\end{matrix}\right) u^K\right) \times \left(\sum_{L\in\mathbb{N}} \Phi_L^{n_2,m_2}\left(\begin{matrix}\{f/e_p\}_{n_2}\\ \{z_p\}_{n_2}\end{matrix}\middle|\begin{matrix}\{(f/d_s)w_s\}_{m_2}\\ \{f w_s\}_{m_2}\end{matrix}\right)\left(\frac{f^{n_2}u}{DE}\right)^L\right),$$

under the restriction $AB/c^{m_1} = DE/f^{n_2}$. By virtue of multiple Euler transformation (2.1), we obtain the following formal power series identity of variable u under the condition above:

$$\begin{aligned}&\left(\sum_{K\in\mathbb{N}} \Phi_K^{n_1,m_1}\left(\begin{matrix}\{a_i\}_{n_1}\\ \{x_i\}_{n_1}\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_{m_1}\\ \{c y_k\}_{m_1}\end{matrix}\right) u^K\right)\\ &\quad\times \left(\sum_{L\in\mathbb{N}} \Phi_L^{n_2,m_2}\left(\begin{matrix}\{f/e_p\}_{n_2}\\ \{z_p\}_{n_2}\end{matrix}\middle|\begin{matrix}\{(f/d_s)w_s\}_{m_2}\\ \{f w_s\}_{m_2}\end{matrix}\right)\left(\frac{f^{n_2}u}{DE}\right)^L\right)\\ &= \left(\sum_{K\in\mathbb{N}} \Phi_K^{m_1,n_1}\left(\begin{matrix}\{c/b_k\}_{m_1}\\ \{y_k\}_{m_1}\end{matrix}\middle|\begin{matrix}\{(c/a_i)x_i\}_{n_1}\\ \{c x_i\}_{n_1}\end{matrix}\right)\left(\frac{c^{m_1}u}{AB}\right)^K\right)\\ &\quad\times \left(\sum_{L\in\mathbb{N}} \Phi_L^{m_2,n_2}\left(\begin{matrix}\{d_s\}_{m_2}\\ \{w_s\}_{m_2}\end{matrix}\middle|\begin{matrix}\{e_p z_p\}_{n_2}\\ \{f z_p\}_{n_2}\end{matrix}\right) u^L\right).\end{aligned}$$

By taking the coefficient of u^N in the identity above, we arrive at the following theorem.

Theorem 3.1

(3.1)

$$\begin{aligned}&\sum_{K\in\mathbb{N}} \Phi_K^{n_1,m_1}\left(\begin{matrix}\{a_i\}_{n_1}\\ \{x_i\}_{n_1}\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_{m_1}\\ \{c y_k\}_{m_1}\end{matrix}\right)\Phi_{N-K}^{n_2,m_2}\left(\begin{matrix}\{f/e_p\}_{n_2}\\ \{z_p\}_{n_2}\end{matrix}\middle|\begin{matrix}\{(f/d_s)w_s\}_{m_2}\\ \{f w_s\}_{m_2}\end{matrix}\right)\left(\frac{f^{n_2}}{DE}\right)^{N-K}\\ &= \sum_{L\in\mathbb{N}} \Phi_L^{m_1,n_1}\left(\begin{matrix}\{c/b_k\}_{m_1}\\ \{y_k\}_{m_1}\end{matrix}\middle|\begin{matrix}\{(c/a_i)x_i\}_{n_1}\\ \{c x_i\}_{n_1}\end{matrix}\right)\Phi_{N-L}^{m_2,n_2}\left(\begin{matrix}\{d_s\}_{m_2}\\ \{w_s\}_{m_2}\end{matrix}\middle|\begin{matrix}\{e_p z_p\}_{n_2}\\ \{f z_p\}_{n_2}\end{matrix}\right)\left(\frac{c^{m_1}}{AB}\right)^L\end{aligned}$$

under the condition

$$AB/c^{m_1} = DE/f^{n_2}. \tag{3.2}$$

Hereafter, we call (3.1) the *master formula* and (3.2) the *totally balancing condition*.

4 Bilinear Transformation Formulas

In this section, we present three transformation formulas for bilinear sums of basic hypergeometric series.

One of the most remarkable and fundamental features of multiple hypergeometric series is that the homogeneous part of multiple hypergeometric series can be expressed in terms of (multiple) very well-poised hypergeometric series.

Lemma 4.1

$$\Phi_N^{2,m}\left(\begin{matrix} \{a_i\}_2 \\ \{x_i\}_2 \end{matrix} \middle| \begin{matrix} \{b_k y_k\}_m \\ \{c y_k\}_m \end{matrix} \right) = \frac{(a_2)_N}{(q)_N} \frac{(a_2 x_2/x_1)_N}{(x_2/x_1)_N} \prod_{1\le k\le m} \frac{(b_k x_2 y_k)_N}{(c x_2 y_k)_N} \tag{4.1}$$

$$\times {}_{2m+6}W_{2m+5}\Big[q^{-N}/x_2 \,;\, a_1, \{b_k y_k\}_m, a_2/x_2, \{(q^{1-N}/x_2) c^{-1} y_k^{-1}\}_m, q^{-N} \,;\, q \,;\, \frac{c^m q}{a_1 a_2 B} \Big]$$

and

$$\Phi_N^{1,m}\left(\begin{matrix} a \\ \cdot \end{matrix} \middle| \begin{matrix} \{b_k y_k\}_m \\ \{c y_k\}_m \end{matrix} \right) = \frac{(a)_N}{(q)_N} \prod_{1\le k\le m} \frac{(b_k y_k)_N}{(c y_k)_N}.$$

Proof It is not hard to see that in the case when $n = 1$, $\Phi_N^{1,m}$ is the coefficient of u^N in the (1-dimensional) basic hypergeometric series

$${}_{m+1}\phi_m\left[\begin{matrix} a, \{b_k y_k\}_m \\ \{c y_k\}_m \end{matrix} \,;\, q \,;\, u \right].$$

Equation (4.1) can be obtained by setting $\gamma_2 = N - \gamma_1$ and elementary series manipulations. ∎

We mention that the homogeneous part of A_{n+1} basic hypergeometric series $\Phi_N^{n+1,m}$ can be expressed in terms of A_n very well-poised basic hypergeometric series. This fact first appeared as [16, Lemma 1.22] with a slightly different notation. Readers interested in the elliptic case might also have a look at [12, Proposition 3.2] for A_n elliptic hypergeometric series.

By using Lemma 4.1 and some replacements of parameters, we obtain the following transformations for bilinear sums of very well-poised basic hypergeometric series $\Phi_N^{n,m}$ from the master formula (3.1).

The $n_1 = m_1 = n_2 = m_2 = 2$ case of (3.1), *i.e.*,

$$\sum_{K\in\mathbb{N}} \frac{(b/t, c/t, d_1/t, d_2/t, \sigma q, \epsilon, \phi, q^{-N})_K}{(q, 1/t, q/e, q/f, \sigma q/\beta, \sigma q/\gamma, \sigma q/\delta_1, \sigma q/\delta_2)_K} q^K \\ \times {}_{10}W_9\Big[tq^{-K} ; b, c, d_1, d_2, eq^{-K}, fq^{-K}, q^{-K} ; q ; \frac{t^3 q^3}{bcd_1d_2ef}\Big] \\ \times {}_{10}W_9\Big[\sigma q^K ; \beta, \gamma, \delta_1, \delta_2, \epsilon q^K, \phi q^K, q^{K-N} ; q ; \frac{\sigma^3 q^{N+3}}{\beta\gamma\delta_1\delta_2\epsilon\phi}\Big] \\ = \phi^N \frac{(\sigma q/\delta_1\phi, \sigma q/\delta_2\phi, \epsilon, \sigma q/\gamma\phi, \sigma q, \sigma q/\beta\phi)_N}{(\sigma q/\delta_1, \sigma q/\delta_2, \epsilon/\phi, (\sigma q/\gamma, \sigma q/\phi, \sigma q/\beta)_N} \\ \times \sum_{L\in\mathbb{N}} \frac{(tq/cf, tq/bf, tq/d_1f, tq/d_2f, q^{1-N}\epsilon/\phi, q^{-N}\phi/\sigma, \phi, q^{-N})_L}{(q, e/f, tq/f, q/f, q^{-N}\gamma\phi/\sigma, q^{-N}\beta\phi/\sigma, q^{-N}\delta_1\phi/\sigma, q^{-N}\delta_2\phi/\sigma)_L} q^L \\ \times {}_{10}W_9\Big[q^{-L}f/e ; \begin{matrix} tq/ce, tq/be, tq/d_1e, \\ tq/d_2e, fq^{-L}/t, fq^{-L}, q^{-L} \end{matrix} ; q ; \frac{bcd_1d_2efq^{-1}}{t^3}\Big] \\ \times {}_{10}W_9\Big[q^{L-N}\phi/\epsilon ; \begin{matrix} \sigma q/\gamma\epsilon, \sigma q/\beta\epsilon, \sigma q/\delta_1\epsilon, \sigma q/\delta_2\epsilon, \\ q^{-N+L}\phi/\sigma, q^L\phi, q^{L-N} \end{matrix} ; q ; \frac{\beta\gamma\delta_1\delta_2\epsilon\phi q^{-N-1}}{\sigma^3}\Big],$$

while provided with the "totally balancing condition" (3.2), after the parameter substitution, becomes

$$t^3\sigma^3 q^{N+4} = bcd_1d_2ef\beta\gamma\delta_1\delta_2\epsilon\phi.$$

The $n_1 = m_1 = n_2 = 2$ and $m_2 = 1$ case of (3.1),

$$\sum_{K\in\mathbb{N}} \frac{(b/t, c/t, d_1/t, d_2/t, \sigma q, \phi, q^{-N})_K}{(q, 1/t, q/e, q/f, \sigma q/\beta, \sigma q/\delta_1, \sigma q/\delta_2)_K} q^K \\ \times {}_{10}W_9\Big[tq^{-K} ; b, c, d_1, d_2, eq^{-K}, fq^{-K}, q^{-K} ; q ; \frac{t^3 q^3}{bcd_1d_2ef}\Big] \\ \times {}_8W_7\Big[\sigma q^K ; \beta, \delta_1, \delta_2, \phi q^K, q^{K-N} ; q ; \frac{\sigma q^{N+2}}{\beta\delta_1\delta_2\phi}\Big] \\ = \phi^N \frac{(\sigma q, \sigma q/\beta\phi, \sigma q/\delta_1\phi, \sigma q/\delta_2\phi)_N}{(\sigma q/\beta, \sigma q/\phi, \sigma q/\delta_1, \sigma q/\delta_2)_N} \\ \times \sum_{L\in\mathbb{N}} \frac{(tq/bf, tq/cf, tq/d_1f, tq/d_2f, q^{-N}\phi/\sigma, \phi, q^{-N})_L}{(q, e/f, tq/f, q/f, q^{-N}\phi\beta/\sigma, q^{-N}\delta_1\phi/\sigma, q^{-N}\delta_2\phi/\sigma)_L} q^L \\ \times {}_{10}W_9\Big[fq^{-L}/e ; \begin{matrix} tq/ce, tq/be, tq/d_1e, tq/d_2e, \\ fq^{-L}/t, fq^{-L}, q^{-L} \end{matrix} ; q ; \frac{bcd_1d_2efq^{-1}}{t^3}\Big]$$

when combined with the "totally balancing condition" becomes

$$t^3\sigma^2 q^{N+3} = bcd_1d_2ef\beta\delta_1\delta_2\phi.$$

Furthermore, the $n_1 = n_2 = 2$ and $m_1 = m_2 = 1$ case of (3.1) becomes

$$\sum_{K\in\mathbb{N}} \frac{(b/t, d_1/t, d_2/t, \sigma q, \phi, q^{-N})_K}{(q, 1/t, q/e, \sigma q/\beta, \sigma q/\delta_1, \sigma q/\delta_2)_K}$$
$$\times {}_8W_7\Big[\, tq^{-K}\,;\, b, d_1, d_2, eq^{-K}, q^{-K}\,;\, q\,;\, \frac{t^2q^2}{bd_1d_2e}\Big]$$
$$\times {}_8W_7\Big[\, \sigma q^{K}\,;\, \beta, \delta_1, \delta_2, \phi q^{K}, q^{K-N}\,;\, q\,;\, \frac{\sigma^2q^{N+2}}{\beta\delta_1\delta_2\phi}\Big]$$
$$= \phi^N \frac{(\sigma q, \sigma q/\beta\phi, \sigma q/\delta_1\phi, \sigma q/\delta_2\phi)_N}{(\sigma q/\beta, \sigma q/\phi, \sigma q/\delta_1, \sigma q/\delta_2)_N}$$
$$\times {}_6\phi_5\Big[\begin{matrix} tq/be, tq/d_1e, tq/d_2e, q^{-N}\phi/\sigma, \phi, q^{-N} \\ tq/e, q/e, q^{-N}\beta\phi/\sigma, q^{-N}\delta_1\phi/\sigma, q^{-N}\delta_2\phi/\sigma \end{matrix}\,;\, q\,;\, q\Big]$$

together with the "totally balancing condition"

$$t^2\sigma^2q^{N+2} = bd_1d_2e\beta\delta_1\delta_2\phi.$$

Finally, we note that in the case when $n_1 = m_1 = n_2 = m_2 = 1$, (3.1) reduces to the Whipple–Sears transformation formula (1.1).

We propose to consider bilinear transformations as natural generalizations of multivariate hypergeometric transformations that extend Whipple–Sears transformations (1.1). In this sense, the hypergeometric transformations and summations that we have obtained in [8, Sections 5 and 6] (see also [10]), such as multiple Whipple–Watson type transformations between ${}_8W_7$ series and terminating balanced ${}_4\phi_3$ series, Dougall–Jackson summation for terminating balanced ${}_8W_7$ series, and Bailey type transformations for terminating balanced ${}_{10}W_9$ series, can be interpreted as natural generalizations of q-Pfaff–Saalschütz summation formula for terminating balanced ${}_3\phi_2$ series. Besides those bilinear transformations, the master formula (3.1) involves various types of transformation formulas for bilinear and linear sums of multivariate basic hypergeometric series. Furthermore, we can obtain another master formula of alternate type and q-Pfaff–Saalschütz summation type and can deduce a family of transformations for bilinear and linear sums of very well-poised hypergeometric series from them in the similar manner as in this paper. Details and other transformation formulas, including linear sums and multivariate generalizations, will be given elsewhere.

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