Mathematical Structures in Computer Science (2021), **31**, pp. 535–552
doi:10.1017/S0960129521000220

CAMBRIDGE
UNIVERSITY PRESS

PAPER

Metric monads

Jiří Rosický*

Department of Mathematics and Statistics, Masaryk University, Faculty of Sciences, Kotlářská 2, 611 37 Brno, Czech Republic
*Corresponding author. Email: rosicky@math.muni.cz

(Received 3 March 2021; revised 7 August 2021; accepted 12 August 2021; first published online 10 September 2021)

Abstract

We develop universal algebra over an enriched category $\mathcal{K}$ and relate it to finitary enriched monads over $\mathcal{K}$. Using it, we deduce recent results about ordered universal algebra where inequations are used instead of equations. Then we apply it to metric universal algebra where quantitative equations are used instead of equations. This contributes to understanding of finitary monads on the category of metric spaces.

Keywords: Equational theory; monad; enriched category; metric space; poset

1. Introduction

Classical universal algebra specifies algebras by means of finitary operations and equations. Since Bloom (1984) ordered universal algebra, i.e., universal algebra over posets, that uses inequations $\leq$ instead of equations. Our aim is to get a better understanding of metric universal algebra, i.e., universal algebra over metric spaces. Motivated by probabilistic programming, metric universal algebra was introduced in Mardare *et al.* (2016) where quantitative equations $=_\varepsilon$ were used instead of equations. We will show that both cases are instances of a categorical universal algebra where only operations and equations are used.

To explain this, one has to start with Lawvere theories Lawvere (2004) using (X, Y)-ary operations ω interpreted on a set A as mappings $\omega_A : A^X \to A^Y$. Since Y is the coproduct $Y \cdot 1$ of Y copies of the one-element set 1, these operations are nothing else that Y-tuples of X-ary operations. Over posets, one uses finite posets as arities and interprets A^X either as a set of monotone mappings $X \to A$ or as a poset of these monotone mappings. Here, Y is a colimit of copies of the terminal poset 1 and the two-element chain 2, which replaces equations of (X, Y)-ary operations by inequations of X-ary operations. When A^X is taken as a poset and operations ω_A are monotone, the resulting algebras are called coherent in Adámek *et al.* (2011*b*). Similarly, over metric spaces, one uses finite metric spaces as arities and, for a metric space A, interprets A^X as the set of nonexpanding mappings $X \to A$. Now, Y is a colimit of copies of the terminal space 1 and the space 2_ε having two points of distance ε, which replaces equations of (X, Y)-ary operations by quantitative equations of X-ary operations. This corresponds to metric algebras of Weaver (1995). To get the quantitative algebras of Mardare *et al.* (2016), one has to interpret A^X as the metric space of nonexpanding mappings $X \to A$ and take $\omega_A : A^X \to A^Y$ as nonexpanding.

The idea of using (X, Y)-ary operations where arities are objects X and Y of a category $\mathcal{K}$ is due to Linton (1969) who showed that the category of algebras for a monad $T : \mathcal{K} \to \mathcal{K}$ can be given

*Supported by the Grant Agency of the Czech Republic under the grant 19-00902S

by equations between such operations. Here, A^X is the set $\mathcal{K}(X, A)$. The enriched case, where $\mathcal{V}$ is a symmetric monoidal closed category, $\mathcal{K}$ is a $\mathcal{V}$-category, and A^X is the object $\mathcal{K}(X, A)$ of $\mathcal{V}$, was done in Dubuc (1970) where $\mathcal{V}$-equational theories over a $\mathcal{V}$-category were introduced.

To do universal algebra over a category $\mathcal{K}$, one needs a concept of finiteness in $\mathcal{K}$ to be able to speak about finitary operations. The optimal case is when $\mathcal{K}$ is locally finitely presentable, and finite arities are finitely presentable objects in $\mathcal{K}$. In the enriched case, one needs that $\mathcal{V}$ is locally finitely presentable as a closed category, and $\mathcal{K}$ is a locally finitely presentable $\mathcal{V}$-category (see Kelly (1982*b*)). Since the category **Pos** of posets with monotone mappings is locally finitely presentable and cartesian closed, we do not have any problem with finiteness here. But the category **Met** of metric spaces (distance ∞ is allowed because, otherwise, the category would not be complete, or cocomplete), and nonexpanding mappings is symmetric monoidal closed and only locally $\aleph_1$-presentable. So, finite presentability cannot be taken as finiteness, which makes metric universal algebra much more difficult. But finite metric spaces can be characterized as those that are finitely generated w.r.t. isometries. This idea goes back to Gabriel and Ulmer (1971) and was further developed in Adámek and Rosický (1991) and Di Liberti and Rosický (2009). Let us add that the importance of allowing distance ∞ was stressed in Lawvere (1973).

Classical universal algebra can be also described by finitary monads T on the category **Set** of sets. Here, a monad is finitary if T preserves filtered colimits. Indeed, Linton (1966) showed that finitary monads on **Set** correspond to Lawvere theories. For a symmetric monoidal closed category $\mathcal{V}$, which is locally finitely presentable as a closed category, Power (1999) introduced enriched Lawvere theories and showed that they correspond to enriched finitary monads on $\mathcal{V}$. In Nishizawa and Power (2009), this was extended to finitary monads on a locally finitely presentable $\mathcal{V}$-category. This immediately generalizes to every regular cardinal λ and describes λ-ary enriched monads, i.e., enriched monads preserving λ-filtered colimits, on every locally λ-presentable $\mathcal{V}$-category $\mathcal{K}$ provided that $\mathcal{V}$ is locally λ-presentable as a closed category. The resulting enriched Lawvere theories from Nishizawa and Power (2009) do not fit our aim to do universal algebra over a $\mathcal{V}$-category $\mathcal{K}$. Recently, Bourke and Garner (2019) introduced $\mathcal{A}$-pretheories for every small dense subcategory $\mathcal{A}$ of $\mathcal{K}$ and related them to $\mathcal{A}$-nervous enriched monads on $\mathcal{K}$. Their pretheories perfectly suit our needs and describe λ-ary monads on every locally λ-presentable $\mathcal{V}$-category $\mathcal{K}$ provided that $\mathcal{V}$ is locally λ-presentable as a closed category. Indeed, if $\mathcal{A}$ is the full subcategory of λ-presentable objects of $\mathcal{K}$ the $\mathcal{A}$-pretheory of Bourke and Garner (2019) is given by (X, Y)-ary operations and equations; here, X and Y are λ-presentable objects of $\mathcal{K}$. Over **Met**, we can apply Di Liberti and Rosický (2009) and relate equational theories whose operations have finite metric spaces as arities to monads preserving directed colimits of isometries.

Both ordered algebras and quantitative algebras form computationally relevant structures making possible an algebraic approach to the semantics of programming languages and computational effects. Since Moggi (1991), monads play an important role here (see the survey Hyland and Power (2007) and the overview Plotkin and Power (2004)). Add that an important paper relating enriched monads and theories is Lucyshyn-Wright (2016).

For the comfort of (some) readers, we will do the unenriched case at first. In Section 3, we show that λ-ary pretheories of Bourke and Garner (2019) can be presented as equational theories over a category $\mathcal{K}$ introduced in Rosický (1981) (motivated by Linton (1969)). We show how it can be used to prove the result from Bourke and Garner (2019) that λ-ary pretheories over a locally λ-presentable category $\mathcal{K}$ correspond to λ-ary monads on $\mathcal{K}$. Then we apply it to the case when $\mathcal{K}$ is not locally finitely presentable but only locally finitely generated in the sense of Di Liberti and Rosický (2009), which is the case of metric spaces. In Section 4, we do the enriched case and show that it yields some recent results from Adámek *et al.* (2011*b*) and (2011a). In Section 5, we discuss monads on metric spaces in more detail. In particular, we show that monads given by finitary unconditional equational theories of Mardare *et al.* (2017) preserve not only filtered colimits but also sifted ones. The same property has monads given by some $\otimes$-theories, for instance the monad given by normed spaces. But a characterization of monads on **Met** preserving filtered (or sifted) colimits remains unknown.

2. Preliminaries about Categories and Metric Spaces

All needed facts about locally presentable categories can be found in Adámek and Rosický (1994), Gabriel and Ulmer (1971), or Makkai and Paré (1989). In particular, we will freely use that fact that λ-filtered colimits can be substituted by λ-directed ones (see (Adámek and Rosický, 1994, Remark 1.21)). A monad on a locally presentable category will be called λ-*ary* if it preserves λ-filtered colimits, which is equivalent to the preservation of λ-directed colimits. We will also need $\mathscr{M}$-locally generated categories $\mathscr{K}$ where $(\mathscr{E}, \mathscr{M})$ is a λ-convenient factorization system on $\mathscr{K}$ (see Di Liberti and Rosický (2009)). Here, an object A of $\mathscr{K}$ is λ-generated w.r.t. $\mathscr{M}$ if $\mathscr{K}(A, -) : \mathscr{K} \to \mathbf{Set}$ preserves λ-directed colimits of morphisms from $\mathscr{M}$. Now, a cocomplete category $\mathscr{K}$ is $\mathscr{M}$-locally λ-presentable if it has a set $\mathscr{A}$ of λ-generated objects w.r.t. $\mathscr{M}$ such that every object of $\mathscr{K}$ is a λ-directed colimit of objects from $\mathscr{A}$ and morphisms from $\mathscr{M}$. Every $\mathscr{M}$-locally λ-generated category is locally μ-presentable for some $\mu \geq \lambda$. But it does not need to be locally λ-presentable.

All needed facts about enriched categories can be found in Borceux (1994), or Kelly (1982*a*). Like in Kelly (1982*b*), λ-filtered colimits are conical ones. Kelly (1982*b*) is also a reference for locally presentable $\mathscr{V}$-categories. Recall that an object A of a $\mathscr{V}$-category $\mathscr{K}$ is λ-presentable in the enriched sense if its enriched hom-functor $\mathscr{K}(A, -) : \mathscr{K} \to \mathscr{V}$ preserves λ-filtered colimits. And a $\mathscr{V}$-category $\mathscr{K}$ is locally λ-presentable in the enriched sense if it has weighted colimits and a set of λ-presentable objects in the enriched sense such that every object of $\mathscr{K}$ is a λ-filtered colimit of them. Enriched $\mathscr{M}$-locally λ-generated categories were introduced in Di Liberti and Rosický (2009). Here, $(\mathscr{E}, \mathscr{M})$ should be a $\mathscr{V}$-factorization system on a $\mathscr{V}$-category $\mathscr{K}$ and an object A is λ-generated in the enriched sense if $\mathscr{K}(A, -) : \mathscr{K} \to \mathscr{V}$ preserves λ-directed colimits of $\mathscr{M}$-morphisms. Now, a $\mathscr{V}$-category $\mathscr{K}$ is $\mathscr{M}$-locally λ-generated if it has weighted colimits and a set $\mathscr{A}$ of λ-generated objects w.r.t. $\mathscr{M}$ in the enriched sense such that every object of $\mathscr{K}$ is a λ-directed colimit of objects from $\mathscr{A}$ and $\mathscr{M}$-morphisms.

We denote by **Met** the category of (generalized) metric spaces and nonexpanding mappings. This means that the metric d takes values in $[0, \infty]$ and satisfies the usual axioms. A mapping $f : A \to B$ is nonexpanding if $d(x, y) \geq d(fx, fy)$ for every $x, y \in A$. The category **Met** is complete and cocomplete with products given by the sup-metric (max-metric for finite products). Moreover, **Met** is locally $\aleph_1$-presentable (see (Lieberman and Rosický, 2017, Examples 4.5(3))) and $\aleph_1$-presentable metric spaces are precisely those having a cardinality $\leq \aleph_0$ (see (Adámek and Rosický, 2006, Proposition 2.6)). The only $\aleph_0$-presentable object in **Met** is the empty space (see (Adámek and Rosický, 2006, Remark 2.7(1))).

A metric space is *discrete* if all nonzero distances are ∞. Discrete metric spaces are precisely copowers of the terminal metric space 1. The metric space 2_ε has two points of distance ε. In particular, $2_\infty = 1 + 1$. Finite metric spaces 1 and 2_ε form a dense subcategory of **Met**, i.e., every metric space is a canonical colimit of copies of 1 and 2_ε.

Following (Adámek and Rosický, 2006, Remark 2.5(2))), every finite metric space A is finitely generated w.r.t. isometries, which means that $\mathbf{Met}(A, -) : \mathbf{Met} \to \mathbf{Set}$ preserves λ-directed colimits of isometries. In **Met**, we have (surjective, isometry) factorizations (see (Adámek and Rosický, 2006, Examples 3.16)) and, following (Adámek and Rosický, 2006, Remark 2.5(2))), this factorization system is $\aleph_0$-convenient in the sense of Di Liberti and Rosický (2009). Since every metric space is a directed colimit of its finite subspaces, **Met** is isometry-locally finitely generated.

The category **Met** is symmetric monoidal closed with the tensor product $X \otimes Y$ putting the $+$-metric $(d \otimes d)((x, y), (x', y')) = d(x, x') + d(y, y')$ on $X \times Y$. The tensor unit I is the terminal metric space 1. The internal hom equips the hom-set $\mathbf{Met}(X, Y)$ with the sup-metric $d(f, g)$ given by $\sup\{d(fx, gx) \mid x \in X\}$. The category **Met** is locally $\aleph_1$-presentable as a closed category. Since $\mathbf{Met}(I, -) : \mathbf{Met} \to \mathbf{Set}$ creates $\aleph_1$-directed colimits, $\aleph_1$-presentability in the enriched and ordinary sense coincide.

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

Let **PMet** be the category of (generalized) pseudometric spaces and nonexpansive maps. (The difference is just that for a pseudometric we do not require $d(x, y) > 0$ if $x \neq y$.) **Met** is a full reflective subcategory of **PMet**.

Lemma 2.1. *The reflector* $F : \mathbf{PMet} \to \mathbf{Met}$ *preserves finite products.*

Proof. The reflection FX is the metric quotient of X where we identify $x, y \in X$ such that $d(x, y) = 0$. Since the distance on a product is given by the max-metric, $d((x, x'), (y, y')) = 0$ iff $d(x, x') = d(y, y') = 0$. Hence, we get the result. □

This was observed in Adámek *et al.* (2015, Corollary 3.10) for 1-bounded metric spaces.

Proposition 2.2. *In* **Met**, *finite products commute with filtered colimits.*

Proof. Following 2.1, it suffices to prove that finite products commute with filtered colimits in **PMet**. The proof is the same as that in Adámek *et al.* (2015) for 1-bounded metric spaces. In fact, let X be a pseudometric space and $D : \mathscr{D} \to \mathbf{PMet}$ a filtered diagram. Then it suffices to realize that $X \times - : \mathbf{PMet} \to \mathbf{PMet}$ preserves filtered colimits because the diagonal $D \to D \times D$ is final. □

Remark 2.3. General finite limits do not commute with filtered colimits in **Met**. For instance, filtered colimits of monomorphisms are not necessarily monomorphisms. It suffices to take the identity maps $2_\infty \to 2_{\frac{1}{n}}$ whose colimit is $2_\infty \to 1$.

Corollary 2.4. *Let A be a finite discrete metric space. Then its hom-functor*

$$\mathbf{Met}\,(A, -) : \mathbf{Met} \to \mathbf{Met}$$

preserves filtered colimits. In particular, A is finitely presentable in the enriched sense.

Remark 2.5. Nondiscrete finite metric spaces are not finitely presentable in the enriched sense. Indeed, given a finite space (A, d) of more than one element, let d_n be the metric on A given by $d_n(x, y) = d(x, y) + \frac{1}{n}$ for $n = 1, 2, 3, \ldots$. This defines an ω-chain

$$(A, d_1) \xrightarrow{\text{id}} (A, d_2) \xrightarrow{\text{id}} (A, d_3) \cdots$$

with colimit (A, d). Obviously, the identity morphism of (A, d) does not factorize through any of the colimit maps $\text{id} \colon (A, d_n) \to (A, d)$. Hence, (A, d) is not finitely presentable in the enriched sense.

Consequently, finite nondiscrete cotensors do not commute with filtered colimits in **Met**.

3. Algebras in General Categories

The following definition captures the Lawvere–Linton approach to equational theories. Given a category $\mathscr{K}$ and a small dense subcategory $\mathscr{A}$ of $\mathscr{K}$ (this includes that $\mathscr{A}$ is a full subcategory), the $\mathscr{A}$*-nerve* functor $N_{\mathscr{A}} : \mathscr{K} \to \mathbf{Set}^{\mathscr{A}^{\text{op}}}$ assigns to every object K the domain restriction of $\mathscr{K}(-, K)$ on $\mathscr{A}^{\text{op}}$. Given X in $\mathscr{A}$ and K in $\mathscr{K}$, we will denote the set $\mathscr{K}(X, K)$ as K^X. Similarly, $K^f = \mathscr{K}(f, K)$ and $h^X = \mathscr{K}(X, h)$.

Definition 3.1. (Bourke and Garner (2019)). *Let $\mathscr{K}$ be a category and $\mathscr{A}$ a small dense subcategory of $\mathscr{K}$. An $\mathscr{A}$-pretheory is an identity-on-objects functor $J : \mathscr{A}^{\text{op}} \to \mathscr{T}$.*

*A $\mathscr{T}$-algebra is an object A of $\mathscr{K}$ together with a functor $\hat{A} : \mathscr{T} \to$ **Set** such that $\hat{A} \cdot J = N_{\mathscr{A}}(A)$.*

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

A homomorphism of $\mathscr{T}$-algebras $(A, \hat{A}) \to (B, \hat{B})$ is a morphisms $h : A \to B$ together with a natural transformation $\hat{h} : \hat{A} \to \hat{B}$ such that $\hat{h}J = N_{\mathscr{A}}(h)$.

The resulting category of $\mathscr{T}$-algebras is denoted as **Alg** $(\mathscr{T})$.

Remark 3.2. This definition can be unpacked as follows. Objects of $\mathscr{A}$ are *arities*. Morphisms $\omega : J(X) \to J(Y)$ of $\mathscr{T}$ are (X, Y)-ary operations where X is the *input arity* and Y is the *output arity* of ω. In particular, a morphism $f : Y \to X$ gives the (X, Y)-ary operation $J(f)$. A $\mathscr{T}$-algebra is an object A of $\mathscr{K}$ together with mappings $\omega_A : A^X \to A^Y$ for every $\omega : X \to Y$ in $\mathscr{T}$ such that

(1) $J(f)_A = A^f$, and
(2) $(\tau\omega)_A = \tau_A\omega_A$.

Homomorphism $h : A \to B$ of $\mathscr{T}$-algebras are morphisms $h : A \to B$ such that $h^Y\omega_A = \omega_B h^X$ for every morphism $\omega : X \to Y$ of $\mathscr{T}$.

Let $\mathscr{K} =$ **Set** and $\mathscr{A}$ consist of finite sets. Then a $\mathscr{A}$-pretheories coincide with Lawvere theories. Morphism $J(n) \to J(m)$ are m-tuples of n-ary operations. In the language of universal algebra, Lawvere theories correspond to clones of finitary operations where compositions are finitary multiple compositions, i.e., superpositions of operations. Mappings $1 \to m$ provide variables $x_1, \ldots, x_m$.

In the same way as clones are presented by equational theories, we can present pretheories by equational theories over $\mathscr{K}$.

Definition 3.3. (Rosický (1981)). *Let $\mathscr{K}$ be a category and $\mathscr{A}$ a full subcategory of $\mathscr{K}$. An $\mathscr{A}$-ary type t over $\mathscr{K}$ is a class Ω equipped with a mapping $t : \Omega \to \mathrm{ob}\,(\mathscr{A}) \times \mathrm{ob}\,(\mathscr{A})$ such that $\Omega^{X,Y} = t^{-1}(X, Y)$ are sets for every $X, Y \in \mathrm{ob}\,(\mathscr{A})$. Elements of $\Omega^{X,Y}$ are called (X, Y)-ary operation symbols of type t.*

Terms of type t are inductively defined as follows:

(1) Every $\omega \in \Omega$ is a $t(\omega)$-ary term,
(2) Every morphism $f : Y \to X$ of $\mathscr{A}$ determines an (X, Y)-ary term x_f,
(3) If p is an (X, Y)-ary term and q an (Y, Z)-ary term then qp is an (X, Z)-ary term,
(4) $x_{fg} = x_g x_f$,
(5) $(pq)r = p(qr)$, and
(6) If p is an (X, Y)-ary term then $x_{\mathrm{id}_Y} p = p = p x_{\mathrm{id}_X}$.

Equations $p = q$ of type t are pairs (p, q) of terms of type t. An equational theory of type t is a class E of equations of type t.

Definition 3.4. *An algebra of type t is an object A of $\mathscr{K}$ equipped with mappings $\omega_A : A^X \to A^Y$ for every (X, Y)-ary operation symbol ω of t.*

Terms are interpreted in A as follows:

(1) $(x_f)_A = A^f$, and
(2) $(qp)_A = q_A p_A$.

An algebra A satisfies an equation $p = q$ if $p_A = q_A$. It satisfies a theory E if it satisfies all equations of E.

A homomorphism $h : A \to B$ of t-algebras is a morphism $h : A \to B$ such that $h^Y\omega_A = \omega_B h^X$ for every $\omega \in \Omega^{X,Y}$.

Alg (E) *will denote the category of E-algebras and $U_E :$* **Alg** $(E) \to \mathscr{K}$ *will be the forgetful functor.*

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

Remark 3.5. The definition of an equational theory in Rosický (1981) does not contain conditions (5) and (6). But it does not influence the concept of an algebra because the composition of mappings is associative and x_{id} are interpreted as identities.

Let $\mathscr{T}$ be an $\mathscr{A}$-pretheory and take Ω as the set of morphisms of $\mathscr{T}$ and $t(\omega) = (X, Y)$ for $\omega : J(X) \to J(Y)$. The resulting $\mathscr{A}$-ary type t has the same algebras as $\mathscr{T}$ and its terms coincide with operation symbols.

Conversely, consider an $\mathscr{A}$-ary type t. Let $\tilde{\Omega}$ denote the category of terms of type t where $\tilde{\Omega}(X, Y)$ consists of (X, Y)-ary terms. Compositions are given by (3) and identities are x_{id}. Our definition makes $\tilde{\Omega}$ a category and $x_- : \mathscr{A}^{\mathrm{op}} \to \tilde{\Omega}$ is a functor which is an identity on objects. Then an algebra A is equipped with a functor $\tilde{\Omega} \to \mathbf{Set}$ whose composition with x_- is $\mathscr{K}(-, A)$. Hence, t-algebras coincide with $\tilde{\Omega}$-algebras.

Given an equational theory E of type t, let $\tilde{\Omega}_E = \tilde{\Omega}/E$ be the quotient of $\tilde{\Omega}$ given by E. Again, E-algebras coincide with $\tilde{\Omega}/E$-algebras. Thus, our equational theories and algebras coincide with pretheories and algebras in the sense of Bourke and Garner (2019).

Definition 3.6. *Let $\mathscr{K}$ be a locally λ-presentable category and $\mathscr{K}_\lambda$ denote the (representative) small full subcategory consisting of λ-presentable objects. Then $\mathscr{K}_\lambda$-ary types will be called λ-ary. Similarly, equational theories of λ-ary types are called λ-ary.*

Examples 3.7. (1) Let **Pos** be the category of posets and monotone mappings. **Pos** is a locally finitely presentable category. Consider a type t over **Pos** and $\omega \in \Omega^{X,Y}$ be its (X, Y)-ary operation. Elements of Y correspond to monotone mappings $f : 1 \to Y$. Then terms $x_f\omega$ are usual X-ary operations. For a t-algebra A, A^X is the set of monotone mapping $a : X \to A$. Then $\omega_A : A^X \to A^Y$ is a mapping. We can replace ω by an Y-tuple of X-ary terms $x_f\omega, f : 1 \to Y$. But we have to force that $\omega_A(a) : Y \to A$ is monotone for every monotone $a : X \to A$. This means that $x_f\omega(a) \leq x_g\omega(a)$ for every $f \leq g$, $f, g : 1 \to Y$ and every $a \in A^X$. But this is the same as $x_f\omega \leq x_g\omega$ for every $f \leq g$, $f, g : 1 \to Y$. Of course, $f \leq g$ iff the corresponding elements $y, z \in Y$ satisfy $y \leq z$.

Consequently, we can reduce (X, Y)-ary operations to X-ary operations, but we have to replace equations in equational theories by inequations $p \leq q$ of terms. Conversely, every such inequational theory can be replaced by an equational theory with (X, Y)-ary operations. In fact, given X-ary terms p and q, then an inequation $p \leq q$ corresponds to the existence of a $(X, 2)$-ary operation r such that $x_0 r = p$ and $x_1 r = q$ where 2 is a two-element chain $\{0, 1\}$. Here $0 : 1 \to 2$ has the value 0 and $1 : 1 \to 2$ has the value 1.

Hence, our finitary equational theories over **Pos** correspond to usual inequational theories for ordered algebras (see Adámek *et al.* (2011*b*)).

(2) Consider a type t over **Met**. Analogously to (1), we can reduce (X, Y)-ary operations to X-ary ones, but we have to ensure that $\omega_A(a)$ is nonexpanding for every nonexpanding $a : X \to A$. This means that

$$d(x_f\omega(a), x_g\omega(a)) \leq d(f, g)$$

for every $f, g : 1 \to Y$ and every $a \in A^X$. This is the same as

$$d(x_f\omega, x_g\omega) \leq d(f, g).$$

Of course, $d(f, g)$ is the distance $d(y, z)$ of the corresponding elements.

Consequently, we can reduce (X, Y)-ary operations to X-ary operations but we have to replace equations in equational theories by "quantitative equations" $p =_\varepsilon q$ of terms. Conversely, every such metric equational theory can be replaced by an equational theory with (X, Y)-ary operations. In fact, given X-ary terms p and q, then a quantitative equation $p =_\varepsilon q$ corresponds to the existence of a $(X, 2_\varepsilon)$-ary operation r such that $x_0 r = p$ and $x_1 r = q$.

Hence, our equational theories over **Met** correspond to "metric equational" theories for metric algebras. The special case when the input arities X are finite discrete metric spaces and the output

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

arities Y are finite metric spaces was considered in Weaver (1995), or Hino (2016) where the resulting categories **Alg** (E) were called varieties of metric algebras. If the arities Y are also discrete, usual equations are used.

Theorem 3.8. *Let $\mathcal{K}$ be a locally λ-presentable and E be a λ-ary equational theory over $\mathcal{K}$. Then* **Alg** (E) *is locally λ-presentable and U_E is monadic and preserves λ-filtered colimits.*

Proof. Let Ω consist of a single (X, Y)-ary operation symbol where X and Y are λ-presentable. Then **Alg** (Ω) is the inserter Ins $(-^X, -^Y)$ which is an accessible category (see (Adámek and Rosický, 1994, Theorem 2.72)). Clearly, **Alg** (Ω) is complete and U_Ω preserves limits and λ-directed colimits. Thus, U_Ω has a left adjoint F (see (Adámek and Rosický, 1994, Theorem 1.66)). Since U is conservative and faithful, FZ, Z λ-presentable, form a strong generator of **Alg** (Ω) consisting of λ-presentable objects. Following (Adámek and Rosický, 1994, Theorem 1.20 and Corollary 2.47), **Alg** (Ω) is locally λ-presentable. Since U_Ω creates coequalizers of U_Ω-absolute pairs, U_Ω is monadic.

For a general λ-ary type, **Alg** (Ω) is a multiple pullback of $U_{\{f\}}$, $f \in \Omega$. Since $U_{\{f\}}$ are transportable, this multiple pullback is a multiple pseudopullback (see (Makkai and Paré, 1989, Proposition 5.1.1)). Hence, **Alg** (Ω) is locally λ-presentable (see (Adámek and Rosický, 1994, Theorem 2.72)). Again, U_Ω preserves limits, λ-filtered colimits and creates coequalizers of U_Ω-absolute pairs. Hence, it is monadic.

Finally, **Alg** (E) forms a full reflective subcategory of **Alg** (Ω) closed under λ-filtered colimits. Hence, it is locally λ-presentable (see (Adámek and Rosický, 1994, Corollary 2.48)). The rest is obvious. □

Remark 3.9. We could shorten the proof by using an unpublished Bird (1984) where 2.14 shows that the inserters Ins $(-^X, -^Y)$ are locally λ-presentable.

Corollary 3.10. *Let $\mathcal{K}$ be a locally λ-presentable and E be a λ-ary equational theory over $\mathcal{K}$. Then the corresponding monad T_E is λ-ary.*

The following observation is crucial.

Observation 3.11. Let $U : \mathcal{A} \to \mathcal{K}$ be a faithful functor, i.e., $\mathcal{A}$ is *concrete* over $\mathcal{K}$. For $X, Y \in$ ob $(\mathcal{K})$, let $\Omega^{X,Y}$ consist of natural transformations $U^X \to U^Y$ where $U^X = \mathcal{K}(X, U-)$. Since the collections $\Omega^{X,Y}$ do not need to be sets, the resulting "type" is not legitimate. Ignoring this, we get terms where $x_f = U^f$ and $\omega\omega'$ is the composition. Observe that $\tilde{\Omega}(X, Y) = \Omega^{X,Y}$. This yields the "equational theory" E_U and the functor $H_U : \mathcal{A} \to$ **Alg** (E_U) such that $H_U(A)$ is A equipped with components ω_A as operations.

Assume that U has a left adjoint F. Then the natural transformations $\omega : U^X \to U^Y$ correspond to natural transformations $\mathcal{A}(FX, -) \to \mathcal{A}(FY, -)$ and thus to morphisms $FY \to FX$. Hence, we get a type t_U and an equational theory E_U.

This is due to Linton (1969) and it goes back to Lawvere (2004). The fundamental result of Linton (1969) is that if U is monadic then the comparison functor $H_U : \mathcal{K}^T \to$ **Alg** (E_U) from the category of T-algebras is an equivalence. Moreover, given a T-algebra, elements $a \in A^X$ correspond to morphisms $FX \to A$.

Observe that Linton's result exactly says that the full subcategory consisting of free algebras is dense in the category of algebras. This can be proved as follows. Consider the canonical presentation

$$FUFUA \underset{FU\varepsilon_A}{\overset{\varepsilon_{FUA}}{\rightrightarrows}} FUA \xrightarrow{\varepsilon_A} A$$

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

of a T-algebra A. Let $\varphi_a : FX \to B$ be a cocone from the canonical diagram of free T-algebras to A; here, $a : FX \to A$. Let $\tilde{a} : X \to UA$ be the transpose of a. There is a unique $t : A \to B$ such that $t\varepsilon_A = \varphi_{\varepsilon_A}$. Since

$$ta = t\varepsilon_A F\tilde{a} = \varphi_{\varepsilon_A} F\tilde{a} = \varphi_a,$$

A is a colimit of its canonical diagram.

Theorem 3.12. *Let $\mathcal{K}$ be a locally λ-presentable category and T be a λ-ary monad on $\mathcal{K}$. Then there is a λ-ary equational theory E such that the concrete categories (over $\mathcal{K}$) $\mathcal{K}^T$ and* **Alg** *(E) are equivalent.*

Proof. Following 3.11, $\mathcal{K}^T$ is equivalent to **Alg** (E_U) where $U : \mathcal{K}^T \to \mathcal{K}$ is the forgetful functor. Moreover, (X, Y)-ary operations of type t_U correspond to morphisms $FY \to FX$. Let E be the subtheory of E_U where operations are λ-ary, i.e., X and Y are λ-presentable. We get the reduct functor $R :$ **Alg** $(E_U) \to$ **Alg** (E) forgetting operations which are not λ-ary.

Given $X, Y \in \mathrm{ob}(\mathcal{K})$, we express them as λ-directed colimits $(x_i : X_i \to X)_{i\in I}$ and $(y_j : Y_j \to Y)_{j\in J}$ of λ-presentable objects in $\mathcal{K}$. This yields λ-directed colimits $(Fx_i : FX_i \to FX)_{i\in I}$ and $(Fy_j : FY_j \to FY)_{j\in J}$ in $\mathcal{K}^T$. Since U preserves λ-directed colimits, F preserves λ-presentable objects. Consequently, every $f : FY \to FX$ is a λ-directed colimit of $f_{ji} : FY_j \to FX_{i_j}$. Indeed, given $j \in J$ and $i \in I$, since FY_j is λ-presentable, there is $i_j \geq i$ such that $fFy_j = F(x_{i_j})f_{ij}$. This implies that R is an equivalence.

Indeed, let ω be an (X, Y)-ary operation corresponding to $f : FY \to FX$ and ω_{ij} be (X_{i_j}, Y_j)-ary operations corresponding to $f_{ij} : FY_j \to FX_{i_j}$. We have $A^Y = \lim_j A^{Y_j}$. Since

$$(\omega_{ij})_A A^{x_{i_j}} : A^X \to A^{Y_j}$$

form a cone, there is a unique $\omega_A : A^X \to A^Y$ such that

$$A^{y_j}\omega_A = (\omega_{ij})_A A^{x_{i_j}}.$$

This clearly makes A an E_U-algebra. □

Remark 3.13. (1) We have got a one-to-one correspondence between λ-monads on $\mathcal{K}$ and λ-ary equational theories over $\mathcal{K}$. As explained in the introduction, this result follows from Bourke and Garner (2019), Corollary 21, and Proposition 23.

(2) 3.12 and 3.8 imply that $\mathcal{K}^T$ is locally λ-presentable, which is already in Gabriel and Ulmer (1971).

Definition 3.14. *We say that a type t over $\mathcal{K}$ is (λ', λ)-ary if all arities X are λ'-presentable and all arities Y of its operation symbols are λ-presentable.*

Proposition 3.15. *Let $\mathcal{K}$ be a locally λ-presentable category, $\lambda' \leq \lambda$ be a regular cardinal and E be a (λ', λ)-ary equational theory over $\mathcal{K}$. Then U_E preserves λ'-directed colimits. Hence, the corresponding monad is λ'-ary.*

Proof. Consider a λ-directed diagram $(a_{ij} : A_i \to A_j)_{i\leq j\in I}$ in **Alg** (E). Take $U_E A = \mathrm{colim}\, U_E A_i$ with a colimit cocone $a_i : U_E A_i \to U_E A$ and an (X, Y)-ary t-operation ω. We have a bijection $u : \mathrm{colim}\,(U_E A_i)^X \to (\mathrm{colim}\, U_E A_i)^X$ and a mapping $v : \mathrm{colim}\,(U_E A_i)^Y \to (\mathrm{colim}\, U_E A_i)^Y$. We define $\omega_A : A^X \to A^Y$ as the composition $v \cdot \mathrm{colim}\, \omega_{A_i} \cdot u^{-1}$. It is easy to see that we get a colimit $(\bar{a}_i : A_i \to A)$ in **Alg** (E) with $U\bar{a}_i = a_i$ for every $i \in I$. □

Examples 3.16. (1) Over **Set**, (X, Y)-ary operations coincide with Y-tuples of usual X-ary operations, i.e. $(X, 1)$-ary in our terminology. Thus, we get the well-known correspondence between Lawvere theories and finitary monads in **Set** due to Linton (1966).

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

(2) Since **Pos** is locally finitely presentable, 3.10 and 3.12 imply that finitary monads on **Pos** correspond to finitary inequational theories, which was recently proved in Adámek *et al.* (2011*b*).

(3) Since **Met** is locally $\aleph_1$-presentable, monads preserving $\aleph_1$-directed colimits on **Met** correspond to $\aleph_1$-ary equational theories.

Remark 3.17. Examples 3.16 can be extended. Let E be a λ-ary equational theory over a locally λ-presentable category $\mathscr{K}$ and $\mathscr{A} \subseteq \mathscr{K}_\lambda$ be a dense subcategory consisting of (some) λ-presentable objects. Let E' be a subtheory of E consisting of (X, Y)-ary operations where $Y \in \mathscr{A}$. Then the reduct functor $\mathbf{Alg}\,(E) \to \mathbf{Alg}\,(E')$ is an equivalence.

It suffices to express Y as a colimit $\delta_d : Dd \to Y$ of its canonical diagram $D : \mathscr{D} \to \mathscr{A}$. Then, for an E'-algebra A and an (X, Y)-ary operation ω of E,

$$A^Y = A^{\operatorname{colim} D} \cong \lim A^D$$

with projections $A^{\delta_d} : A^Y \to A^{Dd}$. This yields a unique $\omega_A : A^X \to A^Y$ making A a E-algebra.

In 3.16(1), one takes $\mathscr{A} = \{1\}$, in 3.16(2), $\mathscr{A} = \{1, 2\}$ and, in 3.16(3), $\mathscr{A} = \{1, 2_\varepsilon\}$ with $\varepsilon > 0$.

Definition 3.18. *Let $\mathscr{K}$ be a $\mathscr{M}$-locally λ-generated category in the sense of Di Liberti and Rosický (2009). We say that a type t over $\mathscr{K}$ is λ-ary if all arities X and Y of its operation symbols are λ-generated w.r.t. $\mathscr{M}$. A theory is λ-ary if its type is λ-ary.*

Theorem 3.19. *Let $\mathscr{K}$ be $\mathscr{M}$-locally λ-generated and E be a λ-ary equational theory over $\mathscr{K}$. Then* **Alg** (E) *is locally λ-generated and U_E is both monadic and a morphism of locally λ-generated categories.*

Proof. Following Di Liberti and Rosický (2009), $\mathscr{K}$ is locally μ-presentable for some $\mu \geq \lambda$. The theory E is then μ-ary. Hence, **Alg** (E) is locally μ-presentable and U_E is monadic (see 3.8).

Since $\mathscr{K}$ is $\mathscr{M}$-locally λ-generated, $\mathscr{K}$ is equipped with a λ-convenient factorization system $(\mathscr{E}, \mathscr{M})$. Put $\mathscr{M}' = U_E^{-1}(\mathscr{M})$. Like in the proof of (Di Liberti and Rosický, 2009, Theorem 2.23), $(\operatorname{colim} F(\mathscr{E}), \mathscr{M}')$ is a λ-convenient factorization system in **Alg** (T); here F is left adjoint to U_E. Since U_E is faithful and conservative, F maps a strong generator $\mathscr{G}$ of $\mathscr{K}$ consisting of λ-generated objects w.r.t. $\mathscr{M}$ to a strong generator $F(\mathscr{G})$ in **Alg** (T). Since U_E sends λ-directed colimits of $\mathscr{M}'$-morphisms to λ-directed colimits of $\mathscr{M}$-morphisms, objects of $F(\mathscr{G})$ are λ-generated w.r.t. $\mathscr{M}'$ (see the proof of (Di Liberti and Rosický, 2009, Lemma 3.11)). Following (Di Liberti and Rosický, 2009, Theorem 2.22), **Alg** (T) is $\mathscr{M}'$-locally λ-generated. Clearly, U_E is a morphisms of locally λ-generated categories. □

Remark 3.20. The resulting monad $T_E = U_E F$ is μ-ary. But we do not know that it preserves λ-directed colimits of $\mathscr{M}$-morphisms. This means that it sends λ-directed colimits of $\mathscr{M}$-morphisms to λ-directed colimits. For this, we would need that F sends $\mathscr{M}$-morphisms to $\mathscr{M}'$-morphisms. Then T_E would also preserve $\mathscr{M}$-morphisms.

Theorem 3.21. *Let $\mathscr{K}$ be $\mathscr{M}$-locally λ-generated and locally μ-presentable for $\lambda \leq \mu$. Let T be a μ-ary monad on $\mathscr{K}$ preserving $\mathscr{M}$-morphisms and λ-directed colimits of $\mathscr{M}$-morphisms. Then there is a λ-ary equational theory E such that $\mathscr{K}^T$ and* **Alg** (E) *are equivalent (as concrete categories over $\mathscr{K}$).*

Proof. The proof is analogous to that of 3.12. Only, given $X, Y \in \operatorname{ob}(\mathscr{K})$, we express them as λ-directed colimits $(x_i : X_i \to X)_{i \in I}$ and $(y_j : Y_j \to Y)_{j \in J}$ of λ-generated objects w.r.t. $\mathscr{M}$ and $\mathscr{M}$-morphisms in $\mathscr{K}$. Then $(Fx_i : FX_i \to FX)_{i \in I}$ and $(Fy_j : FY_j \to FY)_{j \in j}$ are λ-directed colimits of $\mathscr{M}'$-morphisms. Since FY_j are λ-generated w.r.t. $\mathscr{M}'$, $f : FY \to FX$ is a λ-directed colimit of $f_i : FY_{j_i} \to X_i$. □

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

Remark 3.22. Since **Met** is isometry-locally finitely generated, every $\aleph_1$-ary monad $T:\mathbf{Met}\to\mathbf{Met}$ preserving isometries and $\aleph_0$-directed colimits of isometries is given by an $\aleph_0$-ary equational theory over **Met**. This means that arities are finite metric spaces. Conversely, every $\aleph_0$-ary equational theory yields an $\aleph_1$-ary monad $T:\mathbf{Met}\to\mathbf{Met}$. But we do not know whether it also preserves isometries and $\aleph_0$-directed colimits of isometries. The next example shows that it does not need to happen.

Example 3.23. Let X be the metric space having points a, b, c with distances $(a, b) = d(b, c) = 1$ and $d(a, c) = 2$ and $Y = 2_1$ be the metric space with two points $0, 1$ having the distance 1. Let $f, g : Y \to X$ be given as follows: $f(0) = a, f(1) = b = g(0)$ and $g(1) = c$. Let E be given by $x_f = x_g$. This equation corresponds to the formula

$$(\forall x, y, z)(d(x, y) \leq 1 \wedge d(x, z) \leq 1) \Rightarrow y = z)\,,$$

which is equivalent to

$$(\forall x, y)(d(x, y) \leq 1 \Rightarrow x = y)\,.$$

Hence, E-algebras are metric spaces having distances of distinct points > 1 and the monad T_E is given by the reflection to this reflective subcategory. Consider the isometry $m : 2_2 \to X$, where 2_2 has two points having the distance 2. Then $T_E(m)$ is the constant mapping. Hence, T_E does not preserve isometries although it is given by a finitary equational theory. Moreover, T_E preserves directed colimits of isometries, but it does not preserve directed colimits.

4. Enriched Equational Theories

Let $\mathscr{V}$ be a symmetric monoidal closed category and $\mathscr{K}$ a $\mathscr{V}$-category. Given objects A and X in $\mathscr{K}$, we denote the $\mathscr{V}$-object $\mathscr{K}(X, A)$ as A^X. Similarly, $K^f = \mathscr{K}(f, K)$ and $h^X = \mathscr{K}(X, h)$ are morphisms in $\mathscr{V}$. Let $\mathscr{K}_0$ be the underlying category of $\mathscr{K}$.

At first, we will define enriched algebras over unenriched equational theories from 3.3.

Definition 4.1. *Let t be a type over $\mathscr{K}_0$. A $\mathscr{V}$-algebra of type t is an object A of $\mathscr{K}_0$ equipped with morphisms $\omega_A : A^X \to A^Y$ in $\mathscr{V}_0$ for every (X, Y)-ary operation symbol of t. Terms are interpreted in A as in 3.3, just the meaning of A^f is changed.*

A $\mathscr{V}$-algebra A satisfies an equation $p = q$ if $p_A = q_A$. It satisfies a theory E if it satisfies all equations of E.

A homomorphism $h : A \to B$ of $\mathscr{V}$-algebras of type t are morphisms $h : A \to B$ such that $h^Y\omega_A = \omega_B h^X$ for every $\omega \in \Omega^{X,Y}$.

Alg (E) *will denote the category of $\mathscr{V}$-algebras of E and $U_E :$* **Alg** $(E) \to \mathscr{K}_0$ *will be the forgetful functor.*

Remark 4.2. A $\mathscr{V}$-algebra A is equipped with a functor $\tilde{\Omega} \to \mathscr{V}_0$ whose composition with x_- is $\mathscr{K}_0(-, A)$.

Enriched equational theories are defined as follows (they correspond to $\mathscr{V}$-pretheories of Bourke and Garner (2019)).

Definition 4.3. Bourke and Garner (2019). *A $\mathscr{V}$-equational theory over $\mathscr{K}$ is a $\mathscr{V}$-category Θ together with an identity-on-objects $\mathscr{V}$-functor $x_- : \mathscr{A}^{\mathrm{op}} \to \Theta$ where $\mathscr{A}$ is a full subcategory of $\mathscr{K}$. A Θ-algebra is an object A of $\mathscr{K}$ together with a $\mathscr{V}$-functor $\hat{A} : \Theta \to \mathscr{V}$ whose composition with x_- is $\mathscr{K}(-, A)$. A homomorphism of Θ-algebras $(A, \hat{A}) \to (B, \hat{B})$ is a morphism $h : A \to B$ together with a $\mathscr{V}$-natural transformation $\hat{h} : \hat{A} \to \hat{B}$ such that $\hat{h}J$ is $\mathscr{K}(-, h)$ restricted on $\mathscr{A}^{\mathrm{op}}$.* **Alg** (Θ) *will denote the $\mathscr{V}$-category of Θ-algebras and $U_\Theta :$* **Alg** $(\Theta) \to \mathscr{V}$ *will be the forgetful $\mathscr{V}$-functor.*

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

Remark 4.4. A $\mathscr{V}$-equational theory Θ over $\mathscr{K}$ determines an equational theory Θ_0 over $\mathscr{K}_0$ and every Θ-algebra is a $\mathscr{V}$-algebra of Θ_0.

We will show that, over **Pos** or **Met**, enriched equational theories can be reduced to unenriched ones having the same enriched algebras.

$\mathscr{V}$ is locally λ-presentable as a closed category if it is locally λ-presentable, the tensor unit I is λ-presentable and the tensor product of two λ-presentable objects is also λ-presentable. An object X of a $\mathscr{V}$-category $\mathscr{K}$ is λ-presentable (in the enriched sense) if $\mathscr{K}(X, -) : \mathscr{K} \to \mathscr{V}$ preserves λ-directed colimits. Since the set-valued hom-functor $\mathscr{V}(X, -)$ is the composition of the $\mathscr{V}$-hom-functor $\mathscr{V}(X, -)$ and the set-valued hom-functor $\mathscr{V}(I, -)$, λ-presentability in the enriched sense implies λ-presentability in the ordinary sense. If $\mathscr{V}(I, -) : \mathscr{V} \to$ **Set** creates λ-directed colimits, the both concepts are equivalent.

Definition 4.5. *We say that a type t over $\mathscr{K}$ is λ-ary if all arities X and Y of its operation symbols are λ-presentable in $\mathscr{V}$. A theory is λ-ary if its type is λ-ary. Similarly, a λ-ary $\mathscr{V}$-equational theory is a $\mathscr{V}$-category Θ together with an identity-on-objects $\mathscr{V}$-functor $x_- : \mathscr{K}_\lambda^{\mathrm{op}} \to \Theta$.*

Theorem 4.6. Bourke and Garner (2019). *Let $\mathscr{V}$ be a locally λ-presentable as a closed category, $\mathscr{K}$ a locally λ-presentable $\mathscr{V}$-category and Θ be a λ-ary $\mathscr{V}$-equational theory over $\mathscr{K}$. Then* **Alg** (Θ) *is a locally λ-presentable $\mathscr{V}$-category and U_E is monadic and preserves λ-directed colimits. Hence, the corresponding $\mathscr{V}$-monad T_E on $\mathscr{K}$ is λ-ary.*

Observation 4.7. Let $U : \mathscr{A} \to \mathscr{V}$ be a faithful $\mathscr{V}$-functor, i.e. $\mathscr{A}$ is $\mathscr{V}$-*concrete*. For $X, Y \in \mathrm{ob}\,(\mathscr{V})$, let $\Omega^{X,Y}$ consist of natural transformations $U^X \to U^Y$ where $U^X = \mathscr{V}(X, U-)$. Like in 3.11, we get a $\mathscr{V}$-equational theory E_U and the $\mathscr{V}$-functor $H_U : \mathscr{A} \to$ **Alg** (E_U) such that $H_U(A)$ is A equipped with components ω_A as operations.

Assume that U has a left $\mathscr{V}$-adjoint F. Then $\mathscr{V}$-natural transformations $\omega : U^X \to U^Y$ correspond to $\mathscr{V}$-natural transformations $\mathscr{A}(FX, -) \to \mathscr{A}(FY, -)$ and thus to morphisms $FY \to FX$. Hence, we get a type t_U and a $\mathscr{V}$-equational theory E_U. Again, the comparison $\mathscr{V}$-functor $H_U : \mathscr{K}^T \to$ **Alg** (T_U) is an equivalence of $\mathscr{V}$-categories. Moreover, given a T-algebra, elements $a \in A^X$ correspond to morphisms $FX \to A$.

This exactly says that the full subcategory consisting of free algebras is $\mathscr{V}$-dense in the category of algebras, which follows from Kelly (1982a, Theorem 5.1) because $\mathscr{K}^T$ is cotensored.

Theorem 4.8. Bourke and Garner (2019). *Let $\mathscr{V}$ be a locally λ-presentable as a closed category, $\mathscr{K}$ be a locally λ-presentable $\mathscr{V}$-category and T be a λ-ary $\mathscr{V}$-monad on $\mathscr{K}$. Then there is a λ-ary $\mathscr{V}$-equational theory E over $\mathscr{V}$ such that the $\mathscr{V}$-concrete categories $\mathscr{K}^T$ and* **Alg** (E) *are equivalent.*

Using 4.7, the proof is analogous to that of 3.12.

Remark 4.9. Let $\mathscr{V}$ be locally λ-presentable as a closed category. Then we have a one-to-one correspondence between λ-ary $\mathscr{V}$-monads on $\mathscr{V}$ and λ-ary $\mathscr{V}$-equational theories over $\mathscr{V}$. This was proved for $\lambda = \aleph_0$ in Power (1999) and the passage from $\aleph_0$ to λ does not create any difficulties. This also follows from the more general result 7.7 in Lack and Rosický (2011) and, of course, from Bourke and Garner (2019), Corollary 21 and Proposition 23.

Remark 4.10. Like in 3.15, if E is (λ, λ')-ary, $\lambda \leq \lambda'$, then **Alg** (E) is locally λ'-presentable and U_E preserves λ-directed colimits.

Examples 4.11. (1) The category **Pos** is locally finitely presentable as a (cartesian) closed category. Hence, finitary-enriched monads on **Pos** correspond to finitary-enriched equational theories over **Pos**. Since $\mathscr{V}(I, -) :$ **Pos** $\to$ **Set** creates directed colimits, finitely presentable objects in the enriched sense and in the ordinary sense coincide and are equal to finite posets. Given a type t

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

over **Pos** and an enriched t-algebra A in the sense of 4.1, then A^X is the set of monotone mapping $a : X \to A$. But $\omega_A : A^X \to A^Y$ is a monotone mapping for every term ω. Like in 3.7(1), we can reduce (X, Y)-ary operations to X-ary operations but we have to replace equations by inequation. In the terminology of Adámek *et al.* (2011*b*), the resulting algebras are *coherent*, i.e. operations $A^X \to A$ are monotone mappings. In this way, an equational theory over **Pos** can be extended to an enriched equational theory over **Pos** having the same enriched algebras.

As in 3.7(1) an inequation $p \leq q$ of X-ary terms is expressed by factorizing the $(X, 2_0)$-ary term (p, q) through a $(X, 2)$-ary operation r as $(p, q) = x_u r$ where $u : 2_0 \to 2$ is the identity map from the two-element antichain 2_0 to the two-element chain 2. Hence we get that finitary-enriched monads on **Pos** correspond to finitary inequational theories for coherent algebras, which was proved in Adámek *et al.* (2011*b*).

In **Pos**, finite products commute with reflexive coinserters (see Bourke (2010)). Every finite poset is a reflexive coinserter of finite discrete posets and hom-functors $\mathbf{Pos}(A, -) : \mathbf{Pos} \to \mathbf{Pos}$ of finite discrete posets A preserve reflexive coinserters. An argument analogous to 3.15 implies that the **Pos**-monad T_E given by an enriched finitary equational theory over **Pos** with (X, Y)-ary operations, where X is discrete preserves not only filtered colimits but also reflexive coinserters. Adámek *et al.* (2011*a*) showed that, conversely, every enriched monad T on **Pos** preserving filtered colimits and reflexive coinserters is given by such an enriched equational theory. In our setting, this can be proved as follows.

Following 4.8, $\mathcal{K}^T$ is equivalent to $\mathbf{Alg}(E_U)$ where E_U is a finitary-enriched equational theory whose (X, Y)-ary operations correspond to morphisms $f : FY \to FX$; here $U : \mathcal{K}^T \to \mathbf{Pos}$ is the forgetful functor and F is its enriched left adjoint. Express Y and X as reflexive coinserters of finite discrete posets

$$Y_1 \underset{p_1}{\overset{p_0}{\rightrightarrows}} Y_0 \xrightarrow{\ p\ } Y$$

and

$$X_1 \underset{q_1}{\overset{q_0}{\rightrightarrows}} X_0 \xrightarrow{\ q\ } X$$

where Y_0, X_0 give elements and Y_1, X_1 give $\leq$ (see (Adámek *et al.*, 2011*a*, Remark 2.3)). Consider $f : FY \to FX$. Since U preserves reflexive coinserters, $\mathcal{K}^T(FY_0, -)$ preserves reflexive coinserters. Since

$$FX_1 \underset{Fq_1}{\overset{Fq_0}{\rightrightarrows}} FX_0 \xrightarrow{\ Fq\ } FX$$

is a reflexive coinserter,

$$\mathcal{K}^T(FY_0, FX_1) \underset{\mathcal{K}^T(FY_0, Fq_1)}{\overset{\mathcal{K}^T(FY_0, Fq_0)}{\rightrightarrows}} \mathcal{K}^T(FY_0, FX_0) \xrightarrow{\mathcal{K}^T(FY_0, Fq)} \mathcal{K}^T(FY_0, FX)$$

is a reflexive coinserter. Hence, $\mathcal{K}^T(FY_0, Fq)$ is surjective. Thus, there exists

$$f_0 : FY_0 \to FX_0$$

such that $F(q)f_0 = fF(p)$. Since $p_0 \leq p_1$, we have $Fp_0 \leq Fp_1$ and thus $f_0F(p_0) \leq f_0F(p_1)$.

Like in the proof of 3.12, let E be the subtheory of E_U where the input arities X are discrete and the output arities Y are arbitrary. Consider an E-algebra A. Then

$$A^Y \xrightarrow{\ A^p\ } A^{Y_0} \underset{A^{p_1}}{\overset{A^{p_0}}{\rightrightarrows}} A^{Y_1}$$

is a coreflexive inserter. Let ω be an (X, Y)-ary operation corresponding to $f : FY \to FX$ and ω_0 an (X_0, Y_0)-ary operation corresponding to f_0. We get $(\omega_0)_A : A^{X_0} \to A^{Y_0}$. Since $f_0F(p_0) \leq f_0F(p_1)$, we have $A^{p_0}(\omega_0)_A \leq A^{p_1}(\omega_0)_A$. Thus, there is a unique $\tilde{f} : A^{X_0} \to A^Y$ such that $A^p\tilde{f} = \tilde{f_0}$. Put $\omega_A = \tilde{f}A^q$. This clearly makes A an E_U-algebra and proves that the reduct functor $R : \mathbf{Alg}\,(E_U) \to \mathbf{Alg}\,(E)$ is an equivalence.

(2) Since **Met** is locally $\aleph_1$-presentable as a closed category, enriched monads preserving $\aleph_1$-directed colimits on **Met** correspond to $\aleph_1$-ary enriched equational theories over **Met**. Analogously to (1), we can reduce them to equational theories over **Met** without changing enriched algebras. Enriched $\aleph_1$-equational theories where the input arities X are discrete metric spaces correspond to unconditional equational theories of Mardare *et al.* (2017). Following 2.4, every finite discrete metric space is finitely presentable in the enriched sense. Hence, enriched monads given by finitary unconditional equational theories are finitary.

The basic equational theories of Mardare *et al.* (2017) are $\aleph_1$-ary enriched equational theories, where all $(X, 1)$-ary operations symbols are induced by those having X discrete. This means that every $(X, 1)$-ary operation symbol ω is equal to $\omega' x_u$ where $u : X_0 \to X$ is the identity on the underlying set of X and X_0 is the discrete space on this set.

Notation 4.12. Let $\mathcal{K}$ be an $\mathcal{M}$-locally λ-generated $\mathcal{V}$-category (see Di Liberti and Rosický (2009)). We say that a type t over $\mathcal{K}$ is λ*-ary* if all arities X and Y of its operation symbols are λ-generated w.r.t. $\mathcal{M}$ in the enriched sense. A theory is λ-ary if its type is λ-ary.

We say that $\mathcal{V}$ is $\mathcal{M}$*-locally* λ*-generated as closed category* if it is $\mathcal{M}$-locally λ-generated in the enriched sense, the tensor unit I is λ-generated w.r.t. $\mathcal{M}$ and the tensor product of two λ-generated objects w.r.t. $\mathcal{M}$ is λ-generated w.r.t. $\mathcal{M}$.

Remark 4.13. Let $\mathcal{K}$ be $\mathcal{M}$-locally λ-generated as a closed category. Following (Di Liberti and Rosický, 2009, Theorem 2.16), there is a regular cardinal $\mu \geq \lambda$ such that $\mathcal{K}$ is locally μ-presentable and every λ-generated object w.r.t. $\mathcal{M}$ is μ-presentable. Since we can assume that $\lambda \lhd \mu$, (Adámek and Rosický, 1994, Remark 2.15) implies that every μ-presentable object A is a retract of a μ-small λ-directed colimit of λ-generated objects w.r.t. $\mathcal{M}$. Since tensor product preserves colimits and retracts, $\mathcal{K}$ is locally μ-presentable as a closed category. This argument can be found in (Kelly and Lack, 2001, Proposition 2.4).

Theorem 4.14. *Let $\mathcal{V}$ be $\mathcal{M}$-locally λ-generated as a closed category, $\mathcal{K}$ be an $\mathcal{M}$-locally λ-generated $\mathcal{V}$-category and E be a λ-ary enriched equational theory over $\mathcal{K}$. Then* **Alg** (E) *is an $\mathcal{M}$-locally λ-generated $\mathcal{V}$-category and U_E is both $\mathcal{V}$-monadic and a morphism of locally λ-generated $\mathcal{V}$-categories.*

Proof. The proof is analogous to that of 3.19. Following 4.13 and 4.6, **Alg** (E) is locally μ-presentable for some $\mu \geq \lambda$. The factorization system $(F(\mathcal{E}), \mathcal{M}')$ on **Alg** (E) is a $\mathcal{V}$-factorization system because $\mathcal{M}'$ is closed under cotensors (see (Lucyshyn-Wright, 2014, Theorem 5.7)). Indeed, **Alg** (E) is cotensored as a locally presentable $\mathcal{V}$-category and U_E preserves cotensors because it has a left $\mathcal{V}$-adjoint. Thus, it suffices to use (Di Liberti and Rosický, 2009, Theorem 4.17). □

Theorem 4.15. *Let $\mathcal{V}$ be locally λ-generated as a closed category and $\mathcal{K}$ be an $\mathcal{M}$-locally λ-generated and locally μ-presentable $\mathcal{V}$-category for $\lambda \leq \mu$. Let T be a μ-ary $\mathcal{V}$-monad on $\mathcal{K}$ preserving $\mathcal{M}$-morphisms and λ-directed colimits of $\mathcal{M}$-morphisms. Then there is a λ-ary $\mathcal{V}$-equational theory E such that $\mathcal{K}^T$ and* **Alg** (E) *are equivalent (as $\mathcal{V}$-concrete categories over $\mathcal{K}$).*

The proof is analogous to that of 3.21.

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

5. More about Metric Monads

At first, we summarize what we have shown in Sections 3 and 4. Since finitely generated metric spaces w.r.t. isometries coincide with finite metric spaces, our finitary equational theories are precisely equational theories whose operations have finite metric spaces as arities. The corresponding monads preserve $\aleph_1$-directed colimits and send directed colimits of isometries to directed colimits. Conversely, if a monad preserves $\aleph_1$-directed colimits, isometries and sends directed colimits of isometries to directed colimits then it is given by a finitary equational theory. But monads given by finitary equational theories do not need to preserve isometries. Finite metric spaces also coincide with metric spaces finitely generated w.r.t. isometries in the enriched sense and **Met** is isometry-locally finitely generated as a closed category. Hence, our finitary enriched equational theories are precisely enriched equational theories whose operations have finite metric spaces as arities. The corresponding enriched monads preserve $\aleph_1$-directed colimits and send directed colimits of isometries to directed colimits. This follows from 4.14 and 4.15. Since the equational theory from 3.23 is enriched, enriched monads given by finitary enriched equational theories do not need to preserve isometries.

Now, we will look in more detail to finitary enriched equational theories over **Met** having the input arities discrete. Recall that a *reflexive coequalizer* is a coequalizer of a reflexive pair, that is, parallel pair of split epimorphisms having a common splitting. Reflexive coequalizers are important in universal algebra because they commute with finite products in **Set** (see Adámek *et al.* (2011)).

Proposition 5.1. *In* **Met***, finite products commute with reflexive coequalizers.*

Proof. Following 2.1, it suffices to prove that finite products commute with reflexive coequalizers in **PMet**. Like as in 2.2, it suffices to show that the functor

$$X \times - : \mathbf{PMet} \to \mathbf{PMet}$$

preserves reflexive coequalizers because the diagonal $D \to D \times D$ is final for the diagram scheme $\mathcal{D}$ for a reflexive coequalizer.

Let $h : B \to C$ be a coequalizer of a reflexive pair $f, g : A \to B$ with $t : B \to A$. This means that $ft = gt = \mathrm{id}_B$. We have to show that $X \times h : X \times B \to X \times C$ is a coequalizer of the reflexive pair $X \times f, X \times g : X \times A \to X \times B$. Let Z be its coequalizer. For $c_1, c_2 \in C$, we have

$$d(c_1, c_2) = \inf d(b_1, b_2)$$

where the infimum is taken over all pairs $b_1, b_2 \in B$ with $hb_1 = c_1$ and $hb_2 = c_2$. Since the forgetful functor $U : \mathbf{PMet} \to \mathbf{Set}$ preserves all limits and colimits (see (Adámek and Rosický, 2006, Examples 2.3(4))), the pseudometric spaces $X \times C$ and Z have the same underlying set $X \times UC$. In the pseudometric space $X \times C$, we have distances

$$d((x_1, c_1), (x_2, c_2)) = \max\{d(x_1, x_2\}, d(c_1, c_2)\} = \max\{d(x_1, x_2), \inf d(b_1, b_2)\}.$$

while in Z

$$d((x_1, c_1), (x_2, c_2)) = \inf \max\{d(x_1, x_2), d(b_1, b_2)\}.$$

It is easy to see that these distances are equal. In fact, if $d(x_1, x_2) \geq d(b_1, b_2)$ for some b_1, b_2, then the both distances are equal to $d(x_1, x_2)$. If $d(x_1, x_2) \leq d(b_1.b_2)$ for all b_1, b_2 then the both distances are equal to $\inf d(b_1, b_2)$. □

Corollary 5.2. *Let A be a finite discrete metric space. Then its hom-functor*

$$\mathbf{Met}\,(A, -) : \mathbf{Met} \to \mathbf{Met}$$

preserves reflexive coequalizers.

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

Recall that a *finitary unconditional equational theories* of Mardare *et al.* (2017) coincide with our finitary enriched equational theories where the input arities X are discrete.

Corollary 5.3. *Enriched monads on* **Met** *induced by finitary unconditional equational theories preserve filtered colimits and reflexive coequalizers.*

Proof. It follows from 2.4 and 5.2. □

Sifted colimits are those which commute in **Set** with finite products (see Adámek *et al.* (2011)). Following (Adámek *et al.*, 2011, Theorem 7.7), a functor **Met** → **Met** preserving filtered colimits and reflexive coequalizers preserve sifted colimits.

Lemma 5.4. *A colimit commutes with finite products in* **Met** *iff is is sifted.*

Proof. Following 2.2 and 5.1, filtered colimits and reflexive coequalizers commute with finite products. For a general sifted colimit, it again suffices to show that the functor $X \times - : \mathbf{Met} \to \mathbf{Met}$ preserves sifted colimits because the diagonal $D \to D \times D$ is final for the diagram scheme $\mathscr{D}$ for a sifted colimit (see (Adámek *et al.*, 2011, Theorem 2.15)). But this follows from (Adámek *et al.*, 2011, Theorem 7.7).

The converse follows from the fact that discrete metric spaces form a coreflective full subcategory of **Met** which is closed under products. Hence, every colimit commuting with finite products in metric spaces does it in discrete metric spaces, i.e. in **Set**. Thus, it is sifted. □

Remark 5.5. Discrete metric spaces do not form an enriched coreflective subcategory of **Met**. Hence, **Met** might contain weighted colimits commuting with finite products.

In **Pos**, the situation is analogous for discrete posets. And, indeed, reflexive coinserters commute with finite products (see Bourke (2010)).

Hence, finitary unconditional equational theories yield monads preserving sifted colimits.

Example 5.6. The Kantorovich monad on **CMet** is given by a quantitative equational theory for barycentric algebras (see Fritz and Perrone (2019)). Hence, it preserves filtered colimits and reflexive coequalizers. The first fact was proved in Adámek *et al.* (2015).

But monads on **Met** preserving sifted colimits can also appear otherwise.

Lemma 5.7. *Tensor product in a symmetric monoidal closed category* $\mathscr{V}$ *commutes with filtered colimits and reflexive coequalizers.*

Proof. See (Rezk, 1996, Lemma 2.3.2). □

A monoid in a symmetric monoidal closed category $\mathscr{V}$ is an object M of $\mathscr{V}$ equipped with a morphism $m : M \otimes M \to M$ satisfying the associativity and unit axioms. Let **Mon** ($\mathscr{V}$) denote the category of monoids in $\mathscr{V}$ and V : **Mon** ($\mathscr{V}$) → $\mathscr{V}$ the forgetful functor.

Proposition 5.8. *Let* $\mathscr{V}$ *be a symmetric monoidal closed category. Then* **Mon** ($\mathscr{V}$) *has limits, filtered colimits, and reflexive coequalizers, and these limits and colimits are preserved by the forgetful functor.*

Proof. The existence and preservation of limits is evident. For filtered colimits and reflexive coequalizers, it follows 5.7. □

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

Proposition 5.9. Porst (2008). *Let $\mathcal{V}$ be locally λ-presentable as a closed category. Then,* **Mon** $(\mathcal{V})$ *is locally λ-presentable and V is monadic.*

Following 5.8, the corresponding monad preserves sifted colimits.

Example 5.10. (1) Let **Norm** be the category of generalized normed spaces (i.e., norm ∞ is allowed) and linear maps of norm ≤ 1. Analogously as in (Rosický, 2011, Theorem 2.2), we show that the forgetful functor $V : \mathbf{Norm} \to \mathbf{Met}$ is monadic. In fact, normed spaces are monoids in **Met** equipped with unary operations $c \cdot -$, for $|c| \leq 1$, satisfying the appropriate axioms. The reason is that $+ : VA \otimes VA \to VA$ and $c \cdot - : VA \to VA$ are nonexpanding. It does not seem that **Norm** can be given by an equational theory over **Met**.

(2) Let **Ban** be the category of (complex) Banach spaces and linear maps of norm ≤ 1. Following Rosický (2011), the unit ball functor $U : \mathbf{Ban} \to \mathbf{Met}$ is monadic. Since U is an enriched functor, we get an $\aleph_1$-enriched monad on **Met** (see (Rosický, 2011, Remark 4.2(3))). The left adjoint $F : \mathbf{Met} \to \mathbf{Ban}$ sends 1 to the Banach space $\mathbb{C}$ of complex numbers. Since

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 2_\varepsilon \end{array}$$

is an ε-pushout for every $\varepsilon > 0$ and ε-pushouts are weighted colimits (see (Adámek and Rosický, 2006, Examples 4.1)), $F2_\varepsilon$ is an ε-pushout

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & F2_\varepsilon \end{array}$$

in **Ban**. These ε-pushouts are described in (Di Liberti and Rosický, 2009, Lemma 6.5).

Remark 5.11. Following Rezk (1996), every monad given by an operad on **Met** preserves sifted colimits. This captures both monoids and generalized normed spaces.

The category **CMet** of complete generalized metric spaces is a $\mathcal{V}$-reflective full subcategory of **Met**. It is locally $\aleph_1$-presentable (see (Adámek and Rosický, 2006, Examples 2.3(2)) and symmetric monoidal closed. The only $\aleph_0$-presentable object in **CMet** is the empty space (see (Adámek and Rosický, 2006, Remark 2.7(1))). Since the reflector $\mathbf{Met} \to \mathbf{CMet}$ preserves finite products, finite products commute with filtered colimits, and reflexive coequalizers in **CMet**. Hence, every finitary unconditional equational theory over **CMet** yields an enriched monad on **CMet** preserving filtered colimits and reflexive coequalizers. Since 5.4 is also valid in **CMet**, it amounts to the preservation of sifted colimits.

Example 5.12. Consider the $\otimes$-theory from 5.10(1) over **CMet** given by addition $+ : Z \otimes Z \to Z$ and scalar multiplications $c \cdot - : Z \to Z$ for $c \in \mathbb{C}$, $|c| \leq 1$. Following (Rosický, 2011, Theorem 2.2), $\mathbf{Alg}(\mathcal{T}) = \mathbf{Ban}_\infty$ is the category of generalized Banach spaces with the forgetful functor $U_\mathcal{T} : \mathbf{Ban}_\infty \to \mathbf{CMet}$. The corresponding monad on **CMet** preserves sifted colimits.

Unlike **Met**, **CMet** is not isometry-locally finitely generated (see (Adámek and Rosický, 2006, Proposition 5.19)). But **Ban** is isometry-locally finitely generated as a **CMet**-category (see (Adámek and Rosický, 2006, Remark 7.8)). In the same time, **Ban** is a monoidal category with the projective tensor product satisfying $\| x \otimes_p y \| = \| x \| \| y \|$ (see (Borceux, 1994, Examples 6.1.9h)). Hence, one can use our results to study monads on **Ban**.

https://doi.org/10.1017/S0960129521000220 Published online by Cambridge University Press

Acknowledgements. The author is grateful to Jiří Adámek for inspiring discussions, to John Bourke for a feedback concerning pretheories and to the referees for their valuable comments.

Competing interests. The author declare none.

References

Adámek, J., Dostál, M. and Velebil, J. (2011a). *A categorical view of varieties of ordered algebras*, arXiv:2011.13839.

Adámek, J., Ford, C., Milius, S. and Schröder, L. (2011b). *Finitary monads on the category of posets*, arXiv:2011.14796.

Adámek, J., Milius, S., Moss, L. S. and Urbat, H. (2015). On finitary functor and their presentations. *J. Comp. Syst. Sci.* **81** 813–833.

Adámek, J. and Rosický, J. (1994). *Locally Presentable and Accessible Categories*, Cambridge University Press.

Adámek, J. and Rosický, J. (1991). What are locally generated categories? In *Proc. Categ. Conf. Como 1990*. Lecture Notes in Math. 1488, 14–19.

Adámek, J. and Rosický, J. (2006). *Approximate injectivity and smallness in metric-enriched categories*, arXiv:2006.01399.

Adámek, J., Rosický, J. and Vitale, E. M. (2011). *Algebraic Theories*, Cambridge University Press.

Bird, G. J. (1984). *Limits in 2-categories of locally-presented categories*, Sydney Category Seminar Report.

Bloom, S. (1976). Varieties of ordered algebras. *J. Comput. System Sci.* **13** 200–212.

Borceux, F. (1994). *Handbook of Categorical Algebra 2*, Cambridge University Press.

Bourke, J. (2010). *Codescent objects in 2-dimensional universal algebra*, PhD Thesis, University of Sydney.

Bourke, J. and Garner, R. (2019). Monads and theories. *Adv. Math.* **351** 1024–1071.

Dubuc, E. J. (1970). Enriched semantics-structure (meta) adjointness. *Rev. Un. Mat. Argentina* **25** 5–26.

Di Liberti, I. and Rosický, J. (2009). *Enriched locally generated categories*, arXiv:2009.10980.

Fritz, T. and Perrone, P. (2019). A probability monad as the colimit of spaces of finite samples. *Theory Appl. Categ.* **34** 170–220.

Gabriel, P. and Ulmer, F. (1971). *Lokal präsentierbare Kategorien*, Lecture Notes in Math., 221, Springer.

Hino, W. (2016). *Varieties of metric and quantitative algebras*, arXiv:1612.06054.

Hyland, M. and Power, J. (2007). The category theoretic understanding of universal algebra, Lawvere theories and monads. *Electron. Notes Theor. Comput. Sci.* **172** 437–458.

Kelly, G. M. (1982a). *Basic Concepts of Enriched Category Theory*, Cambridge University Press.

Kelly, G. M. (1982b). Structures defined by finite limits in the enriched context I. *Cah. Topol. Gém. Différ. Catég.* **23** 3–42.

Kelly, G. M. and Lack, S. (2001). $\mathscr{V}$-CAT is locally presentable or locally bounded if $\mathscr{V}$ is so. *Theory Appl. Categ.* **8** 555–575.

Lack, S. and Rosický, J. (2011). Notions of Lawvere theory. *Appl. Categ. Structures* **19** 363–391.

Lawvere, F.W. (2004). Functorial semantics of algebraic theories. *Dissertation, Columbia University 1963; Reprints in Theory Appl. Categ.* **5** 23–107.

Lawvere, F. W. (1973). Metric spaces, generalized logic, and closed categories. *Rend. Sem. Mat. Fis. Milano* **43** 135–166.

Lieberman, M. and Rosický, J. (2017). Metric abstract elementary classes as accessible categories. *J. Symbolic Logic* **82** 1022–1040.

Linton, F. E. J. (1966). Some aspects of equational categories. In: *Conf. Categ. Algebra (La Jolla 1965)*, Springer, 84–94.

Linton, F. E. J. (1969). *An Outline of Functorial Semantics*. Lect. Notes in Math. 80, Springer, 7–52.

Lucyshyn-Wright, R. B. B. (2014). Enriched factorization systems. *Theory Appl. Categ.* **29** 475–495.

Lucyshyn-Wright, R. B. B. (2016). Enriched algebraic theories and monads for a system of arities. *Theory Appl. Categ.* **31** 01–137.

Makkai, M. and Paré, R. (1989). Accessible categories: The foundation of Categorical Model Theory. *Contemp. Math.* 104, AMS.

Mardare, R., Panangaden, P. and Plotkin, G. (2016). *Quantitative algebraic reasoning*, In: Proc. LICS 2016, 700–709.

Mardare, R., Panangaden, P. and Plotkin, G. (2017). On the axiomatizability of quantitative algebras. In: *Proc. LICS 2017*.

Moggi, E. (1991). Notions of computation and monads. *Inf. Comp.* **93** 55–92.

Nishizawa, K. and Power, J. (2009). Lawvere theories enriched over a general base. *J. Pure Appl. Algebra* **213** 377–386.

Plotkin, G. and Power, J. (2004). *Computational effects and operations: An overview*. *Electron. Notes Theor. Comp. Sci.* **73** 149–163.

Porst, H.-E. (2008). On categories of monoids, comonoids, and bimonoids. *Quest. Math.* **31** 127–139.

Power, J. (1999). Enriched Lawvere theories. *Theory Appl. Categ.* **6** 83–93.

Rezk, C. W. (1996). *Spaces of algebra structures and cohomology of operads*, Thesis MIT.

Rosický, J. (1981). On algebraic categories. In: *Coll. Math. Soc. J. Bolyai*, 29. (Universal Algebra, Esztergom 1977), Budapest 1981, 662–690.
Rosický, J. (2011). *Are Banach spaces monadic?*, arXiv:2011.07543, to appear in Comm. Algebra.
Rosický, J. and Tholen, W. (2018). Approximate injectivity. *Appl. Categ. Structures* **26** 699–716.
Weaver, N. (1995). Quasi-varieties of metric algebras. *Algebra Universalis* **33** 1–9.

Cite this article: Rosický J (2021). Metric monads. *Mathematical Structures in Computer Science* **31**, 535–552. https://doi.org/10.1017/S0960129521000220