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Hill Stability of Configurations in the Full N-Body Problem

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Abstract. Rigorous results on Hill Stability for the classical N-body problem are in general unknown for $N \geqslant 3$, due to the complex interactions that may occur between bodies and the many different outcomes which may occur. However, the addition of finite density for the bodies along with a rigidity assumption on their mass distribution allows for Hill stability to be easily established. In this note we generalize results on Hill stability developed for the Full 3-body problem and show that it can be applied to the Full N-body problem. Further, we find that Hill Stability concepts can be applied to identify types of configurations which can escape and types which cannot as a function of the system energy.

1. Introduction

The current paper further explores the model introduced in Scheeres (2012) for the Full N-body problem, distinguished from the traditional N-body problem in that each body has a finite density, and hence two bodies cannot come arbitrarily close to each other. The finite-sized bodies are assumed to be rigid, and only exhibit contact forces between each other of friction and coefficient of restitution. This means that relative equilibria must also include resting configurations, with all relative motion between components zeroed. This both results in new possible relative equilibrium configurations of an N-body system, with components resting on each other, and also places lower limits on the gravitational potential energy. This model has been developed to describe the relative mechanics and dynamics of self-gravitating rubble pile asteroids, where the relative forces between components are weak enough for the rigidity of the components to not be compromised. Scheeres (2012) shows how this simple change provides a drastic modification of the stable states of the 2 and 3 body problems, with the main focus on the equal mass, spherical, Full N-body problem.

Using this model, in Scheeres (2014) it is shown that a rigorous constraint for Hill Stability in the 3-body problem can be developed, which in general is unknown in the classical point-mass 3-body problem. In the current paper this concept is extended to the N-body problem as a function of system energy. In addition, we show that as a system's energy is raised from values that ensure Hill Stability, some configurations can escape while others cannot. These limits are detailed for the N-body problem for $N = 2, 3, 4, 5$. The analysis for higher values of $N > 5$ require more research into the resting equilibrium of the Full N-body problem.

2. Problem Definition

The main results are established by using the minimum energy function defined in Scheeres (2012), which can be derived from Sundman's Inequality:

$$\mathcal{E} = \frac{1}{2}\frac{H^2}{I_H} + \mathcal{U} \leqslant E \tag{2.1}$$

where E and H are the total energy and angular momentum of the system, including translational and rotational motion, I_H is the total moment of inertia of the system about the fixed total angular momentum vector of the system, and $\mathcal{U}$ is the gravitational potential energy of the system. The function $\mathcal{E}$ is intimately related to Smale's Amended Potential (Smale 1970) and can be used to discover all of the previously mentioned relative equilibria as outlined in Scheeres (2012).

If we assume that the N bodies are of equal mass and density (note that we do not need to make this assumption, but that doing so makes the following discussion much simpler and hence is adopted for this short note) the system can be normalized such that

$$I_H = \frac{N}{10} + \frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} r_{ij}^2 \tag{2.2}$$

$$\mathcal{U} = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{1}{r_{ij}} \tag{2.3}$$

where r_{ij} denotes the distance between bodies i and j. For a finite density system made of equal-sized spheres the distance between the bodies has a lower limit such that $r_{ij} \geqslant 1$. This implies that the potential energy and moment of inertia have lower limits. For $N \leqslant 4$ it is always possible to pack the grains so that they are all mutually equidistant and touching. For $N \geqslant 5$, however, it is no longer possible for all of the grains to touch, meaning that some of the distances must take on a value > 1. Still, by just evaluating all of the grains at the mutually equal distance a conservative lower bound can be found. We also note that for N bodies there are $N(N-1)/2$ mutual distances that appear in these expressions. Applying these observations we can find that

$$I_H \geqslant \frac{N}{10} + \frac{N-1}{2} \tag{2.4}$$

$$\mathcal{U} \geqslant -\frac{1}{2}N(N-1) \tag{2.5}$$

where we note that the inequalities are only sharp for $N \leqslant 4$.

For definiteness in the following we will sometimes denote the potential and moment of inertia to be a function of the number of bodies, N, with the notation $I_H(N)$ and $\mathcal{U}(N)$. In all cases we recall that the moment of inertia and potential energy are always functions of the mutual distances between the bodies. We then introduce the notation $\mathcal{U}_m(N) = \min_{r_{ij}} \mathcal{U}(N)$, which denotes the potential energy configuration for N bodies minimized over all possible relative configurations. As noted above, for $N \leqslant 4$ this lower limit can be achieved with all of the grains resting on each other, with mutual distances $r_{ij} = 1$. Thus, we find that $\mathcal{U}_m(2) = -1$, $\mathcal{U}_m(3) = -3$ and $\mathcal{U}_m(4) = -6$. For systems with $N \geqslant 5$ a lower limit on the minimum potential is found to be $\mathcal{U}_m(N) > -\frac{1}{2}N(N-1)$, as indicated above.

3. Hill Stability Definitions

We now define the concept of Hill Stability used in this paper.

Definition 1. *An N-body system is Hill Stable if there exists a positive constant C such that $r_{ij} < C < \infty$ for all indices and for all time, both future and past.*

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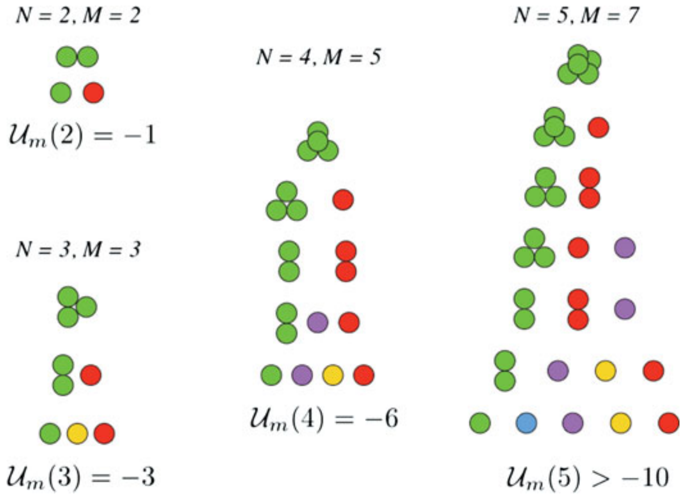

Figure 1. All unique configurations of the equal mass and size N-body problem for $N = 2, 3, 4, 5$. The number M denotes the number of unique configurations at each value of N. Also shown is the minimum potential energy as a function of N. The information in this figure, combined with the theorems, is sufficient to generate the following figures.

We also introduce a weaker form of Hill Stability, tied to certain configurations. First we define what we mean by a configuration, and then we introduce a modified concept we call Configuration Hill Stability.

Definition 2. *A configuration of the N-body problem, denoted as $\mathcal{C}_j$, is defined as a unique grouping of the N bodies into P groups, with each group having a set number of bodies $q_1 \geqslant q_2 \ldots \geqslant q_P \geqslant 1$ and $\sum_{i=1}^{P} q_i = N$. We denote a given grouping as $\mathcal{C}_j = \{N, P, Q = (q_1, q_2, \ldots, q_P)\}$. We note that there can be several different configurations as a function of N, or $j = 1, 2, \ldots, M$, where each will either have a different P or different Q.*

For example, every group of N bodies has a single, unique grouping into N groups, with $P = N$ and $q_i = 1$. Each group of N bodies also has a single, unique grouping into 1 group, with $P = 1$ and $q_1 = N$. Figure 1 shows all possible configurations for $N = 2, 3, 4, 5$.

Definition 3. *A configuration $\mathcal{C}_j = \{N, P, Q = (q_1, q_2, \ldots, q_P)\}$ is called Configuration Hill Stable if it is impossible for the groups in this particular configuration to mutually escape from each other to infinity, either forwards or backwards in time, while each sub-group q_i remains bound.*

4. Hill Stability Theorems

We first establish a general result on Hill Stability for the Full N-Body Problem.

Theorem 4.1. *An equal size and density, spherical Full N-body system is Hill Stable if $E < \mathcal{U}_m(N-1)$.*

Proof. Assume that $E < \mathcal{U}_m(N-1)$ but that the system is not Hill Stable, and escape of at least one particle can occur. If a body escapes, $I_H \to \infty$ and $\mathcal{U}(N) \to$

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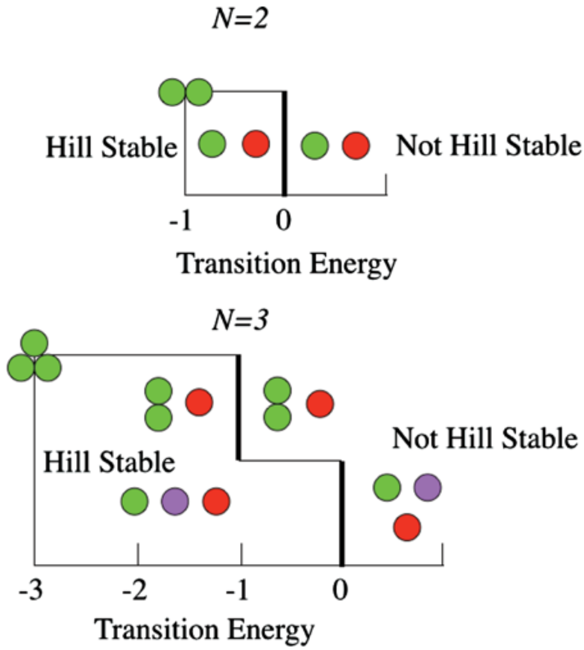

Figure 2. Graph displaying transition from Hill Stability to non-Hill Stability for different energy levels for the $N = 2$ and $N = 3$ problem.

$\mathcal{U}(N-1)$, as the mutual distances involving the escaped body all go to ∞. However, Sundman's Inequality still holds, and after the escape gives $\mathcal{E} = \mathcal{U}(N-1) \leqslant E < \mathcal{U}_m(N-1)$. However, this cannot be true by definition, and thus the contention that the system is not Hill Stable is untrue. Applying the converse, we see that the system is Hill Stable. □

Now, we introduce a second Theorem that allows us to determine whether a given configuration $\mathcal{C}_j$ is Hill Stable.

THEOREM 4.2. *A given configuration of the Full N-Body Problem with equal sized bodies,* $\mathcal{C}_j = \{N, P, Q = (q_1, q_2, \ldots, q_P)\}$, *is Configuration Hill Stable if* $E < \sum_{i=1}^{P} \mathcal{U}_m(q_i)$.

Proof. Assume that $E < \sum_{i=1}^{P} \mathcal{U}_m(q_i)$ but that the system is not Configuration Hill Stable for these components. Thus we can assume that all of these components mutually escape each other. As the distances between these components go to ∞, $I_H \to \infty$ and $\mathcal{U}(N) \to \sum_{i=1}^{P} \mathcal{U}(q_i)$. Applying the Sundman Inequality then gives us $\mathcal{E} = \sum_{i=1}^{P} \mathcal{U}(q_i) \leqslant E < \sum_{i=1}^{P} \mathcal{U}_m(q_i)$. This is a contradiction, meaning that the given configuration is Configuration Hill Stable. □

In this context, Configuration Hill Stability means that not all of the configurations can mutually escape each other. If instead only some configurations escape, but others are bound to each other, this is equivalent to a configuration with fewer components being Hill Unstable, and corresponds to one of the configurations at a lower value of energy in Figs. 2 and 3 being Hill Unstable.

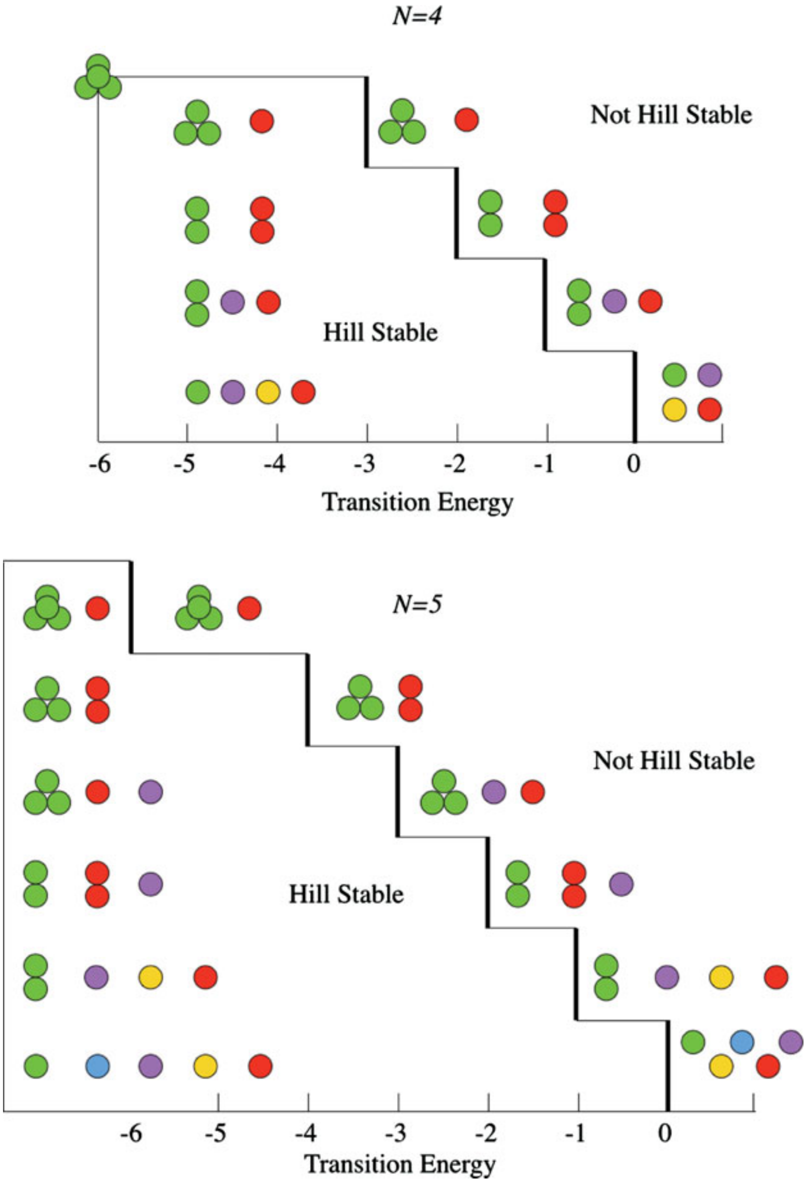

Figure 3. Graph displaying transition from Hill Stability to non-Hill Stability for different energy levels for the $N = 4$ and $N = 5$ problem.

5. Configuration Hill Stability for $N \leqslant 5$

Using these theorems we can compute Hill Stability and Configuration Hill Stability for systems up to $N = 5$. First, in Fig. 1 we show all possible configurations as a function of N, also giving the minimum potential energy for each value of N. Using these results, we can immediately develop the Configuration Hill Stability graphs in Figs. 2 - 3. These graphs show the energy levels at which different configurations lose their Hill Stability, and thus are able to mutually escape at higher levels of energy. A clear pattern is that

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as the energy increases the configurations that can escape involve more bodies. In the limit, and for all cases, when the total energy is positive it is then possible for all of the bodies to mutually escape from each other.

Similar results can be found if we relax the equal size and density and the spherical assumptions. However, in these cases the number of possible outcomes increases significantly and the analysis of these systems becomes more difficult.

6. Discussion

The Theorems proved above add an interesting aspect to the Full N-body problem. They show that as the energy of an N-body system is increased that some configurations are allowed to disrupt, but not others. Eventually, when the energy is positive, it always allowed for the entire system to disperse. It is important that the energy used here is due both to body rotation and to translational motion. Thus, a system could be made initially motionless but be given an energy due to rotation of the individual bodies only – yet the outcome and Hill Stability is unchanged.

Another interesting result is the discrete level of the energy values when new configurations lose their Hill Stability. This is solely due to the ability of the grains to mutually touch for $N \leqslant 4$. For bodies with $N \geqslant 6$ the energies at which Configuration Hill Stability is lost will not take on integer values for configurations that involve more than 5 bodies. We do not consider them here, but will explore these systems in the future.

It is physically relevant to compare these energy levels with the energy levels at which stable resting configurations fission. These are discussed for the equal mass case in Scheeres (2016) for $N = 2, 3, 4$. In it we find that the $N = 2$ and $N = 3$ minimum energy resting configurations both fission at energies less than 0 for $N = 2$ and less than -1 for $N = 3$. This means that once these systems fission they are still Hill Stable for all of their possible configurations. The same is not true for the $N = 4$ case, as 3 different minimum energy resting configurations fission at energies above -3 but below -2. Thus in this case fission can lead to mutual escape of a single body, leaving the remaining three bound into orbit about each other. A test calculation with a particle type of resting configuration for $N = 5$ shows that fission for this body occurs at an energy greater than -3, meaning that three different types of configurations do not have Hill Stability here. More research on this case will occur in the future.

Another question for future study is what the possible levels of energy and angular momentum of the separated components are following such an escape. This will necessitate an estimate of the angular momentum left in the different components, highlighting the interesting fact that the angular momentum does not enter any of the computations given here for Hill Stability. Developing such limits and constraints will be of interest for tying these results to the observed dynamics of rubble pile asteroid systems.

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