Proceedings of the Edinburgh Mathematical Society (1993) **36**, 447–461 ©

ON THE COMPLEX OSCILLATION THEORY OF $f^{(k)}+Af=F$

by CHEN ZONG-XUAN

(Received 9th December 1991)

In this paper, we investigate the complex oscillation theory of

$$f^{(k)}+Af=F(z), \qquad k\geqq 1$$

where $A, F\not\equiv 0$ are entire functions, and obtain general estimates of the exponent of convergence of the zero-sequence and of the order of growth of solutions for the above equation.

1991 *Mathematics subject classification*: 34A20, 30D35.

1. Introduction and results

For convenience in our statement, we first explain the notation used in this paper. We will use respectively the notation $\lambda(f)$ and $\bar{\lambda}(f)$ to denote the exponent of convergence of the zero-sequence and of the sequences of distinct zeros of $f(z)$, $\sigma(f)$ to denote the order of growth of $f(z)$, $v_f(r)$ to denote the central index of the entire function $f(z)$. By the Wiman–Valiron theory, we have

$$\sigma(f)=\overline{\lim_{r\to\infty}}\frac{\log v_f(r)}{\log r}.$$

In addition, other notation of the Nevanlinna theory is standard (e.g. see [**4,5**]). Other notation will be shown when it appears.

In 1982, S. Bank and I. Laine investigated the complex oscillation theory of homogeneous linear differential equation, and proved in [**1**]:

Theorem A. *Let $A(z)$ be a nonconstant polynomial of degree n, and let $f(z)\not\equiv 0$ be a solution of the equation*

$$f''+A(z)f=0. \tag{1.1}$$

Then

(a) *the order of growth of f is $(n+2)/2$,*

(b) *if n is odd, the exponent of convergence of the zero-sequence of f is $(n+2)/2$,*

(c) *if n is even, and if f_1 and f_2 are two linearly independent solutions of* (1.1), *then at*

https://doi.org/10.1017/S0013091500018538 Published online by Cambridge University Press

least one of f_1, f_2, has the property that the exponent of convergence of its zero-sequence is $(n+2)/2$.

Afterwards, I. Laine showed in [**6**].

Theorem B. *Let a_0, P_0, $P_1 \not\equiv 0$ be polynomials such that* $\deg a_0 = n$, $\deg p_0 < 1 + n/k$. *Consider the equation*

$$f^{(k)} + a_0 f = P_1 e^{P_0} \qquad (k \geqq 2). \tag{1.2}$$

(a) *If* $\deg P_1 < n$, *then all solutions of* (1.2) *satisfy*

$$\bar{\lambda}(f) = \lambda(f) = \sigma(f) = 1 + \frac{n}{k}. \tag{1.3}$$

(b) *If* $\deg P_1 \geqq n$, *then, apart from one possible exception, all solutions satisfy* (1.3). *The possible exceptional solution is of the form $f_0 = Qe^{P_0}$, where Q is a polynomial of degree* $\deg Q = \deg P_1 - n$.

Gao Shi-an [**3**] had earlier addressed the case when $n = 2$ in Theorem B.

In this paper, we consider the differential equation (DE)

$$f^{(k)} + Af = F(z), \tag{1.4}$$

where $A(z)$ and $F(z)$ are both entire functions of finite order, and where $A(z)$ is transcendental, or $A(z)$ is a polynomial, $F(z) \not\equiv 0$ is an entire function of finite order with infinitely many zeroes.

We will prove the following theorems in this paper.

Theorem 1. *Let A and $F(z) \not\equiv 0$ be both entire functions of finite order, where A is transcendental. Then:*

(a) *All solutions f of* (1.4) *satisfy*

$$\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \infty, \tag{1.5}$$

except for at most one possible exceptional solution f_0 of finite order.

(b) *The exceptional solution f_0 satisfies*

$$\sigma(f_0) \leqq \max\{\sigma(A), \sigma(F), \bar{\lambda}(f_0)\}.$$

Furthermore, if $\sigma(A) \neq \sigma(F)$, $\bar{\lambda}(f_0) < \sigma(f_0)$, then $\sigma(f_0) = \max\{\sigma(A), \sigma(F)\}$.

Remark. If $\sigma(A) = \infty$, or $\sigma(F) = \infty$, then Theorem 1 does not hold.

Example 1. $f=e^{\sin z}$ solves

$$f''+\sin z\, f=\cos^2 z\, e^{\sin z},$$

and

$$f''+(\sin z-\cos^2 z+e^{-\sin z+z})f=e^z,$$

there $\lambda(f)=0$, $\sigma(f)=\infty$.

Theorem 2. *Let $A(z)$ be a transcendental entire function with $\sigma(A)\neq 1$, $\sigma(A)<\infty$, let $F(z)\not\equiv 0$ be an entire function with $\sigma(F)<\infty$, and $\alpha(>0)$, a constant, and let $f(z)$ be a solution of the DE*

$$f^{(k)}+e^{-\alpha z}f'+Af=F(z). \tag{1.6}$$

Then:

(a) *all solutions f of* (1.6) *satisfy* (1.5), *except for at most one possible exceptional solution f_0 of finite order,*

(b) *the exceptional solution f_0 satisfies*

$$\sigma(f_0)\leqq\max\{\sigma(A),\sigma(F),\bar{\lambda}(f_0),1\},$$

Furthermore, if $\sigma(A)\neq\sigma(F)$, $\bar{\lambda}(f_0)<\sigma(f_0)$, then $\sigma(f_0)=\max\{\sigma(A),\sigma(F),1\}$.

Theorem 3. *Let $A(z)$ be a polynomial with $\deg A=n\geqq 1$, let $F(z)\not\equiv 0$ be an entire function with infinitely many zeros, and let $f(z)$ be a solution of the DE* (1.4). *If $(n+k)/k<\sigma(F)=\beta<\infty$, then*

(a) $\sigma(f)=\beta$,

(b) *if $\lambda(F)=\beta$, then every solution satisfies $\lambda(f)=\beta$,*

(c) *if $\lambda(F)<\beta$, then all solutions f of* (1.4) *satisfy*

$$\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta,$$

except for at most one exceptional one f_0 with $\lambda(f_0)=\lambda(F)$.

Theorem 4. *Let $A(z)$ be a nonconstant polynomial of degree n, and let $F(z)\not\equiv 0$ be an entire function with infinitely many zeros and $\sigma(F)=\beta$. Then:*

(a) *if $\beta<(n+k)/k$, then all solutions f of* (1.4) *satisfy*

$$\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\frac{n+k}{k},$$

except for at most one exceptional one f_0 *with* $\sigma(f_0)=\beta$,

(b) *if* $\beta=(n+k)/k$, *then all solutions* f *of* (1.4) *satisfy* $\sigma(f)=(n+k)/k$, $\lambda(f)\geqq\lambda(F)$.

2. Lemmas

Lemma 1. *Let* $A(z)$ *be a transcendental entire function of finite order, Then every solution* $g\not\equiv 0$ *to the DE*

$$g^{(k)}+Ag=0 \tag{2.1}$$

satisfies $\sigma(g)=\infty$.

Proof. If $\sigma(g)<\infty$, then by $A=-g^{(k)}/g$, we have

$$m(r,A)=m\left(r,\frac{g^{(k)}}{g}\right)=O(\log r),$$

and this contradicts the hypothesis that $A(z)$ is transcendental.

Lemma 2. *Let* $A(z)$ *be a transcendental entire function with* $1\not\equiv\sigma(A)<\infty$ *and let* $\alpha(>0)$ *be a constant. Then every solution* $g\not\equiv 0$ *to the DE*

$$g^{(k)}+e^{-\alpha z}g'+A(z)g=0 \tag{2.2}$$

satisfies $\sigma(g)=\infty$.

Proof. Using the same proof as in the proof of Theorem 1 in [**2**], we have $\sigma(g)=\infty$.

Lemma 3 (Wiman–Valiron). *Let* $g(z)$ *be a transcendental entire function and let* z *be a point with* $|z|=r$ *at which* $|g(z)|=M(r,g)$. *Then for all* $|z|$ *outside a set* E *of* r *of finite logarithmic measure, we have*

$$\text{(a)}\quad \frac{g^{(k)}(z)}{g(z)}=\left(\frac{v_g(r)}{z}\right)^k(1+o(1)) \qquad (k \text{ is an integer}, r\notin E), \tag{2.3}$$

$$\text{(b)}\quad \varlimsup_{\substack{r\to\infty\\ r\in[0,\infty)}}\frac{\log v_g(r)}{\log r}=\varlimsup_{\substack{r\to\infty\\ r\in[0,\infty)-E}}\frac{\log v_g(r)}{\log r},$$

where $v_g(r)$ *is the central index of* $g(z)$ (*see* [**5, 7, 8**]).

Proof. (a) This is the Wiman–Valiron theory (see [**5, 7, 8**]).

https://doi.org/10.1017/S0013091500018538 Published online by Cambridge University Press

(b) We clearly have

$$\varlimsup_{\substack{r\to\infty\\ r\in[0,\infty)}}\frac{\log v_g(r)}{\log r}\geqq\varlimsup_{\substack{r\to\infty\\ r\in[0,\infty)-E}}\frac{\log v_g(\gamma)}{\log r}.$$

On the other hand, from $v_g(r)$ is the central index of $g(z)$, we have that $v_g(r)>0$ and $v_g(r)$ is a nondecreasing function on $[0,+\infty)$. Setting $\int_E dr/r=\log\delta<\infty$ for a given $\{r'_n\}$, $r'_n\in[0,+\infty)$, $r'_n\to\infty$, there exists a point $r_n\in[r'_n,(\delta+1)r'_n]-E$. From

$$\frac{\log v_g(r'_n)}{\log r'_n}\leqq\frac{\log v_g(r_n)}{\log r'_n}\leqq\frac{\log v_g(r)}{\log r_n+\log\dfrac{1}{\delta+1}}=\frac{\log v_g(r)}{\log r_n\cdot(1+o(1))},$$

it follows that

$$\varlimsup_{r'_n\to\infty}\frac{\log v_g(r'_n)}{\log r'_n}\leqq\varlimsup_{r_n\to\infty}\frac{\log v_g(r_n)}{\log r_n}\leqq\varlimsup_{\substack{r\to\infty\\ r\in[0,\infty)-E}}\frac{\log v_g(r)}{\log r}.\tag{2.4}$$

Since $\{r'_n\}$ is arbitrary, we have

$$\varlimsup_{\substack{r\to\infty\\ r\in[0,\infty)}}\frac{\log v_g(r)}{\log r}\leqq\varlimsup_{\substack{r\to\infty\\ r\in[0,\infty)-E}}\frac{\log v_g(r)}{\log r}.$$

This proves Lemma 3(b).

From Lemma 3, we can deduce the following:

Lemma 4. *Let $A(z)$ be a nonconstant polynomial with $\deg A=n$. Then every solution $f\not\equiv 0$ to the DE*

$$f^{(k)}+A(z)f=0\tag{2.5}$$

satisfies $\sigma(f)=(n+k)/k$.

Lemma 5. *Let $A(z)$ be a polynomial with $\deg A=n\geqq 1$, $F(z)\not\equiv 0$ be an entire function with $\sigma(F)=\beta<\infty$. Let f be a solution of the DE*

$$f^{(k)}+Af=F(z).\tag{2.6}$$

Then

(a) *if $\beta\geqq(n+k)/k$, then $\sigma(f)=\beta$,*

(b) *if $\beta<(n+k)/k$, then all solutions f of* (2.6) *satisfy $\sigma(f)=(n+k)/k$, except for at most one possible exceptional one f_0 with $\sigma(f_0)=\beta$.*

Proof. It is easy to see that $\sigma(f)\geqq\sigma(F)=\beta$ from (2.6). On the other hand we assume that $\{f_1,\ldots,f_k\}$ is a fundamental solution set of (2.5) that is the corresponding homogeneous differential equation of (2.6). By Lemma 4, we have $\sigma(f_j)=(n+k)/k\,(j=1,\ldots,k)$.

By variation of parameters, we can write

$$f=B_1(z)f_1+\cdots+B_k(z)f_k,$$

where $B_1(z),\ldots,B_k(z)$ are determined by

$$\begin{cases} B_1'f_1+\cdots+B_k'f_k=0 \\ B_1'f_1'+\cdots+B_k'f_k'=0 \\ \cdots\cdots \\ B_1'f_1^{(k-1)}+\cdots+B_k'f_k^{(k-1)}=F. \end{cases}$$

Noting that the Wronskian $W(f_1,\ldots,f_k)$ is a differential polynomial in $f_1,\ldots,f_k$ with constant coefficients, it is easy to deduce that $\sigma(W)\leqq\sigma(f_j)=(n+k)/k$. Set

$$W_i=\begin{array}{c} \\ \left|\begin{array}{ccccc} f_1, & \ldots, & \overset{(i)}{O}, & \ldots, & f_k \\ \vdots & & & & \vdots \\ f_1^{(k-1)}, & \ldots, & F, & \ldots, & f_k^{(k-1)} \end{array}\right| \end{array}=F\cdot g_i \qquad (i=1,\ldots,k).$$

where g_i are differential polynomials in $f_1,\ldots,f_k$ with constant coefficients. So

$$\sigma(g_i)\leqq\sigma(f_j)=\frac{n+k}{k},\ B_i'=\frac{W_i}{W}=\frac{F\cdot g_i}{W},$$

and

$$\sigma(B_i)=\sigma(B_i')\leqq\max\left\{\sigma(F),\frac{n+k}{k}\right\}.$$

$$\sigma(f)\leqq\max\left\{\beta,\frac{n+k}{k}\right\}.$$

Therefore,

(a) if $\beta\geqq\dfrac{n+k}{k}$, then $\sigma(f)=\beta$,

(b) if $\beta<\dfrac{n+k}{k}$, then $\beta\leqq\sigma(f)\leqq\dfrac{n+k}{k}$.

We affirm that the DE (2.6) can only possess at most one exceptional solution f_0 with $\beta\leqq\sigma(f_0)<(n+k)/k$. In fact, if f^* is a second solution with $\beta\leqq\sigma(f^*)<(n+k)/k$, then $\sigma(f_0-f^*)<(n+k)/k$. But f_0-f^* is a solution of the corresponding homogeneous equation (2.5) of (2.6). This contradicts Lemma 4.

Now we prove that the exceptional solution f_0 satisfies $\sigma(f_0)=\beta$. We assume $\beta<\sigma(f_0)<(n+k)/k$. Let z be a point with $|z|=r$ at which $|f_0(z)|=M(r,f_0)$. From Lemma 3(a)

$$\frac{f_0^{(k)}(z)}{f_0(z)}=\left(\frac{v_{f_0}(r)}{z}\right)^k(1+o(1))\qquad r\notin E. \tag{2.7}$$

holds for all $|z|$ outside a set E of r of finite logarithmic measure. For sufficiently large $|z|$, we have $A=az^n(1+o(1))$ ($a\neq 0$ is constant). Substituting (2.7) into (2.6), we have

$$\left(\frac{v_{f_0}(r)}{z}\right)^k(1+o(1))+az^n(1+o(1))=\frac{F(z)}{f_0(z)}\qquad r\notin E. \tag{2.8}$$

Now for a given ε $(0<3\varepsilon<\sigma(f_0)-\beta)$, there exists $\{\bar r_m\}$ $(\bar r_m\to\infty)$ such that $M(\bar r_m,f_0)>\exp\{\bar r_m^{\sigma(f_0)-\varepsilon}\}$. Setting $\int_E dr/r=\log\delta<\infty$, there exists a point $r_m\in[\bar r_m,(\delta+1)\bar r_m]-E$. At such points r_m, we have

$$M(r_m,f_0)\geqq M(\bar r_m,f_0)>\exp\{\bar r_m^{\sigma(f_0)-\varepsilon}\}$$

$$\geqq \exp\left\{\frac{r_m^{\sigma(f_0)-\varepsilon}}{(\delta+1)^{\sigma(f_0)}}\right\} = \exp\left\{r_m^{\sigma(f_0)-2\varepsilon}\cdot\frac{r_m^{\varepsilon}}{(\delta+1)^{\sigma(f_0)}}\right\}$$

$$\geqq \exp\{r_m^{\sigma(f_0)-2\varepsilon}\}.$$

In addition for sufficiently large r_m, we have

$$|F(z)| \leqq M(r_m, F) < \exp\{r_m^{\beta+\varepsilon}\}.$$

So

$$\left|\frac{F(z)}{f_0(z)}\right| = \frac{|F(z)|}{M(r_m, f_0)} < \exp\{r_m^{\beta+\varepsilon} - r_m^{\sigma(f_0)-2\varepsilon}\} \to 0 (r_m \to \infty).$$

Therefore, at such points $|z_m| = r_m (r_m \notin E, |f_0(z_m)| = M(r_m, f_0))$, from (2.8) we have

$$\left(\frac{v_{f_0}(\gamma_m)}{z_m}\right)^k (1+o(1)) + a z_m^n (1+o(1)) = o(1). \tag{2.9}$$

From the Wiman–Valiron theory, we obtain

$$\overline{\lim_{r_m\to\infty}} \frac{\log v_{f_0}(r_m)}{\log r_m} = \frac{n+k}{k}.$$

This contradicts that $\sigma(f_0) < (n+k)/k$. Hence $\sigma(f_0) = \beta$.

Lemma 6. *Let A_{k-j} $(j = 1, \ldots, k)$, $B \not\equiv 0$ be entire functions. If f is a solution of the DE*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = B, \tag{2.10}$$

and $\max\{\sigma(B), \sigma(A_0), \ldots, \sigma(A_{k-1})\} = \beta < \sigma(f)$, *then* $\bar{\lambda}(f) = \lambda(f) = \sigma(f)$.

Proof. We can write from (2.10)

$$\frac{1}{f} = \frac{1}{B}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_0\right). \tag{2.11}$$

If f has a zero at z_0 of order $\alpha(>k)$, then B must have a zero at z_0 of order $\alpha - k$. Hence,

$$n\left(r, \frac{1}{f}\right) \leqq k\bar{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{B}\right)$$

and

$$N\left(r,\frac{1}{f}\right)\leqq k\bar{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{B}\right). \tag{2.12}$$

From (2.11), we have

$$m\left(r,\frac{1}{f}\right)\leqq m\left(r,\frac{1}{B}\right)+\sum_{j=1}^{k} m(r,A_{k-j})+O(\log T(r,f)+\log r) \qquad r\notin E \tag{2.13}$$

holds for all r outside a set E of r of finite linear measure (if $\sigma(f)<\infty$, then $E=\phi$). (2.12) and (2.13) give

$$\begin{aligned}T(r,f)&=T\left(r,\frac{1}{f}\right)+O(1)\\ &\leqq k\bar{N}\left(r,\frac{1}{f}\right)+T\left(r,\frac{1}{B}\right)+\sum_{j=1}^{k} T(r,A_{k-j})+O(\log T(r,f)+\log r)\\ &=k\bar{N}\left(r,\frac{1}{f}\right)+T(r;B)+\sum_{j=1}^{k} T(r,A_{k-j})+O(\log T(r,f)+\log r)\,(r\notin E) \end{aligned} \tag{2.14}$$

and there exists $\{r_n'\}\,(r_n'\to\infty)$ such that

$$\lim_{r_n'\to\infty}\frac{\log T(r_n',f)}{\log r_n'}=\sigma(f).$$

Setting $mE=\delta<\infty$, then there exists a point $r_n\in[r_n', r_n'+\delta+1]-E$. For such r_n, we have

$$\frac{\log T(r_n,f)}{\log r_n}\geqq\frac{\log T(r_n',f)}{\log(r_n'+\delta+1)}=\frac{\log T(r_n',f)}{\log r_n'+\log\left(1+\dfrac{\delta+1}{r_n'}\right)}.$$

So

$$\varliminf_{r_n\to\infty}\frac{\log T(r_n,f)}{\log r_n}\geqq\lim_{r_n'\to\infty}\frac{\log T(r_n',f)}{\log r_n'+\log\left(1+\dfrac{\delta+1}{r_n'}\right)}=\sigma(f) \qquad (r_n\notin E)$$

and

$$\lim_{r_n\to\infty}\frac{\log T(r_n,f)}{\log r_n}=\sigma(f).$$

For a given c such that $\beta<c<\sigma(f)$, we have

$$T(r_n,f)\geqq r_n^c$$

for sufficiently large r_n. On the other hand, for a given ε $(0<\varepsilon<c-\beta)$, we have

$$T(r_n,B)<r_n^{\beta+\varepsilon},\ T(r_n,A_{k-j})<r_n^{\beta+\varepsilon}\qquad (j=1,\dots,k).$$

Therefore

$$T(r_n,B)\leqq\frac{1}{k+3}\,T(r_n,f), \tag{2.15}$$

$$T(r_n,A_{k-j})\leqq\frac{1}{k+3}\,T(r_n,f)\qquad (j=1,\dots,k) \tag{2.16}$$

hold for sufficiently large r_n. Since

$$O\{\log T(r_n,f)+\log r_n\}=o\{T(r_n,f)\},$$

for sufficiently large r_n

$$O\{\log T(r_n,f)+\log r_n\}\leqq\frac{1}{k+3}\,T(r_n,f) \tag{2.17}$$

holds. (2.14) and (2.15), (2.16), (2.17) give

$$T(r_n,f)\leqq(k+3)K\bar{N}\left(r_n,\frac{1}{f}\right).$$

So

$$\sigma(f)=\lim_{r_n\to\infty}\frac{\log T(r_n,f)}{\log r_n}\leqq\overline{\lim_{r_n\to\infty}}\frac{\log\bar{N}\left(r_n,\dfrac{1}{f}\right)}{\log r_n}\leqq\bar{\lambda}(f).$$

Therefore $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.

Lemma 7. *Let F be the same as in Theorem 3, let Q be the canonical product formed with the nonzero zeros of F, with $\sigma(Q)<\beta=\sigma(F)$, let $m(\geqq 0)$ be an integer, and let $b_{k-r},\dots,b_0$ be polynomials with $\deg b_{k-i}=i\ (\beta-1)$. Then the DE*

$$g^{(k)}+b_{k-1}g^{(k-1)}+\cdots+b_0g=z^mQ \tag{2.18}$$

https://doi.org/10.1017/S0013091500018538 Published online by Cambridge University Press

may have at most one exceptional solution g_0 with $\lambda(g_0)=\sigma(g_0)=\sigma(Q)=\lambda(Q)$, and all the other solutions g of (2.18) *satisfy*

$$\bar{\lambda}(g)=\lambda(g)=\sigma(g)=\beta.$$

Proof. It is not difficult to see that all solutions of (2.18) and its corresponding homogeneous equation

$$g^{(k)}+b_{k-1}g^{(k-1)}+\cdots+b_0g=0 \tag{2.19}$$

are entire functions. For the DE (2.19), from Lemma 3(a), we have basic formulas

$$\frac{g^{(j)}(z)}{g(z)}=\left(\frac{v_g(r)}{z}\right)^j(1+o(1)) \qquad r\notin E, (j=1,\dots,k) \tag{2.20}$$

where $|z|=r$, $|g(z)|=M(r,g)$, $\int_E dr/r<\infty$, $v_g(r)$ denotes the central index of g. As $r\to\infty$ set $b_{k-i}=d_{k-i}z^{i(\beta-1)}(1+o(1))$ $d_{k-1},\dots,d_0$ are nonzero constants). Substituting them and (2.20) into (2.19), we have

$$\left(\frac{v_g(r)}{z}\right)^k(1+o(1))+d_{k-1}z^{\beta-1}\left(\frac{v_g(r)}{z}\right)^{k-1}(1+o(1))+\cdots+d_0z^{k(\beta-1)}(1+o(1))=0 \qquad (r\notin E). \tag{2.21}$$

By the reasoning in [**7**, pp. 106–108] for sufficiently large r, we have $v_g(r)\sim cz^\alpha$ ($|z|=r\notin E$, $c\neq 0$ a constant), substituting it into (2.21), it is easy to see that the degrees of all terms of (2.21) are respectively

$$k(\alpha-1),\ j(\beta-1)+(k-j)(\alpha-1) \qquad (j=1,\dots,k-1),\ k(\beta-1).$$

From the Wiman–Valiron theory (see [**5**, pp. 227–229], [**7, 8**]), we see that $\alpha=\beta$ is the only possible value. Therefore, all solutions of (2.19) satisfy $\sigma(g)=\beta$.

Using the same proof (variation of parameters) as in the proof of Lemma 5, we have that all solution g of the DE (2.18) satisfy $\sigma(g)\leqq\beta$.

Using the same proof as in the proof of Lemma 5, it is easy to see that the DE (2.18) may have at most one exceptional solution g_0 with $\sigma(g_0)<\beta$.

Next we are going to work out the order of the exceptional solution g_0. By the above proof and (2.18), we have $\sigma(Q)\leqq\sigma(g_0)<\beta$. We will now prove that $\sigma(Q)<\sigma(g_0)<\beta$ fails.

Suppose that $\sigma(Q)<\sigma(g_0)<\beta$. From the Wiman–Valiron theory, we have basic formulas

$$\frac{g_0^{(j)}(z)}{g_0(z)}=\left(\frac{v_{g_0}(r)}{z}\right)^j(1+o(1)) \qquad r\notin E, j=1,\dots,k \tag{2.22}$$

where $|z|=r$, $|g_0(z)|=M(r,g_0)$, $\int_E dr/r<\infty$. As $r\to\infty$, we set $b_{k-i}=d_{k-i}z^{i(\beta-1)}(1+o(1))\,(d_{k-i}\not\equiv 0$ are constants). Substituting them and (2.22) into (2.18), and using the same proof as in the proof of Lemma 5(b), we can obtain a sequence $\{z_n\}$. The sequence $\{z_n\}$ satisfies $|z_n|=r_n\notin E$, $|g_0(z_n)|=M(r_n,g_0)$ and for sufficiently large r_n

$$\left(\frac{\nu_{g_0}(r_n)}{z_n}\right)^k(1+o(1))+d_{k-1}z_n^{\beta-1}\left(\frac{\nu_{g_0}(r_n)}{z_n}\right)^{k-1}(1+o(1))+\cdots+d_0z_n^{k(\beta-1)}(1+o(1))=o(1). \quad (2.23)$$

Setting $\sigma(g_0)=\delta<\beta$, by the reasoning in [7, pp. 106–108], for sufficiently larger r_n, we have $\nu_{g_0}(r_n)\sim c_1z_n^\delta$ ($|z_n|=r_n\notin E$, $c_1\neq 0$ a constant). It is easy to see that the degrees of all terms of (2.23) are respectively

$$k(\delta-1),\ j(\beta-1)+(k-j)(\delta-1)\,(j=1,\dots,k-1),\ k(\beta-1).$$

Then there is only one term $d_0z_n^{k(\beta-1)}(d_0\neq 0)$ with the degree $k(\beta-1)$ being the highest one in (2.23). This is impossible. Therefore, the order of g_0 can only be $\sigma(g_0)=\sigma(Q)$.

As $\sigma(g_0)=\sigma(Q)$, by [3], we have $\lambda(g_0)\geqq\lambda(z^mQ)=\sigma(Q)$. Hence

$$\lambda(g_0)=\sigma(g_0)=\sigma(Q)=\lambda(Q).$$

As $\sigma(g)=\beta>\sigma(Q)$, by Lemma 6, we have

$$\bar{\lambda}(g)=\lambda(g)=\sigma(g)=\beta.$$

3. Proof of theorems

Proof of Theorem 1. (a) Now assume f_0 is a solution of (1.4) with $\sigma(f_0)<\infty$. If f^* is a second solution with $\sigma(f^*)<\infty$, then $\sigma(f^*-f_0)<\infty$. And f^*-f_0 is a solution of (2.1), that is the corresponding homogeneous differential equation of (1.4). But by Lemma 1, we have $\sigma(f^*-f_0)=\infty$.

Now assume f is a solution of (1.4) with $\sigma(f)=\infty$. Then $\max\{\sigma(A),\sigma(F)\}<\sigma(f)$. By Lemma 6, we have $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\infty$.

(b) Assume f is an exceptional solution of (1.4) with $\sigma(f_0)<\infty$. Using the same proof as in the proof of Lemma 6, we have

$$T(r,f_0)\leqq k\bar{N}\left(r,\frac{1}{f_0}\right)+T(r,F)+T(r,A)+O(\log r). \quad (3.1)$$

Now set $\max\{\sigma,(A),\sigma(F)\}=\bar{\alpha}$. Then for sufficiently large r, we have

$$T(r,F)<r^{\bar{\alpha}+\varepsilon},\ T(r,A)<r^{\bar{\alpha}+\varepsilon}.$$

By (3.1) we have

https://doi.org/10.1017/S0013091500018538 Published online by Cambridge University Press

$$T(r,f_0)<k\bar{N}\left(r,\frac{1}{f_0}\right)+2r^{\bar{\alpha}+\varepsilon}+O(\log r).$$

Therefore

$$\sigma(f_0)\leqq\max\{\bar{\lambda}(f_0),\bar{\alpha}\}=\max\{\bar{\lambda}(f_0),\sigma(A),\sigma(F)\}. \tag{3.2}$$

If $\sigma(A)\not\equiv\sigma(F)$, $\bar{\lambda}(f_0)<\sigma(f_0)$, then from (3.2), we get

$$\sigma(f_0)\leqq\max\{\sigma(A),\sigma(F)\},$$

and by (1,4), we have $\sigma(f_0)\geqq\max\{\sigma(A),\sigma(F)\}$. Therefore,

$$\sigma(f_0)=\max(\sigma(A),\sigma(F)\}.$$

Proof of Theorem 2. By Lemma 2, every solution $g\not\equiv 0$ of (2.2), that is the corresponding homogeneous equation of (1.6), satisfies $\sigma(g)=\infty$. Using the analogous proof to that in Theorem 1, we can prove that Theorem 2 holds.

Proof of Theorem 3. (a) By Lemma 5, we have $\sigma(f)=\beta$.

(b) If $\lambda(F)=\beta$, then by [3] we have $\lambda(f)\geqq\lambda(F)$. Hence $\lambda(f)=\sigma(f)=\beta$.

(c) If $\lambda(F)<\beta$, set $F=Z^mQe^P$, (Q is the canonical product formed with the nonzero zeros of F,P is a polynomial with $\deg P=\beta$). Set $f=ge^P$, then $\lambda(g)=\lambda(f)$, $\bar{\lambda}(g)=\bar{\lambda}(f)$. Substituting $f=ge^P$ into (1.4), we have

$$g^{(k)}+d_{k-1}g^{(k-1)}+\cdots+d_0g=z^mQ. \tag{3.3}$$

To work out the degrees of d_{k-j} for $j=1,\ldots,k$, we need d_{k-j} ($j=1,\ldots,k$) in more detailed form. It is easy to check by induction that we have for $k\geqq 2$ (see [**6**])

$$f^{(k)}=\left\{g^{(k)}+kp'g^{(k-1)}+\sum_{j=2}^{k}[c_k^j(p')^j+H_{j-1}(p')]g^{(k-j)}\right\}e^p \tag{3.4}$$

where $H_{j-1}(p')$ are differential polynomials in p' and its derivatives of total degree $j-1$ with constant coefficients. It is easy to see that the derivatives of $H_{j-1}(p')$ as to z are of the same form $H_{j-1}(p')$. C_k^j is the usual notation for the binomial coefficients. (1.4) and (3.4) give

$$d_{k-1}=kp',\ d_{k-j}=c_k^j(p')^j+H_{j-1}(p') \qquad j=2,\ldots,k-1,$$

$$d_0=(p')^k+H_{k-1}(p')+A.$$

So $\deg d_{k-j}=j(\beta-1)\,(j=1,\ldots,k-1)$. Since $\beta>(n+k)/k$, we have $\deg d_0=k(\beta-1)$. By

Lemma 7, the DE (3.3) may have at most one exceptional solution g_0 with $\lambda(g_0)=\sigma(g_0)=\sigma(Q)=\lambda(Q)$, and all the other solutions g of (3.3) satisfy $\bar{\lambda}(g)=\lambda(g)=\sigma(g)=\beta$.

Therefore, the DE (1.4) may have at most one exceptional solution $f_0=g_0e^p$ with $\lambda(f_0)=\lambda(F)$, and all the other solutions $f=ge^p$ of (1.4) satisfy

$$\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta.$$

Proof of Theorem 4. (a) If $\beta<(n+k)/k$, then by Lemma 5, the DE(1.4) may have at most one exceptional solution f_0 with $\sigma(f_0)=\beta$, and all the other solutions f of (1.4) satisfy $\sigma(f)=(n+k)/k$. By Lemma 6, all the other solutions f of (1.4) satisfy

$$\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\frac{n+k}{k}.$$

(b) If $\beta=(n+k)/k$, then by Lemma 5, all solutions f of (1.4) satisfy $\sigma(f)=(n+k)/k$. From [3], we have $\lambda(f)\geqq\lambda(F)$.

4. Examples of the exceptional solution

Example 2 (concerning the exceptional solution in Theorem 1). $f_0=e^{z^2}$ solves

$$f''+(\sin z-4z^2-2)f=e^{z^2}\sin z,$$

there $\sigma(A)<\sigma(F)$, $\sigma(f_0)=\sigma(F)$, $\lambda(f_0)=0<\sigma(f_0)$.

Example 3 (concerning the exceptional solution in Theorem 2). Let G be a given transcendental entire function with $\sigma(G)\neq 1$, $\sigma(G)<\infty$, Then $f_0=e^{3z}$ solves

$$f''+e^{-z}f'-Gf=e^{3z}(9+3e^{-z}-G),$$

there $\sigma(f_0)=1$, $\bar{\lambda}(f_0)=0$.

Example 4 (concerning the exceptional solution in Theorem 3). The DE

$$f''+(1-6z)f=3z^2(2\cos z+3z^2\sin z)e^{z^3}$$

has exceptional solution $f_0=\sin z\cdot e^{z^3}$, there $\sigma(F)=3>(n+k)/k$, and $\sigma(f_0)=3$, $\lambda(f_0)=1$. Now we prove that $\lambda(F)=1$. For the real function $2\cos x+3x^2\sin x$, $\sin m\pi=0$ and $2\cos m\pi+3(m\pi)^2\cdot\sin m\pi\neq 0$ $(m=\pm 1,\pm 2,\ldots)$, the zeros of $2\cos x+3x^2\sin x$ are zeros of $ctgx+\frac{3}{2}x^2$. $ctgx+\frac{3}{2}x^2$ has zeros $x_m\in(m\pi,(m+1)\pi)(m=\pm 1,\pm 2,\ldots)$. Hence $\lambda(2\cos x+3x^2\sin x)=1$. Therefore $\lambda(F)=1$.

Example 5 (concerning the exceptional solution in Theorem 4). $f_0=\sin z$ solves

https://doi.org/10.1017/S0013091500018538 Published online by Cambridge University Press

$$f''+(z^2+1)f=z^2\sin z,$$

there $(n+k)/k=2>\sigma(F)=1$, $\sigma(f_0)=\sigma(F)$.

Acknowledgement. The author would like to thank the referees for valuable suggestions to improve my paper.

REFERENCES

1. S. Bank and I. Laine, On the oscillation theory of $f''+Af=0$ where A is entire, *Trans. Amer. Math. Soc.* **273** (1982), 351–363.

2. G. Gundersen, On the question of whether $f''+e^{-z}f'+B(z)f=0$ can admit a solution $f\not\equiv 0$ of finite order, *Proc. Roy. Soc. Edinburgh* **102A** (1986), 9–17.

3. Gao shi-an, On the complex oscillation of solutions of non-homogeneous linear differential equations with polynomial coefficients, *Comment. Math. Univ. Sancti Pauli* **38** (1989), 11–20.

4. W. Hayman, *Meromorphic Functions* (Clarendon Press, Oxford, 1964).

5. He Yu-zan and Xiao Xiu-zhi, *Algebraid Functions and Ordinary Differential Equations* (Science Press, 1988, (in Chinese)).

6. I. Laine, A note on the complex oscillation theory of non-homogeneous linear differential equations, *Results in Math.* **18** (1990), 282–285.

7. G. Valiron, *Lectures on the General Theory of Integral Functions* (Cleisea, New York, 1949).

8. G. Valiron, *Functions Analytiques* (Presses Universitaires de France, Paris, 1954).

Department of Mathematics
Jiangxi Normal University
Nanchang
P.R. China