

AN INVESTIGATION ON THE LOGICAL STRUCTURE OF MATHEMATICS (III)⁰⁾

FUNDAMENTAL DEDUCTIONS

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In this Part (III) the proofs in UL are given for fundamental formulas concerning the following dependent variables:

Elementary set	$\{a_1, \dots, a_n\};$
Ordered pair ¹⁾	$\langle a, b \rangle;$
Image ²⁾ by σ of a	$\sigma'a;$
Image ²⁾ by σ of elements of a	$\sigma''a;$
Domain of operator	$D_\sigma;$
Range of operator	$W_\sigma;$
Uniqueness	$Un;$
Bi-uniqueness	$Un_2;$
Inverse operator	$\sigma^{-1};$
One-to-one mapping	$Map_2^{a,b};$
Composition of operators	$\sigma \circ \tau;$
Restriction of operator	$\sigma \upharpoonright a;$
Identical mapping	$\iota.$

The following defining formulas are used for these dependent variables:

$$\begin{aligned}
 u \in \{a_1, \dots, a_n\} &\equiv u = a_1 \vee \dots \vee u = a_n, \\
 u \in \langle ab \rangle &\equiv u = \{a\} \vee u = \{ab\}, \\
 u \in \sigma'a &\equiv \exists x. \langle ax \rangle \in \sigma \wedge u \in x, \\
 u \in \sigma''a &\equiv \exists x. x \in a \wedge \langle xu \rangle \in \sigma, \\
 u \in D_\sigma &\equiv \exists x. \langle ux \rangle \in \sigma, \\
 u \in W_\sigma &\equiv \exists x. \langle xu \rangle \in \sigma, \\
 u \in Un &\equiv \forall xyz. \langle xy \rangle \in u \wedge \langle xz \rangle \in u \rightarrow y = z,
 \end{aligned}$$

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⁰⁾ See foot note ⁰⁾ in Part (IV) published in this same volume.

¹⁾ $\langle a, b \rangle$ is also written as $\langle ab \rangle$.

²⁾ $\sigma'a$ as well as $\sigma''a$ are also written in the same way as a^σ , when the distinction is clear by the context.

$$\begin{aligned}
u \in \text{Un}_2 &\equiv u \in \text{Un} \wedge u^{-1} \in \text{Un}, \\
u \in \sigma^{-1} &\equiv \exists xy. u = \langle xy \rangle \wedge \langle yx \rangle \in \sigma, \\
u \in \text{Map}_2^{a,b} &\equiv u \in \text{Un}_2 \wedge D_u = a \wedge W_u = b, \\
u \in \sigma \circ \tau &\equiv \exists xyz. u = \langle xz \rangle \wedge \langle xy \rangle \in \tau \wedge \langle yz \rangle \in \sigma, \\
u \in \sigma \upharpoonright a &\equiv \exists xy. u = \langle xy \rangle \wedge u \in \sigma \wedge x \in a, \\
u \in \iota &\equiv \exists x. u = \langle xx \rangle.
\end{aligned}$$

Besides these dependent variables, we use the universal constant V and $a \cap b$. All the dependent variables used in the following deductions are the variables listed above, or those which are results of successive substitutions of the above listed dependent variables, such as $\langle aa^2 \rangle$, $(\sigma \circ \tau)'a$, $D_\sigma \upharpoonright a$ etc.

Since the variables defined above belong all to the consistent V-system,³⁾ all the formulas proved in this Part (III) are not only theorems of UL, but also theorems of the consistent V-system of UL. Specifically, the variables Un , Un_2 , $\text{Map}_2^{a,b}$, and also D_m , W_m with independent or dependent variable m are not used as sets but only as concepts in the following deductions.⁴⁾

The formulas proved in this Part (III) are not an arbitrary collection of formulas but are those which are needed as ordinary cuts in the deduction in UL of some branches of mathematics. Hence, most of the fundamental properties, though not all, of these dependent variables are deduced in this part.

Our purpose of these deductions is to construct mathematics, as far as possible, consistently in UL. On the one hand, we know some part of abstract mathematics is consistent. On the other hand, we know also some part of concrete mathematics is consistent. The most essential part of the consistency proof consists in those subsystems of UL in which concrete and abstract mathematics, or our formal and intuitive knowledge, are combined. In virtue of the strong theorem of Gödel on the impossibility⁵⁾ of the proof of consistency

³⁾ Cf. Part (VI).

⁴⁾ What dependent variables are used as sets (see Part (X)) can be seen from each proof, although we do not list up in each case. Such a list is needed, for instance, in the observation of Burali-Forti's contradiction.

⁵⁾ Impossible in the sense precisely formulated and proved by Gödel. Moreover, if there were a criterion for any premise σ of UL to be consistent or inconsistent, then either the proof for the criterion could be, after Gödel mapping, formalized in UL and the premise τ of the formalized proof for the criterion would turn out to be inconsistent by the criterion itself, or UL would be still narrow enough to formalize our logical thought. This would mean almost absolute impossibility of such criterion; or else, the criterion or the proof for it would be expressed by means of such part of intuitive logic or mathematics that would be susceptible of no formalization.

as well as in virtue of the difficulty of the problem shown by the development in these decades, it seems almost hopeless to solve the consistency problem by a “once-and-for-all” method as was originally planned by Hilbert. However, it seems there remains yet a “step-by-step” method by which we gradually increase the parts of mathematics which are proved to be consistent. Hence we should construct *actually* even a small part of mathematics consistently. The deductions in this Part (III) will serve as preparation for this purpose. Another purpose of this Part (III) is to show some examples and the elegance of the actual deductions in UL.⁶⁾

Equality (=)

=*1 $a \subseteq a$

-	=*1
(w) -	$\forall x. x \in a \rightarrow x \in a$
-	$w \in a \rightarrow w \in a$
1	$w \notin a$
	$w \in a$
	(1)

=*2 $a \subseteq b \wedge b \subseteq c \rightarrow a \subseteq c$

-	=*2
1	$a \subseteq b$
2	$b \subseteq c$
-	$a \subseteq c$
(w) -	$\forall x. x \in a \rightarrow x \in c$
3	$w \notin a$
4	$w \in c$
(1) -	$\neg \forall x. x \in a \rightarrow x \in b$
-	$\neg. w \in a \rightarrow w \in b$
$w \in a$	⁵ $w \notin b$
(3)	(2)
	-
	$\neg \forall x. x \in b \rightarrow x \in c$
	-
	$\neg. w \in b \rightarrow w \in c$
$w \in b$	$w \notin c$
(5)	(4)

=*3 $a = a$

-	=*3
(w) -	$\forall x. x \in a \equiv x \in a$
-	$w \in a \equiv w \in a$
1	$w \in a$
	$w \notin a$
	(1)
1	$w \in a$
	$w \notin a$
	(1)

⁶⁾ See Part (IV); Compendium for deductions, contained in this same volume.

= *4 $a = b \rightarrow b = a$

-	= *4		
1	$a \neq b$		
-	$b = a$		
(w)	$\forall x. x \in b \equiv x \in a$		
2	$w \in b \equiv w \in a$		
(1)	$\neg \forall x. x \in a \equiv x \in b$		
3	$w \in a \neq w \in b$		
(2)		4	$w \in b$
4	$w \in a$	5	$w \neq a$
5	$w \neq b$	(3)	$w \neq b$ $w \in a$
(3)	$w \in b$ $w \neq a$	(4)	$w \neq b$ $w \in a$
	(5)	(4)	(5)

= *5 $a = b \wedge b = c \rightarrow a = c$

-	= *5		
-	$\neg \forall x. x \in a \equiv x \in b$		
-	$\neg \forall x. x \in b \equiv x \in c$		
(w)	$\forall x. x \in a \equiv x \in c$		
-	$w \in a \equiv w \in c$		
1	$w \in a \neq w \in b$		
2	$w \in b \neq w \in c$		
3	$w \in a$	3	$w \in c$
4	$w \neq c$	4	$w \neq a$
(1)	$w \neq a$ 5 $w \in b$	(1)	$w \in a$ 5 $w \neq b$
	(3) (2) $w \neq b$ $w \in c$		(4) (2) $w \in b$ $w \neq c$
	(5)	(4)	(5) (3)

= *6 $a = b \rightarrow a \subseteq b \wedge b \subseteq a$

-	= *6		
-	$a \neq b$		
1	$a \subseteq b \wedge b \subseteq a$		
2	$\neg \forall x. x \in a \equiv x \in b$		
(1)	$a \subseteq b$	-	$b \subseteq a$
(w)	$\forall x. x \in a \rightarrow x \in b$	-	$\forall x. x \in b \rightarrow x \in a$
-	$w \in a \rightarrow w \in b$	-	$w \in b \rightarrow w \in a$
3	$w \neq a$	3	$w \neq b$
4	$w \in b$	4	$w \in a$
(2)	$w \in a \neq w \in b$	(2)	$w \in a \neq w \in b$
	$w \in a$ $w \neq b$		$w \neq a$ $w \in b$
	(3)	(4)	(4)
		(3)	

=*7 $a \subseteq b \wedge b \subseteq a \rightarrow a = b$

	- \equiv *7		
	1 $a \not\subseteq b$		
	2 $b \not\subseteq a$		
	- $a = b$		

	- $\forall x. x \in a \equiv x \in b$		
	(w) - $w \in a \equiv w \in b$		

3	$w \in a$	3	$w \in b$
4	$w \not\in b$	4	$w \not\subseteq a$
(2) -	$\neg \forall x. x \in b \rightarrow x \in a$	(1) -	$\neg \forall x. x \in a \rightarrow x \in b$
-	$\neg. w \in b \rightarrow w \in a$	-	$\neg. w \in a \rightarrow w \in b$
	-----		-----
	$w \in b$ (4)	$w \not\subseteq a$ (3)	$w \in a$ (4)
			$w \not\in b$ (3)

=*8 $a = b \equiv a \subseteq b \wedge b \subseteq a$ (From =*6 and =*7.)

=*9 $b \subseteq a \wedge \neg b \subseteq a \rightarrow \neg a = b$

Elementary set (E1)

E1*1 $\langle a_1, \dots, a_m \rangle \subseteq \langle b_1, \dots, b_n \rangle \rightarrow \forall_{i=1}^m \exists_{j=1}^n a_i = b_j$

	- E1*1		
	- $\langle a_1, \dots, a_m \rangle \not\subseteq \langle b_1, \dots, b_n \rangle$		
	1 $\forall_{i=1}^m \exists_{j=1}^n a_i = b_j$		

	- $a_i \in \langle a_1, \dots, a_m \rangle$	2	$a_i \notin \langle b_1, \dots, b_n \rangle$
	$a_i = a_i$:	:
	=*3	:	$a_m \notin \langle b_1, \dots, b_n \rangle$

(1) -	$\exists_{j=1}^n a_i = b_j$...	$\exists_{j=1}^n a_m = b_j$

3	$a_1 = b_1$	Symmetric to	
	:	the left	
	:		
	$a_1 = b_n$		

(2)	$a_1 \neq b_1$ (2)	...	$a_1 \neq b_n$ (4)

E1*2 $\forall_{i=1}^m \exists_{j=1}^n a_i = b_j \rightarrow \langle a_1, \dots, a_m \rangle \subseteq \langle b_1, \dots, b_n \rangle$

	- E1*2		
	1 $\neg \forall_{i=1}^m \exists_{j=1}^n a_i = b_j$		
	-		$\langle a_1, \dots, a_m \rangle \not\subseteq \langle b_1, \dots, b_n \rangle$

-		$\forall x. x \in \{a_1, \dots, a_m\} \rightarrow x \in \{b_1, \dots, b_n\}$
(w)		
2		$w \in \{a_1, \dots, a_m\}$
-		$w \in \{b_1, \dots, b_n\}$
3		$w = b_1$ \vdots $w = b_n$
4		$w = b_n$
(1)		$\bigvee_{j=1}^n \exists a_i = b_j$ \vdots
5		
(2)		
6	$w \neq a_1$	\dots
	$w \neq a_n$	
(5)	$a_i \neq b_1 \dots a_i \neq b_n$ $(3, 6) \quad (1, 6)$	Symmetric to the left
	Cut = *5 Cut = *5	

El*3 $\{a_1, \dots, a_m\} = \{b_1, \dots, b_n\} \equiv \forall_{i=1}^m \exists_{j=1}^n a_i = b_j \wedge \forall_{j=1}^n \exists_{i=1}^m a_i = b_j$

From El*1. El*2 and =*8.

El*4 $\langle a, b \rangle = \langle c, d \rangle \rightarrow [a = c \wedge b = d] \vee [a = d \wedge b = c]$

-	El*4			
1	$\langle a, b \rangle \neq \langle c, d \rangle$			
2	$a = c \wedge b = d$			
3	$a = d \wedge b = c$			
(1)	Cut = *8			
4	$\langle a, b \rangle \neq \langle c, d \rangle$			
5	$\langle c, d \rangle \neq \langle a, b \rangle$			
	Cut El*1			
-	$\bigvee. \langle a, b \rangle \subseteq \langle c, d \rangle \rightarrow [a = c \vee a = d] \wedge [b = c \vee b = d]$			
	$\langle a, b \rangle \subseteq \langle c, d \rangle$ (1)	6 $\bigvee. a = c \vee a = d$ 7 $\bigvee. b = c \vee b = d$	Cut El*1	
-	$\bigvee. \langle c, d \rangle \subseteq \langle a, b \rangle \rightarrow [c = a \vee c = b] \wedge [d = a \vee d = b]$			
	$\langle c, d \rangle \subseteq \langle a, b \rangle$ (5)	8 $\bigvee. c = a \vee c = b$ 9 $\bigvee. d = a \vee d = b$		
(2)				
	10 $a = c$ (3)	10 $b = d$ (3)		
(6)	11 $a = d$ (10)	11 $b = c$ (8)	11 $a = d$ (9)	(7) 11 $b = c$ (11)
	$a \neq c$ $a \neq d$ (10) (11)	$c \neq a$ $c \neq b$ (10) (11)	$d \neq a$ $d \neq b$ (11) (10)	$b \neq c$ $b \neq d$ (11) (10)
	= *4 = *4		= *4 = *4	

El*5 $\langle ab \rangle = \{ \langle a \rangle, \langle a, b \rangle \}$

-	El*5	
(w)	$\forall x. x \in \langle ab \rangle \equiv x \in \{ \langle a \rangle, \langle a, b \rangle \}$	
-	$w \in \langle ab \rangle \equiv w \in \{ \langle a \rangle, \langle a, b \rangle \}$	

$\frac{- \quad w \in \langle ab \rangle}{1 \quad w \in \{\langle a \rangle, \langle a, b \rangle\}}$ <hr style="border-top: 1px dashed black;"/> $\frac{- \quad w = \langle a \rangle \vee w = \langle a, b \rangle}{2 \quad w = \langle a \rangle}$ $3 \quad w = \langle a, b \rangle$ <hr style="border-top: 1px dashed black;"/> $\frac{(1) \quad - \quad \neg . w = \langle a \rangle \vee w = \langle a, b \rangle}{w \neq \langle a \rangle \quad w \neq \langle a, b \rangle}$ <p style="text-align: center; margin-top: -10px;"> $\quad \quad \quad (2) \quad \quad \quad (3)$ </p>	$\frac{- \quad w \in \{\langle a \rangle, \langle a, b \rangle\}}{1 \quad w \in \langle ab \rangle}$ <hr style="border-top: 1px dashed black;"/> $\frac{- \quad w = \langle a \rangle \vee w = \langle a, b \rangle}{2 \quad w = \langle a \rangle}$ $3 \quad w = \langle a, b \rangle$ <hr style="border-top: 1px dashed black;"/> $\frac{(1) \quad - \quad \neg . w = \langle a \rangle \vee w = \langle a, b \rangle}{w \neq \langle a \rangle \quad w \neq \langle a, b \rangle}$ <p style="text-align: center; margin-top: -10px;"> $\quad \quad \quad (2) \quad \quad \quad (3)$ </p>
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*El*6 $\langle ab \rangle \in \langle cd \rangle \rightarrow a = c \wedge b = d$

$\frac{- \quad \text{El*6}}{1 \quad \langle ab \rangle \in \langle cd \rangle}$ $2 \quad a = c \wedge b = d \quad \text{Cut El*5}$ <hr style="border-top: 1px dashed black;"/> $3 \quad \langle ab \rangle \neq \{\langle a \rangle, \langle a, b \rangle\}$ $4 \quad \langle cd \rangle \neq \{\langle c \rangle, \langle c, d \rangle\}$ <p style="margin-left: -20px;">(1, 3, 4, =)</p> $5 \quad \{\langle a \rangle, \langle a, b \rangle\} \in \{\langle c \rangle, \langle c, d \rangle\} \quad \text{Cut El*1}$ $6 \quad \neg . \langle a \rangle \in \langle c \rangle \rightarrow a = c \quad \text{Cut El*1}$ <hr style="border-top: 1px dashed black;"/> $7 \quad \neg . \langle c, d \rangle \in \langle a \rangle \rightarrow a = c \wedge a = d$ $8 \quad \neg . \langle a, b \rangle \in \langle c \rangle \rightarrow c = a \wedge c = b \quad \text{Cut El*4}$ <p style="margin-left: 20px;">(5)</p> $9 \quad \neg . \langle a, b \rangle \in \langle c, d \rangle \rightarrow [a = c \wedge b = d] \vee [a = d \wedge b = c]$ $10 \quad \neg . [\langle a \rangle = \langle c \rangle \wedge \langle a, b \rangle = \langle c, d \rangle] \vee [\langle a \rangle = \langle c, d \rangle \wedge \langle a, b \rangle = \langle c \rangle]$ <p style="margin-left: -20px;">(2)</p> $11 \quad a = c \quad (6)$	$11 \quad b = d \quad (7)$ <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">(*)</p> $12 \quad \langle a \rangle \neq \langle c \rangle$ $\text{Spf.} \quad \dots$ <hr style="border-top: 1px dashed black;"/> $(6) \quad \langle a \rangle \in \langle c \rangle \quad a \neq c \quad (11)$
$12 \quad \langle a \rangle \neq \langle c \rangle$ $13 \quad \langle a, b \rangle \neq \langle c, d \rangle$ <hr style="border-top: 1px dashed black;"/> $(9, 13) \quad - \quad \neg . [a = c \wedge b = d] \vee [a = d \wedge b = c]$ $* \text{Spf.} \quad a \neq c \quad 14 \quad a \neq d$ $b \neq d \quad 15 \quad b \neq c$ <p style="margin-left: -20px;">(11) (6)</p> $\langle a \rangle \in \langle c \rangle \quad a \neq c$ <p style="text-align: center; margin-top: -10px;">(12) (11, 11, 15, =)</p>	$12 \quad \langle a \rangle \neq \langle c, d \rangle$ $13 \quad \langle a, b \rangle \neq \langle c \rangle$ <hr style="border-top: 1px dashed black;"/> $(7) \quad \langle c, d \rangle \in \langle a \rangle \quad \text{Spf.} \quad a \neq c$ $14 \quad a \neq d$ <hr style="border-top: 1px dashed black;"/> $(8) \quad \langle a, b \rangle \in \langle c \rangle \quad c \neq a$ $c \neq b$ <p style="text-align: center; margin-top: -10px;">(11, 11, =)</p>

Remark: Cut =*6 is used at every place indicated by (12) and (13) in the proof of El*6.

*El*7 $\langle ab \rangle = \langle cd \rangle \rightarrow a = c \wedge b = d$ (From *El*6)

El*8 $a=c \wedge b=d \rightarrow \langle ab \rangle \subseteq \langle cd \rangle$

- El*8	
1 $a \neq c$	
2 $b \neq d$	
- $\langle ab \rangle \subseteq \langle cd \rangle$	

(w) $\forall x. x \in \langle ab \rangle \rightarrow x \in \langle cd \rangle$	
- $w \in \langle ab \rangle \rightarrow w \in \langle cd \rangle$	
- $w \notin \langle ab \rangle$	
- $w \in \langle cd \rangle$	

3 $\neg. w = \{a\} \vee w = \{a, b\}$	
- $w = \{c\} \vee w = \{c, d\}$	
- $w = \{c\}$	
- $w = \{c, d\}$	

4 $\forall x. x \in w \equiv x \in \{c\}$	
5 $\forall x. x \in w \equiv x \in \{c, d\}$	

(3) 6 $w \neq \{a\}$ (*)	6 $w \neq \{a, b\}$ (**)

6 (*)	
7 $\neg \forall x. x \in w \equiv x \in \{a\}$	
(r, 4) 8 $r \in w \equiv r \in \{c\}$	
(7) 9 $r \in w \neq r \in \{a\}$	
(8) 10 $r \in w$ - $r \in \{c\}$	10 $r \in \{c\}$ - $r \in w$
- $r \notin \{c\}$	10 $r \neq w$
11 $r \neq c$	11 $r = c$
(9) $r \in w$ - $r \in \{a\}$ (10)	(9) $r \in w$ - $r \notin \{a\}$ (10)
$r = a$ (1, 11)	$r \neq a$ (1, 11)
Cut = *4	Cut = *5
Cut = *5	

6 (**)	
7 $\neg \forall x. x \in w \equiv x \in \{a, b\}$	
(r, 5) 8 $r \in w \equiv r \in \{c, d\}$	
(7) 9 $r \in w \neq r \in \{a, b\}$	
(8) 10 $r \in w$ - $r \in \{c, d\}$	10 $r \in \{c, d\}$ - $r \in w$
- $r \notin \{c, d\}$	10 $r \neq w$
11 $\neg. r = c \vee r = d$	- $r = c \vee r = d$
(9) $r \in w$ - $r \in \{a, b\}$ (10)	11 $r = c$
- $r = a \vee r = b$	12 $r = d$
12 $r = a$	(9) $r \in w$ - $r \notin \{a, b\}$ (10)
13 $r = b$	- $\neg. r = a \vee r = b$
(11) $r \neq c$ (1, 12) $r \neq d$ (2, 13)	$r \neq a$ (1, 11) $r \neq b$ (2, 12)
Cut = *4 Cut = *4	Cut = *5 Cut = *5
Cut = *5 Cut = *5	

El*9 $a=c \wedge b=d \rightarrow \langle ab \rangle = \langle cd \rangle$ (From El*8, =*8)

El*10 $a \in c \wedge b \in c \rightarrow \langle a, b \rangle \in c$

-	El*10
1	$a \notin c$
2	$b \notin c$
-	$\langle a, b \rangle \in c$
(w) -	$\forall x. x \in \langle a, b \rangle \rightarrow x \in c$
-	$w \in \langle a, b \rangle \rightarrow w \in c$
-	$w \notin \langle a, b \rangle$
3	$w \in c$
-	$\neg. w = a \vee w = b$
$w \neq a$ (1, 3, 1)	$w \neq b$ (2, 3, 1)

El*11 $\forall xy[x \in a \wedge y \in a \rightarrow x = y] \wedge p \in a \rightarrow a = \langle p \rangle$

-	El*11
1	$\neg \forall xy. x \in a \wedge y \in a \rightarrow x = y$
2	$p \notin a$
-	$a = \langle p \rangle$
(w) -	$\forall x. x \in a \equiv x \in \langle p \rangle$
-	$w \in a \equiv w \in \langle p \rangle$
3	$w \in a$ - $w \in \langle p \rangle$
-	$w \notin \langle p \rangle$ 3 $w \notin a$
$w \neq p$ (2, 3, 1)	4 $w = p$ (1)
-	$\neg. w \in a \wedge p \in a \rightarrow w = p$
$w \in a$ (3)	$p \in a$ (2)
$w \neq p$ (1)	$w = p$ (1)

Image of Operator (Im)

Im*1 $\sigma \in \text{Un} \wedge \tau \in \sigma \rightarrow \tau \in \text{Un}$

-	Im*1
-	$\sigma \notin \text{Un}$
-	$\tau \notin \sigma$
-	$\tau \in \text{Un}$
1	$\neg \forall xyz. \langle xy \rangle \in \sigma \wedge \langle xz \rangle \in \sigma \rightarrow y = z$
2	$\neg \forall x. x \in \tau \rightarrow x \in \sigma$
-	$\forall xyz. \langle xy \rangle \in \tau \wedge \langle xz \rangle \in \tau \rightarrow y = z$
(r, s, t) -	$\langle rs \rangle \in \tau \wedge \langle rt \rangle \in \tau \rightarrow s = t$
3	$\langle rs \rangle \notin \tau$
4	$\langle rt \rangle \notin \tau$
5	$s = t$

$$\begin{array}{c}
 \text{(2)} \\
 \hline
 - \quad \neg. \langle rs \rangle \in \tau \rightarrow \langle rs \rangle \in \sigma \\
 6 \quad \neg. \langle rt \rangle \in \tau \rightarrow \langle rt \rangle \in \sigma \\
 \hline
 \langle rs \rangle \in \tau \quad \neg \quad \langle rs \rangle \notin \sigma \\
 \text{(3)} \quad \text{(6)} \\
 \hline
 \langle rt \rangle \in \tau \quad \neg \quad \langle rt \rangle \notin \sigma \\
 \text{(4)} \quad \text{(1)} \quad \text{(8)} \\
 \hline
 - \quad \neg. \langle rs \rangle \in \sigma \wedge \langle rt \rangle \in \sigma \rightarrow s = t \\
 \langle rs \rangle \in \sigma \quad \langle rt \rangle \in \sigma \quad s \neq t \\
 \text{(7)} \quad \text{(8)} \quad \text{(5)}
 \end{array}$$

Im*2 $\sigma \in \text{Un} \wedge \langle ab \rangle \in \sigma \rightarrow a^\sigma = b$

$$\begin{array}{c}
 - \quad \text{Im*2} \\
 \hline
 1 \quad \sigma \notin \text{Un} \\
 2 \quad \langle ab \rangle \notin \sigma \\
 - \quad a^\sigma = b \\
 \hline
 \text{(w)} - \quad \forall x. x \in a^\sigma \equiv x \in b \\
 \hline
 - \quad w \in a^\sigma \quad 3 \quad w \in b \\
 3 \quad w \notin b \quad - \quad w \notin a^\sigma \\
 \hline
 - \quad \exists x. \langle ax \rangle \in \sigma \wedge w \in x \quad - \quad \neg \exists x. \langle ax \rangle \in \sigma \wedge w \in x \\
 - \quad \langle ab \rangle \in \sigma \wedge w \in b \quad \text{(b}_1) - \quad \neg. \langle ab_1 \rangle \in \sigma \wedge w \in b_1 \\
 \hline
 \langle ab \rangle \in \sigma \quad w \in b \quad 4 \quad \langle ab_1 \rangle \notin \sigma \\
 \text{(2)} \quad \text{(3)} \quad 5 \quad w \notin b_1 \\
 \hline
 \text{(1)} - \quad \neg \forall xyz. \langle xy \rangle \in \sigma \wedge \langle xz \rangle \in \sigma \rightarrow y = z \\
 - \quad \neg. \langle ab \rangle \in \sigma \wedge \langle ab_1 \rangle \in \sigma \rightarrow b = b_1 \\
 \hline
 \langle ab \rangle \in \sigma \quad \langle ab_1 \rangle \in \sigma \quad b \neq b_1 \\
 \text{(2)} \quad \text{(4)} \quad \text{(2, 5, =)}
 \end{array}$$

Im*3 $\sigma \in \text{Un} \wedge a \in D_\sigma \rightarrow \langle aa^\sigma \rangle \in \sigma$

$$\begin{array}{c}
 - \quad \text{Im*3} \\
 \hline
 1 \quad \sigma \notin \text{Un} \\
 - \quad a \notin D_\sigma \\
 2 \quad \langle aa^\sigma \rangle \notin \sigma \\
 \hline
 - \quad \neg \exists x. \langle ax \rangle \in \sigma \\
 \text{(r)} \quad 3 \quad \langle ar \rangle \notin \sigma \quad \text{Cut Im*2} \\
 \hline
 - \quad \neg. \sigma \in \text{Un} \wedge \langle ar \rangle \in \sigma \rightarrow a^\sigma = r \\
 \hline
 \sigma \in \text{Un} \quad \langle ar \rangle \in \sigma \quad a^\sigma \neq r \\
 \text{(1)} \quad \text{(3)} \quad \text{(2, 3, =, I)}
 \end{array}$$

Im*4 $\sigma \in \text{Un} \wedge a \in D_\sigma \wedge a \in A \rightarrow a^\sigma \in A^\sigma$

$$\begin{array}{c}
 - \quad \text{Im*4} \\
 \hline
 1 \quad \sigma \notin \text{Un} \quad 3 \quad a \notin A \\
 2 \quad a \notin D_\sigma \quad 4 \quad a^\sigma \notin A^\sigma \\
 \text{(4)} \quad - \quad \exists x. x \in A \wedge \langle xa^\sigma \rangle \in \sigma \\
 - \quad a \in A \wedge \langle aa^\sigma \rangle \in \sigma
 \end{array}$$

$$\begin{array}{c}
 \frac{\alpha \in A \quad \langle \alpha \alpha^\sigma \rangle \in \sigma}{\exists x. \sigma \in \text{Un} \wedge \alpha \in D_\sigma \rightarrow \langle \alpha \alpha^\sigma \rangle \in \sigma} \text{Cut Im*3} \\
 \frac{\sigma \in \text{Un} \quad \alpha \in D_\sigma \quad \langle \alpha \alpha^\sigma \rangle \notin \sigma}{\exists x. \sigma \in \text{Un} \wedge \alpha \in D_\sigma \rightarrow \langle \alpha \alpha^\sigma \rangle \in \sigma} \\
 \begin{array}{ccc}
 (3) & & \\
 (1) & & (2) \quad (5)
 \end{array}
 \end{array}$$

Im*5 $A \subseteq B \rightarrow A^\sigma \subseteq B^\sigma$

$$\begin{array}{c}
 \frac{}{\text{Im*5}} \\
 \frac{1 \quad A \notin B}{A^\sigma \subseteq B^\sigma} \\
 \frac{(5) \quad \forall x. x \in A^\sigma \rightarrow x \in B^\sigma}{s \in A^\sigma \rightarrow s \in B^\sigma} \\
 \frac{}{s \notin A^\sigma} \\
 \frac{}{s \in B^\sigma} \\
 \frac{(r) \quad \exists x. x \in A \wedge \langle xs \rangle \in \sigma}{\exists x. x \in B \wedge \langle xs \rangle \in \sigma} \\
 \frac{2}{\exists x. x \in B \wedge \langle xs \rangle \in \sigma} \\
 \frac{}{\exists x. x \in B \wedge \langle xs \rangle \in \sigma} \\
 \frac{3 \quad r \notin A}{\langle rs \rangle \notin \sigma} \\
 \frac{4 \quad \langle rs \rangle \notin \sigma}{r \in B \wedge \langle rs \rangle \in \sigma} \\
 \frac{(2)}{r \in B \wedge \langle rs \rangle \in \sigma} \\
 \frac{5 \quad r \in B \quad \langle rs \rangle \in \sigma}{\exists x. x \in A \rightarrow x \in B} \\
 \frac{(1)}{\exists x. x \in A \rightarrow x \in B} \\
 \frac{}{\exists x. x \in A \rightarrow x \in B} \\
 \frac{r \in A \quad r \notin B}{\exists x. x \in A \rightarrow x \in B} \\
 \begin{array}{ccc}
 (3) & & (5)
 \end{array}
 \end{array}$$

Im*6 $\sigma \in \text{Un} \wedge \alpha \in D_\sigma \rightarrow \alpha^\sigma \in W_\sigma$

$$\begin{array}{c}
 \frac{}{\text{Im*6}} \\
 \frac{1 \quad \sigma \notin \text{Un}}{\alpha \notin D_\sigma} \\
 \frac{}{\alpha \notin D_\sigma} \\
 \frac{}{\alpha^\sigma \in W_\sigma} \\
 \frac{(r) \quad \exists x. \langle \alpha x \rangle \in \sigma}{\exists x. \langle \alpha x^\sigma \rangle \in \sigma} \\
 \frac{2}{\exists x. \langle \alpha x^\sigma \rangle \in \sigma} \\
 \frac{3 \quad \langle \alpha r \rangle \notin \sigma}{\langle \alpha \alpha^\sigma \rangle \in \sigma} \\
 \frac{(2)}{\langle \alpha \alpha^\sigma \rangle \in \sigma} \quad \text{Cut Im*2} \\
 \frac{}{\exists x. \sigma \in \text{Un} \wedge \langle \alpha x \rangle \in \sigma \rightarrow \alpha^\sigma = r} \\
 \frac{\sigma \in \text{Un} \quad \langle \alpha r \rangle \in \sigma \quad \alpha^\sigma \neq r}{\exists x. \sigma \in \text{Un} \wedge \langle \alpha x \rangle \in \sigma \rightarrow \alpha^\sigma = r} \\
 \begin{array}{ccc}
 (1) & (3) & (3, 4, =, I)
 \end{array}
 \end{array}$$

Im*7 $\langle ab \rangle \notin \sigma \rightarrow a \in D_\sigma$

Im*8 $\langle ab \rangle \in \sigma \rightarrow b \in W_\sigma$

Im*9 $W_\sigma = (D_\sigma)^\circ$

$\frac{\text{Im*9}}{\frac{\forall x. x \in W_\sigma \equiv x \in (D_\sigma)^\circ}{s \in W_\sigma \equiv s \in (D_\sigma)^\circ}}$	
$\frac{\frac{s \in W_\sigma}{s \notin (D_\sigma)^\circ}}{\frac{1 \quad \exists x. \langle xs \rangle \in \sigma}{(r) \quad \neg \exists x. x \in D_\sigma \wedge \langle xs \rangle \in \sigma}}{\frac{\neg \exists x. x \in D_\sigma \wedge \langle rs \rangle \in \sigma}{\text{Spf} \quad r \notin D_\sigma}}{\frac{2 \quad \langle rs \rangle \notin \sigma}{(1) \quad \langle rs \rangle \in \sigma}}{\frac{\langle rs \rangle \in \sigma}{(2)}}$	$\frac{\frac{s \in (D_\sigma)^\circ}{s \notin W_\sigma}}{\frac{1 \quad \exists x. x \in D_\sigma \wedge \langle xs \rangle \in \sigma}{(r) \quad \neg \exists x. \langle xs \rangle \in \sigma}}{\frac{2 \quad \langle rs \rangle \notin \sigma}{(1) \quad r \in D_\sigma \wedge \langle rs \rangle \in \sigma}}{\frac{\neg \exists x. \langle rx \rangle \in \sigma}{(2) \quad \langle rs \rangle \in \sigma}}$

Inverse Operator (- 1)

- 1*1 $(\sigma^{-1})^{-1} \subseteq \sigma$

$\frac{- \quad - 1*1}{\frac{a \notin (\sigma^{-1})^{-1}}{a \in \sigma}}$	
$\frac{(r, s) \quad \neg \exists xy. a = \langle xy \rangle \wedge \langle yx \rangle \in \sigma^{-1}}{\neg \exists xy. \langle sr \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma}$	$\frac{\frac{2 \quad a \neq \langle rs \rangle}{\langle sr \rangle \notin \sigma^{-1}}}{\frac{\neg \exists xy. \langle sr \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma}{(r_1, s_1) \quad \neg \exists xy. \langle sr \rangle = \langle s_1 r_1 \rangle \wedge \langle r_1 s_1 \rangle \in \sigma}}$
$\frac{\langle sr \rangle \neq \langle s_1 r_1 \rangle}{\langle r_1 s_1 \rangle \notin \sigma}$ <p style="text-align: center; margin-top: -10px;">(1, 2, =, 1)</p>	

- 1*2 $\sigma \in \text{Un} \rightarrow (\sigma^{-1})^{-1} \in \text{Un}$

$\frac{- \quad - 1*2}{\frac{1 \quad \sigma \notin \text{Un}}{2 \quad (\sigma^{-1})^{-1} \in \text{Un}}}$	
$\frac{\text{Cut Im*1}}{\frac{\neg \exists xy. \langle sr \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma}{\sigma \in \text{Un} \quad (\sigma^{-1})^{-1} \subseteq \sigma \quad (\sigma^{-1})^{-1} \notin \text{Un}}}$	
$\frac{(1) \quad \sigma \in \text{Un}}{- 1*1}$	$\frac{(2) \quad (\sigma^{-1})^{-1} \notin \text{Un}}{- 1*1}$

$$\begin{array}{l}
 -1*3 \quad \sigma \in \text{Un}_2 \rightarrow \sigma^{-1} \in \text{Un}_2 \\
 \begin{array}{c}
 - \\
 - \\
 - \\
 - \\
 1 \\
 2 \\
 3 \\
 (1)
 \end{array}
 \frac{
 \begin{array}{c}
 -1*3 \\
 \sigma \notin \text{Un}_2 \\
 \sigma^{-1} \in \text{Un}_2 \\
 \neg \cdot \sigma \in \text{Un} \wedge \sigma^{-1} \in \text{Un} \\
 \sigma^{-1} \in \text{Un} \wedge (\sigma^{-1})^{-1} \in \text{Un} \\
 \sigma \notin \text{Un} \\
 \sigma^{-1} \notin \text{Un}
 \end{array}
 }{
 \begin{array}{c}
 \sigma^{-1} \in \text{Un} \\
 (\sigma^{-1})^{-1} \in \text{Un} \\
 \neg \cdot \sigma \in \text{Un} \rightarrow (\sigma^{-1})^{-1} \in \text{Un} \\
 \sigma \in \text{Un} \quad (\sigma^{-1})^{-1} \notin \text{Un}
 \end{array}
 }
 \text{Cut } -1*2
 \end{array}$$

$$\begin{array}{l}
 -1*4 \quad D_\sigma \subseteq W_{\sigma^{-1}} \\
 \begin{array}{c}
 - \\
 - \\
 - \\
 (r) \\
 1 \\
 2 \\
 (1) \\
 - \\
 - \\
 - \\
 - \\
 (1)
 \end{array}
 \frac{
 \begin{array}{c}
 -1*4 \\
 a \notin D_\sigma \\
 a \in W_{\sigma^{-1}} \\
 \neg \exists x. \langle ax \rangle \in \sigma \\
 \exists x. \langle xa \rangle \in \sigma^{-1} \\
 \langle ar \rangle \notin \sigma \\
 \langle ra \rangle \in \sigma^{-1} \\
 \exists xy. \langle ra \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma \\
 \langle ra \rangle = \langle ra \rangle \wedge \langle ar \rangle \in \sigma \\
 \langle ra \rangle = \langle ra \rangle \quad \langle ar \rangle \in \sigma
 \end{array}
 }{
 \begin{array}{c}
 \langle ra \rangle = \langle ra \rangle \quad \langle ar \rangle \in \sigma
 \end{array}
 }
 \end{array}$$

$$\begin{array}{l}
 -1*5 \quad W_{\sigma^{-1}} \subseteq D_\sigma \\
 \begin{array}{c}
 - \\
 - \\
 - \\
 (r) \\
 1 \\
 - \\
 - \\
 (r_1, a_1) \\
 - \\
 2 \\
 3 \\
 (1)
 \end{array}
 \frac{
 \begin{array}{c}
 -1*5 \\
 a \notin W_{\sigma^{-1}} \\
 a \in D_\sigma \\
 \neg \exists x. \langle xa \rangle \in \sigma^{-1} \\
 \exists x. \langle ax \rangle \in \sigma \\
 \langle ra \rangle \in \sigma^{-1} \\
 \neg \exists xy. \langle ra \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma \\
 \neg \cdot \langle ra \rangle = \langle r_1 a_1 \rangle \wedge \langle a_1 r_1 \rangle \in \sigma \\
 \langle ra \rangle \notin \langle r_1 a_1 \rangle \\
 \langle a_1 r_1 \rangle \notin \sigma \\
 \langle ar \rangle \in \sigma
 \end{array}
 }{
 \begin{array}{c}
 \langle ar \rangle \in \sigma \\
 (2, 2, =, 1)
 \end{array}
 }
 \end{array}$$

- 1*6 $D_\sigma = W_{\sigma^{-1}}$ (From -1*4, -1*5 and =*7)
- 1*7 $W_\sigma \subseteq D_{\sigma^{-1}}$ (Similar to -1*4)
- 1*8 $D_{\sigma^{-1}} \subseteq W_\sigma$ (Similar to -1*5)

-1*9 $W_\sigma = D_{\sigma^{-1}}$ (From -1*7, -1*8 and =*7)

-1*10 $D_\sigma = (W_\sigma)^{\sigma^{-1}}$

	-1*10
- $r \in D_\sigma$	- $r \in (W_\sigma)^{\sigma^{-1}}$
- $r \notin (W_\sigma)^{\sigma^{-1}}$	- $r \notin D_\sigma$
1 $\exists x. \langle rx \rangle \in \sigma$	1 $\exists x. x \in W_\sigma \wedge \langle xr \rangle \in \sigma^{-1}$
(S) $\neg \exists x. x \in W_\sigma \wedge \langle xr \rangle \in \sigma^{-1}$	(S) $\neg \exists x. \langle rx \rangle \in \sigma$
- $\neg \exists s \in W_\sigma \wedge \langle sr \rangle \in \sigma^{-1}$	(1) $\langle rs \rangle \notin \sigma$
Spf. $s \notin W_\sigma$	- $s \in W_\sigma \wedge \langle sr \rangle \in \sigma^{-1}$
- $\langle sr \rangle \notin \sigma^{-1}$	- $s \in W_\sigma$ - $\langle sr \rangle \in \sigma^{-1}$
(S1r1) $\neg \exists xy. \langle sr \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma$	- $\exists x. \langle xs \rangle \in \sigma$ - $\exists xy. \langle sr \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma$
- $\neg \exists s_1 r_1. \langle sr \rangle = \langle s_1 r_1 \rangle \wedge \langle r_1 s_1 \rangle \in \sigma$	$\langle rs \rangle \in \sigma$ - $\langle sr \rangle = \langle sr \rangle \wedge \langle rs \rangle \in \sigma$
2 $\langle sr \rangle \neq \langle s_1 r_1 \rangle$	(2) $\langle sr \rangle = \langle sr \rangle$ $\langle rs \rangle \in \sigma$
3 $\langle r_1 s_1 \rangle \notin \sigma$	(=) $\langle sr \rangle = \langle sr \rangle$ $\langle rs \rangle \in \sigma$
(1) $\langle rs \rangle \in \sigma$	(2) $\langle rs \rangle \in \sigma$
(2, 3, =, 1)	

-1*11 $\sigma \in \text{Map}_2^{a,b} \rightarrow \sigma^{-1} \in \text{Map}_2^{b,a}$

	-1*11
- $\sigma \notin \text{Map}_2^{a,b}$	
- $\sigma^{-1} \in \text{Map}_2^{b,a}$	
1 $\neg \exists \sigma \in \text{Un}_2 \wedge D_\sigma = a \wedge W_\sigma = b$	
2 $\sigma^{-1} \in \text{Un}_2 \wedge D_{\sigma^{-1}} = b \wedge W_{\sigma^{-1}} = a$	
3 $\sigma \notin \text{Un}_2$	
4 $D_\sigma \neq a$	
5 $W_\sigma \neq b$	
(2) $\sigma^{-1} \in \text{Un}_2$	(4, =) $D_{\sigma^{-1}} = b$
(3, =) $W_{\sigma^{-1}} = a$	
Cut -1*3	Cut -1*9 Cut -1*6

-1*12 $\sigma \in \text{Un}_2 \wedge a \in D_\sigma \rightarrow a^{\sigma\sigma^{-1}} = a$

	-1*12
1 $\sigma \notin \text{Un}$	3 $a \notin D_\sigma$
2 $\sigma^{-1} \notin \text{Un}$	4 $a^{\sigma\sigma^{-1}} = a$ Cut $\circ * 2$
- $\neg \exists \sigma \in \text{Un} \wedge a \in D_\sigma \rightarrow a^{\sigma\sigma^{-1}} = a^{\sigma^{-1}\sigma}$	
$\sigma \in \text{Un}$ $a \in D_\sigma$	5 $a^{\sigma\sigma^{-1}} \neq a^{\sigma^{-1}\sigma}$ Cut $\iota * 12$
(1) (3)	- $\neg \exists \sigma^{-1} \in \text{Un} \rightarrow \sigma^{-1} \circ \sigma = \iota \upharpoonright D_\sigma$
$\sigma^{-1} \in \text{Un}$	6 $\sigma^{-1} \circ \sigma \neq \iota \upharpoonright D_\sigma$ Cut $\uparrow * 8$
(2)	- $\neg \exists a \in D_\sigma \rightarrow a^{\iota \upharpoonright D_\sigma} = a^{\iota}$
$a \in D_\sigma$	7 $a^{\iota \upharpoonright D_\sigma} \neq a^{\iota}$ Cut $\iota * 2$
(3)	$a^{\iota} \neq a$
	(4, 5, 6, 7, =)

-1*13 $\sigma \in \text{Un} \wedge a \in D_\sigma \rightarrow a^{(\sigma^{-1})^{-1}} = a^\sigma$

<p style="text-align: center;">-</p> <p style="text-align: center;">1 $\sigma \notin \text{Un}$</p> <p style="text-align: center;">2 $a \notin D_\sigma$</p> <p style="text-align: center;">- $a^{(\sigma^{-1})^{-1}} = a^\sigma$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">(w) - $\forall x. x \in a^{(\sigma^{-1})^{-1}} \equiv x \in a^\sigma$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $w \in a^{(\sigma^{-1})^{-1}} \equiv w \in a^\sigma$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $w \in a^{(\sigma^{-1})^{-1}}$ 3 $w \in a^\sigma$</p> <p style="text-align: center;">3 $w \notin a^\sigma$ - $w \notin a^{(\sigma^{-1})^{-1}}$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\exists x. \langle ax \rangle \in (\sigma^{-1})^{-1} \wedge w \in x$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\langle aa^\sigma \rangle \in (\sigma^{-1})^{-1} \wedge w \in a^\sigma$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\langle aa^\sigma \rangle \in (\sigma^{-1})^{-1}$ $w \in a^\sigma$</p> <p style="text-align: center;">- $\langle a^\sigma a \rangle \in \sigma^{-1}$ (3)</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">4 $\langle aa^\sigma \rangle \in \sigma$ Cut Im*3</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\nabla. \sigma \in \text{Un} \wedge a \in D_\sigma \rightarrow \langle aa^\sigma \rangle \in \sigma$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">$\sigma \in \text{Un}$ $a \in D_\sigma$ $\langle aa^\sigma \rangle \in \sigma$</p> <p style="text-align: center;">(1) (2) (4)</p>	<p style="text-align: center;">-</p> <p style="text-align: center;">-1*13</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">1 $\sigma \notin \text{Un}$</p> <p style="text-align: center;">2 $a \notin D_\sigma$</p> <p style="text-align: center;">- $a^{(\sigma^{-1})^{-1}} = a^\sigma$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">(b) - $\nabla \exists x. \langle ax \rangle \in (\sigma^{-1})^{-1} \wedge w \in x$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\nabla. \langle ab \rangle \in (\sigma^{-1})^{-1} \wedge w \in b$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\langle ab \rangle \notin (\sigma^{-1})^{-1}$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">(a, b) 4 $w \notin b$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">(a, b) - $\langle ba \rangle \notin \sigma^{-1}$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">5 $\langle ab \rangle \notin \sigma$ Cut Im*2</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\nabla. \sigma \in \text{Un} \wedge \langle ab \rangle \in \sigma \rightarrow a^\sigma = b$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">$\sigma \in \text{Un}$ $\langle ab \rangle \in \sigma$ $a^\sigma \neq b$</p> <p style="text-align: center;">(1) (5, =, 1) (3, 4, =)</p>
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-1*14 $\sigma \in \text{Un}_2 \wedge a \in W_\sigma \rightarrow a^{\sigma^{-1}\sigma} = a$

<p style="text-align: center;">-</p> <p style="text-align: center;">-1*14</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">1 $\sigma \notin \text{Un}_2$</p> <p style="text-align: center;">2 $a \notin W_\sigma$</p> <p style="text-align: center;">3 $a^{\sigma^{-1}\sigma} = a$ Cut -1*12</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\nabla. \sigma^{-1} \in \text{Un}_2 \wedge a \in D_{\sigma^{-1}} \rightarrow a^{\sigma^{-1}(\sigma^{-1})^{-1}} = a$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">$\sigma^{-1} \in \text{Un}_2$ $a \in D_{\sigma^{-1}}$ 4 $a^{\sigma^{-1}(\sigma^{-1})^{-1}} \neq a$ Cut -1*13</p> <p style="text-align: center;">Cut -1*3 Cut -1*9 - $\nabla. \sigma \in \text{Un} \wedge a^{\sigma^{-1}} \in D_\sigma \rightarrow a^{\sigma^{-1}(\sigma^{-1})^{-1}} = a^{\sigma^{-1}\sigma}$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">$\sigma \in \text{Un}$ 5 $a^{\sigma^{-1}} \in D_\sigma$ $a^{\sigma^{-1}(\sigma^{-1})^{-1}} \neq a^{\sigma^{-1}\sigma}$</p> <p style="text-align: center;">(1) (3, 4, =)</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">Cut -1*7</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\nabla. a \in W_\sigma \rightarrow a \in D_{\sigma^{-1}}$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">$a \in W_\sigma$ 6 $a \notin D_{\sigma^{-1}}$ Cut Im*6</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">- $\nabla. \sigma^{-1} \in \text{Un} \wedge a \in D_{\sigma^{-1}} \rightarrow a^{\sigma^{-1}} \in W_{\sigma^{-1}}$</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">$\sigma^{-1} \in \text{Un}$ $a \in D_{\sigma^{-1}}$ $a^{\sigma^{-1}} \notin W_{\sigma^{-1}}$</p> <p style="text-align: center;">(1) (6) (5, =)</p> <hr style="border-top: 1px dashed black;"/> <p style="text-align: center;">Cut -1*6</p>	
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Composition of Operators (◦)

◦*1 $A^{\sigma\tau} = A^{\tau\circ\sigma}$ ($A^{\sigma\tau} = (A^\sigma)^\tau$)

<div style="text-align: center; margin-bottom: 10px;">◦*1</div> <hr/> <p>1 $t \in A^{\sigma\tau}$ - $t \notin A^{\tau\circ\sigma}$</p> <hr/> <p>(r) $\neg \exists x. x \in A \wedge \langle xt \rangle \in \tau \circ \sigma$ - $\neg. r \in A \wedge \langle rt \rangle \in \tau \circ \sigma$</p> <hr/> <p>2 $r \notin A$ - $\langle rt \rangle \notin \tau \circ \sigma$</p> <hr/> <p>(r₁, s, t₁) $\neg \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau$ - $\neg. \langle rt \rangle = \langle r_1 t_1 \rangle \wedge \langle r_1 s \rangle \in \sigma \wedge \langle s t_1 \rangle \in \tau$</p> <hr/> <p>3 $\langle rt \rangle \neq \langle r_1 t_1 \rangle$ 4 $\langle r_1 s \rangle \notin \sigma$ 5 $\langle s t_1 \rangle \notin \tau$</p> <hr/> <p>(1) $\exists x. x \in A^\sigma \wedge \langle xt \rangle \in \tau$ - $s \in A^\sigma \wedge \langle st \rangle \in \tau$ - $s \in A^\sigma$ $\langle st \rangle \in \tau$ (3, 5, =, I)</p> <hr/> <p>$\exists x. x \in A \wedge \langle xs \rangle \in \sigma$ - $r \in A \wedge \langle rs \rangle \in \sigma$</p> <hr/> <p>$r \in A$ $\langle rs \rangle \in \sigma$ (2) (3, 4, =, I)</p>	<div style="text-align: center; margin-bottom: 10px;">◦*1</div> <hr/> <p>1 $t \in A^{\tau\circ\sigma}$ - $t \notin A^{\sigma\tau}$</p> <hr/> <p>(s) $\neg \exists x. x \in A^\sigma \wedge \langle xt \rangle \in \tau$ - $\neg. s \in A^\sigma \wedge \langle st \rangle \in \tau$</p> <hr/> <p>2 $s \notin A^\sigma$ - $\langle st \rangle \notin \tau$</p> <hr/> <p>(r) $\neg \exists x. x \in A \wedge \langle xs \rangle \in \sigma$ - $\neg. r \in A \wedge \langle rs \rangle \in \sigma$</p> <hr/> <p>3 $r \notin A$ 4 $\langle rs \rangle \notin \sigma$</p> <hr/> <p>(1) $\exists x. x \in A \wedge \langle xt \rangle \in \tau \circ \sigma$ - $r \in A \wedge \langle rt \rangle \in \tau \circ \sigma$</p> <hr/> <p>$r \in A$ $\langle rt \rangle \in \tau \circ \sigma$ (3)</p> <hr/> <p>$\exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau$ - $\langle rt \rangle = \langle r_1 t_1 \rangle \wedge \langle r_1 s \rangle \in \sigma \wedge \langle s t_1 \rangle \in \tau$ $\langle r_1 t_1 \rangle = \langle r_1 t \rangle$ $\langle r_1 s \rangle \in \sigma$ $\langle s t_1 \rangle \in \tau$ (=) (4) (2)</p>
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◦*2 $\sigma \notin \text{Un} \wedge a \in D_\sigma \rightarrow a^{\sigma\tau} = a^{\tau\circ\sigma}$ ($a^{\sigma\tau} = (a^\sigma)^\tau$)

<div style="text-align: center; margin-bottom: 10px;">◦*2</div> <hr/> <p>1 $\sigma \notin \text{Un}$ 2 $a \in D_\sigma$ - $a^{\sigma\tau} = a^{\tau\circ\sigma}$</p> <hr/> <p>(w) $\forall x. x \in a^{\sigma\tau} \equiv x \in a^{\tau\circ\sigma}$ - $w \in a^{\sigma\tau} \equiv w \in a^{\tau\circ\sigma}$</p> <hr/> <p>3 $w \in a^{\sigma\tau}$ - $w \notin a^{\tau\circ\sigma}$</p> <hr/> <p>(c) $\neg \exists x. \langle ax \rangle \in \tau \circ \sigma \wedge w \in x$ - $\neg. \langle ac \rangle \in \tau \circ \sigma \wedge w \in c$</p> <hr/> <p>$\langle ac \rangle \notin \tau \circ \sigma$ 4 $w \notin c$</p> <hr/> <p>(a, b, c) $\neg \exists xyz. \langle ac \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau$ - $\neg. \langle ac \rangle = \langle ab \rangle \wedge \langle ab \rangle \in \sigma \wedge \langle bc \rangle \in \tau$</p> <hr/> <p>5 $\langle ab \rangle \notin \sigma$ 6 $\langle bc \rangle \notin \tau$</p> <hr/> <p>(3) $\exists x. \langle a^\sigma x \rangle \in \tau \wedge w \in x$ - $\exists xyz. \langle ac \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau$ - $\langle aa^\sigma \rangle \in \sigma \wedge \langle a^\sigma c \rangle \in \tau$</p>	<div style="text-align: center; margin-bottom: 10px;">◦*2</div> <hr/> <p>3 $w \in a^{\tau\circ\sigma}$ - $w \notin a^{\sigma\tau}$</p> <hr/> <p>(c) $\neg \exists x. \langle a^\sigma x \rangle \in \tau \wedge w \in x$ - $\neg. \langle a^\sigma c \rangle \in \tau \wedge w \in c$</p> <hr/> <p>4 $\langle a^\sigma c \rangle \notin \tau$ 5 $w \notin c$</p> <hr/> <p>(3) $\exists x. \langle ax \rangle \in \tau \circ \sigma \wedge w \in x$ - $\langle ac \rangle \in \tau \circ \sigma \wedge w \in c$</p> <hr/> <p>$\langle ac \rangle \in \tau \circ \sigma$ $w \in c$ (5)</p> <hr/> <p>$\exists xyz. \langle ac \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau$ - $\langle aa^\sigma \rangle \in \sigma \wedge \langle a^\sigma c \rangle \in \tau$</p>
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$\frac{- \quad \langle a^{\circ}c \rangle \in \tau \wedge w \in c}{\tau \quad \langle a^{\circ}c \rangle \in \tau \quad \text{Cut Im}^*2 \quad w \in c \quad (4)}$	$\frac{6 \quad \langle aa^{\circ} \rangle \in \sigma \quad \text{Cut Im}^*3 \quad \langle a^{\circ}c \rangle \in \tau \quad (4)}{- \quad \neg. \sigma \in \text{Un} \wedge a \in D_{\sigma} \rightarrow \langle aa^{\circ} \rangle \in \sigma}$
$\frac{- \quad \neg. \sigma \in \text{Un} \wedge \langle ab \rangle \in \sigma \rightarrow a^{\circ} = b}{\sigma \in \text{Un} \quad \langle ab \rangle \in \sigma \quad a^{\circ} \neq b \quad (1) \quad (5, =, I) \quad (6, \neg, =, I)}$	$\frac{\sigma \in \text{Un} \quad (1) \quad a \in D_{\sigma} \quad (2) \quad \langle aa^{\circ} \rangle \notin \sigma \quad (6)}{- \quad \neg. \sigma \in \text{Un} \wedge a \in D_{\sigma} \rightarrow \langle aa^{\circ} \rangle \in \sigma}$

◦*3 $\sigma \subseteq \tau \rightarrow \sigma \circ \rho \subseteq \tau \circ \rho$

$\frac{- \quad \circ *3}{1 \quad \sigma \not\subseteq \tau \quad \sigma \circ \rho \subseteq \tau \circ \rho}$	$\frac{6 \quad \langle st \rangle \in \sigma \quad (5) \quad \langle st \rangle \notin \tau \quad (2)}{- \quad \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \rho \wedge \langle yz \rangle \in \tau}$
$\frac{- \quad \forall x. x \in \sigma \circ \rho \rightarrow x \in \tau \circ \rho}{(w) \quad w \in \sigma \circ \rho \rightarrow w \in \tau \circ \rho}$	$\frac{- \quad w \notin \sigma \circ \rho \quad 2 \quad w \in \tau \circ \rho}{(r, s, t) \quad \neg \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \rho \wedge \langle yz \rangle \in \sigma}$
$\frac{- \quad w \neq \langle rt \rangle \quad 3 \quad \langle rs \rangle \notin \rho \quad 4 \quad \langle st \rangle \notin \sigma \quad 5}{(1) \quad \neg \forall x. x \in \sigma \rightarrow x \in \tau}$	$\frac{- \quad \neg. w = \langle rt \rangle \wedge \langle rs \rangle \in \rho \wedge \langle st \rangle \in \sigma}{\neg. \langle st \rangle \in \sigma \rightarrow \langle st \rangle \in \tau}$
$\frac{\langle st \rangle \in \sigma \quad (5)}{(2) \quad \langle st \rangle \notin \tau}$	$\frac{- \quad w = \langle rt \rangle \wedge \langle rs \rangle \in \rho \wedge \langle st \rangle \in \tau}{w = \langle rt \rangle \quad \langle rs \rangle \in \rho \quad \langle st \rangle \in \tau \quad (3) \quad (4) \quad (6)}$

◦*4 $\sigma = \tau \rightarrow \sigma \circ \rho = \tau \circ \rho$ (From ◦*3 and =*6, 7)

◦*5 $\sigma = \tau \rightarrow \rho \circ \sigma = \rho \circ \tau$ (Similar to ◦*4)

◦*6 $\rho \circ (\tau \circ \sigma) \subseteq (\rho \circ \tau) \circ \sigma$

$\frac{- \quad \circ *6}{1 \quad a \notin \rho \circ (\tau \circ \sigma) \quad a \in (\rho \circ \tau) \circ \sigma}$	$\frac{- \quad \neg \exists xyz. a = \langle xz \rangle \wedge \langle xy \rangle \in \tau \circ \sigma \wedge \langle yz \rangle \in \rho}{(r, t, u) \quad \neg. a = \langle ru \rangle \wedge \langle rt \rangle \in \tau \circ \sigma \wedge \langle tu \rangle \in \rho}$
$\frac{- \quad a \neq \langle ru \rangle \quad - \quad \langle rt \rangle \notin \tau \circ \sigma \quad 3 \quad \langle tu \rangle \notin \rho}{(r, s, t) \quad \neg \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau}$	$\frac{- \quad \neg. \langle rt \rangle = \langle rt \rangle \wedge \langle rs \rangle \in \sigma \wedge \langle st \rangle \in \tau}{- \quad \neg. \langle rt \rangle = \langle rt \rangle \wedge \langle rs \rangle \in \sigma \wedge \langle st \rangle \in \tau}$

$$\begin{array}{c}
 \hline
 4 \quad \langle rs \rangle \notin \sigma \\
 5 \quad \langle st \rangle \notin \tau \\
 \hline
 (1) \quad - \quad \exists xyz. a = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \rho \circ \tau \\
 \hline
 - \quad a = \langle ru \rangle \wedge \langle rs \rangle \in \sigma \wedge \langle su \rangle \in \rho \circ \tau \\
 \hline
 a = \langle ru \rangle \quad \langle rs \rangle \in \sigma \quad - \quad \langle su \rangle \in \rho \circ \tau \\
 (2) \quad (4, =, 1) \\
 \hline
 - \quad \exists xyz. \langle su \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \tau \wedge \langle yz \rangle \in \rho \\
 \hline
 - \quad \langle su \rangle = \langle su \rangle \wedge \langle st \rangle \in \tau \wedge \langle tu \rangle \in \rho \\
 \hline
 \langle st \rangle \in \tau \quad \langle tu \rangle \in \rho \\
 (5, =, 1) \quad (3)
 \end{array}$$

- *7 $(\rho \circ \tau) \circ \sigma \subseteq \rho \circ (\tau \circ \sigma)$ (Similar to ◦*6)
- *8 $(\rho \circ \tau) \circ \sigma = \rho \circ (\tau \circ \sigma)$ (From ◦*6, ◦*7 and = *7)
- *9 $(\tau \circ \sigma)^{-1} = \sigma^{-1} \circ \tau^{-1}$

$$\begin{array}{c}
 \hline
 \text{◦*9} \\
 \hline
 1 \quad a \in (\tau \circ \sigma)^{-1} \quad 1 \quad a \in \sigma^{-1} \circ \tau^{-1} \\
 - \quad a \notin \sigma^{-1} \circ \tau^{-1} \quad - \quad a \notin (\tau \circ \sigma)^{-1} \\
 \hline
 (t, s, r) \quad - \quad \exists xyz. a = \langle xz \rangle \wedge \langle xy \rangle \in \tau^{-1} \wedge \langle yz \rangle \in \sigma^{-1} \quad (r, t) \quad - \quad \exists xy. a = \langle xy \rangle \wedge \langle yx \rangle \in \tau \circ \sigma \\
 \hline
 - \quad \exists a = \langle tr \rangle \wedge \langle ts \rangle \in \tau^{-1} \wedge \langle sr \rangle \in \sigma^{-1} \quad - \quad \exists a = \langle tr \rangle \wedge \langle rt \rangle \in \tau \circ \sigma \\
 \hline
 2 \quad a \notin \langle tr \rangle \quad 2 \quad a \notin \langle tr \rangle \\
 - \quad \langle ts \rangle \notin \tau^{-1} \quad - \quad \langle rt \rangle \notin \tau \circ \sigma \\
 - \quad \langle sr \rangle \notin \sigma^{-1} \quad (r, s, t) \quad - \quad \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau \\
 \hline
 (t, s) \quad - \quad \exists xy. \langle ts \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \tau \quad - \quad \exists a = \langle tr \rangle \wedge \langle rt \rangle \in \tau \circ \sigma \\
 (s, r) \quad - \quad \exists xy. \langle sr \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma \\
 \hline
 - \quad \exists a = \langle tr \rangle \wedge \langle ts \rangle \in \tau \\
 - \quad \exists a = \langle sr \rangle \wedge \langle rs \rangle \in \sigma \\
 \hline
 3 \quad \langle st \rangle \notin \tau \quad 3 \quad \langle rs \rangle \notin \sigma \\
 4 \quad \langle rs \rangle \notin \sigma \quad 4 \quad \langle st \rangle \notin \tau \\
 \hline
 (1) \quad - \quad \exists xy. a = \langle xy \rangle \wedge \langle yx \rangle \in \tau \circ \sigma \quad (1) \quad - \quad \exists xyz. a = \langle xz \rangle \wedge \langle xy \rangle \in \tau^{-1} \wedge \langle yz \rangle \in \sigma^{-1} \\
 \hline
 - \quad a = \langle tr \rangle \wedge \langle rt \rangle \in \tau \circ \sigma \quad - \quad a = \langle tr \rangle \wedge \langle ts \rangle \in \tau^{-1} \wedge \langle sr \rangle \in \sigma^{-1} \\
 \hline
 a = \langle tr \rangle \quad - \quad \langle rt \rangle \in \tau \circ \sigma \quad - \quad \langle ts \rangle \in \tau^{-1} \quad \langle sr \rangle \in \sigma^{-1} \\
 (2) \quad - \quad \exists xy. \langle ts \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \tau \quad \text{Similar to the left} \\
 \hline
 - \quad \langle ts \rangle = \langle ts \rangle \wedge \langle st \rangle \in \tau \quad (3, =, 1) \\
 \hline
 \langle rs \rangle \in \sigma \quad \langle st \rangle \in \tau \\
 (4, =, 1) \quad (3, =, 1)
 \end{array}$$

- *10 $\sigma \in \text{Un} \wedge \tau \in \text{Un} \rightarrow \tau \circ \sigma \in \text{Un}$

$$\begin{array}{c}
 \hline
 \text{◦*10} \\
 \hline
 1 \quad \sigma \in \text{Un} \\
 2 \quad \tau \in \text{Un} \\
 - \quad \tau \circ \sigma \in \text{Un}
 \end{array}$$

		$\forall xyz. \langle xy \rangle \in \tau \circ \sigma \wedge \langle xz \rangle \in \tau \circ \sigma \rightarrow y = z$
		$\langle rt \rangle \in \tau \circ \sigma \wedge \langle rt' \rangle \in \tau \circ \sigma \rightarrow t = t'$
		$\langle rt \rangle \notin \tau \circ \sigma$
		$\langle rt' \rangle \notin \tau \circ \sigma$
	3	$t = t'$
		$\exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau$
		$\exists xyz. \langle rt' \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau$
		$\neg. \langle rt \rangle = \langle rt' \rangle \wedge \langle rs_1 \rangle \in \sigma \wedge \langle s_1 t \rangle \in \tau$
		$\neg. \langle rt' \rangle = \langle rt' \rangle \wedge \langle rs_2 \rangle \in \sigma \wedge \langle s_2 t' \rangle \in \tau$
	4	$\langle rs_1 \rangle \notin \sigma$
	5	$\langle s_1 t \rangle \notin \tau$
	6	$\langle rs_2 \rangle \notin \sigma$
	7	$\langle s_2 t' \rangle \notin \tau$
(1, 2)		$\neg \forall xyz. \langle xy \rangle \in \sigma \wedge \langle xz \rangle \in \sigma \rightarrow y = z$
		$\neg \forall xyz. \langle xy \rangle \in \tau \wedge \langle xz \rangle \in \tau \rightarrow y = z$
		$\neg. \langle rs_1 \rangle \in \sigma \wedge \langle rs_2 \rangle \in \sigma \rightarrow s_1 = s_2$
	8	$\neg. \langle s_1 t \rangle \in \tau \wedge \langle s_1 t' \rangle \in \tau \rightarrow t = t'$
		$\langle rs_1 \rangle \in \sigma \quad \langle rs_2 \rangle \in \sigma \quad (4, =, 1) \quad (6, =, 1) \quad (8) \quad s_1 \neq s_2$
		$\langle s_1 t \rangle \in \tau \quad \langle s_1 t' \rangle \in \tau \quad t \neq t' \quad (5, =, 1) \quad (7, 9, =, 1) \quad (3)$

◦ *11 $\sigma \in \text{Un}_2 \wedge \tau \in \text{Un}_2 \rightarrow \tau \circ \sigma \in \text{Un}_2$

		$\circ *11$
		$\sigma \notin \text{Un}_2$
		$\tau \notin \text{Un}_2$
	1	$\tau \circ \sigma \in \text{Un}_2$
2		$\sigma \notin \text{Un} \quad 4 \quad \tau \notin \text{Un}$
3		$\sigma^{-1} \notin \text{Un} \quad 5 \quad \tau^{-1} \notin \text{Un}$
(1)		$\tau \circ \sigma \in \text{Un} \quad 6 \quad (\tau \circ \sigma)^{-1} \in \text{Un} \quad \text{Cut } \circ *10$
		$\text{Cut } \circ *10 \quad \neg. \tau^{-1} \in \text{Un} \wedge \sigma^{-1} \in \text{Un} \rightarrow \sigma^{-1} \circ \tau^{-1} \in \text{Un}$
		$\tau^{-1} \in \text{Un} \quad \sigma^{-1} \in \text{Un} \quad 7 \quad \sigma^{-1} \circ \tau^{-1} \notin \text{Un} \quad \text{Cut } \circ *9$
		$(\tau \circ \sigma)^{-1} \neq \sigma^{-1} \circ \tau^{-1} \quad (5) \quad (3) \quad (6, 7, =, 1)$

◦ *12 $W_\sigma \subseteq D_\tau \rightarrow D_{\tau \circ \sigma} = D_\sigma$

		$\circ *12$
		$W_\sigma \subseteq D_\tau$
		$D_{\tau \circ \sigma} = D_\sigma$
1		$\neg \forall x. x \in W_\sigma \rightarrow x \in D_\tau$
(r)		$\forall x. x \in D_{\tau \circ \sigma} \equiv x \in D_\sigma$
		$r \in D_{\tau \circ \sigma} \equiv r \in D_\sigma$

$$\begin{array}{c}
 \begin{array}{c}
 \frac{2 \quad r \in D_{\tau \circ \sigma} \quad - \quad r \in D_{\sigma}}{\exists x. \langle rx \rangle \in \sigma} \\
 \frac{3 \quad \langle rs \rangle \notin \sigma}{\neg \exists x. \langle rx \rangle \in \sigma} \\
 \frac{(I) \quad - \quad \exists x. \langle rx \rangle \in \sigma}{\neg \exists x. \langle rx \rangle \in \sigma} \\
 \frac{- \quad s \in W_{\sigma}}{\exists x. \langle xs \rangle \in \sigma} \\
 \frac{\langle rs \rangle \in \sigma}{(3)} \\
 \frac{- \quad \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau}{\exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau} \\
 \frac{- \quad \langle rt \rangle = \langle rt \rangle \wedge \langle rs \rangle \in \sigma \wedge \langle st \rangle \in \tau}{\langle rs \rangle \in \sigma \quad \langle st \rangle \in \tau} \\
 \frac{\langle rs \rangle \in \sigma}{(3)} \quad \frac{\langle st \rangle \in \tau}{(4)}
 \end{array}
 &
 \begin{array}{c}
 \frac{2 \quad r \in D_{\sigma} \quad - \quad r \in D_{\tau \circ \sigma}}{\exists x. \langle rx \rangle \in \tau \circ \sigma} \\
 \frac{(t) \quad - \quad \exists x. \langle rx \rangle \in \tau \circ \sigma}{\langle rt \rangle \notin \tau \circ \sigma} \\
 \frac{- \quad \langle rt \rangle \notin \tau \circ \sigma}{\neg \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau} \\
 \frac{(r, s, t) \quad - \quad \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau}{\neg \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \tau} \\
 \frac{3 \quad \langle rs \rangle \notin \sigma \quad \text{Spf.} \quad \langle st \rangle \notin \tau}{\exists x. \langle rx \rangle \in \sigma} \\
 \frac{(2) \quad - \quad \exists x. \langle rx \rangle \in \sigma}{\langle rs \rangle \in \sigma} \\
 \frac{\langle rs \rangle \in \sigma}{(3, =, I)}
 \end{array}
 \end{array}$$

- *13 $D_{\tau} \subseteq W_{\sigma} \rightarrow W_{\tau \circ \sigma} = W_{\tau}$ (Similar to ◦ *12)
- *14 $W_{\sigma} = D_{\tau} \rightarrow D_{\tau \circ \sigma} = D_{\sigma} \wedge W_{\tau \circ \sigma} = W_{\tau}$ (From ◦ *12, ◦ *13 and =*6)
- *15 $\sigma \in \text{Map}_2^{a,b} \wedge \tau \in \text{Map}_2^{b,c} \rightarrow \tau \circ \sigma \in \text{Map}_2^{a,c}$

$$\begin{array}{c}
 \frac{\circ *15}{\sigma \in \text{Map}_2^{a,b} \wedge \tau \in \text{Map}_2^{b,c} \rightarrow \tau \circ \sigma \in \text{Map}_2^{a,c}} \\
 \frac{- \quad \exists x. \langle rx \rangle \in \sigma \quad - \quad \exists x. \langle rx \rangle \in \tau}{\exists x. \langle rx \rangle \in \sigma \wedge \langle rx \rangle \in \tau} \\
 \frac{1 \quad \tau \circ \sigma \in \text{Un}_2 \wedge D_{\tau \circ \sigma} = a \wedge W_{\tau \circ \sigma} = c}{\tau \circ \sigma \in \text{Un}_2 \wedge D_{\tau \circ \sigma} = a \wedge W_{\tau \circ \sigma} = c} \\
 \frac{2 \quad \sigma \in \text{Un}_2 \quad 4 \quad D_{\sigma} \neq a \quad 6 \quad W_{\sigma} \neq b}{\sigma \in \text{Un}_2 \quad D_{\sigma} \neq a \quad W_{\sigma} \neq b} \\
 \frac{3 \quad \tau \in \text{Un}_2 \quad 5 \quad D_{\tau} \neq b \quad 7 \quad W_{\tau} \neq c}{\tau \in \text{Un}_2 \quad D_{\tau} \neq b \quad W_{\tau} \neq c} \quad \text{Cut } \circ *14 \\
 \frac{- \quad \exists x. \langle rx \rangle \in \sigma \wedge \langle rx \rangle \in \tau}{\exists x. \langle rx \rangle \in \sigma \wedge \langle rx \rangle \in \tau} \\
 \frac{W_{\sigma} = D_{\tau} \quad (5, 6, =) \quad 8 \quad D_{\tau \circ \sigma} \neq D_{\sigma} \quad 9 \quad W_{\tau \circ \sigma} \neq W_{\tau}}{D_{\tau \circ \sigma} \neq D_{\sigma} \quad W_{\tau \circ \sigma} \neq W_{\tau}} \\
 \frac{(1) \quad \tau \circ \sigma \in \text{Un}_2 \quad \text{Cut } \circ *11 \quad D_{\tau \circ \sigma} = a \quad W_{\tau \circ \sigma} = c}{\tau \circ \sigma \in \text{Un}_2 \quad D_{\tau \circ \sigma} = a \quad W_{\tau \circ \sigma} = c} \\
 \frac{10 \quad - \quad \exists x. \langle rx \rangle \in \sigma \wedge \langle rx \rangle \in \tau}{\exists x. \langle rx \rangle \in \sigma \wedge \langle rx \rangle \in \tau} \\
 \frac{\sigma \in \text{Un}_2 \quad (2) \quad \tau \in \text{Un}_2 \quad (3) \quad \tau \circ \sigma \in \text{Un}_2 \quad (10)}{\sigma \in \text{Un}_2 \quad \tau \in \text{Un}_2 \quad \tau \circ \sigma \in \text{Un}_2}
 \end{array}$$

Restriction of Operator (\uparrow)

* \uparrow *1 $\sigma \uparrow a \subseteq \sigma$

$$\frac{- \quad * \uparrow *1}{(w) \quad \forall x. x \in \sigma \uparrow a \rightarrow x \in \sigma}$$

-	$w \notin \sigma \uparrow a$
1	$w \in \sigma$
(r, s) -	$\neg \exists xy. w = \langle xy \rangle \wedge w \in \sigma \wedge x \in a$
*Spf.	$w \neq \langle rs \rangle$
	$w \notin \sigma$
*Spf.	$r \notin a$ (1)

$\uparrow * 2 \quad a \subseteq b \rightarrow \sigma \uparrow a \subseteq \sigma \uparrow b$

-	$\uparrow * 2$										
-	$a \subseteq b$										
-	$\sigma \uparrow a \subseteq \sigma \uparrow b$										
1	$\neg \forall x. x \in a \rightarrow x \in b$										
(w) -	$\forall x. x \in \sigma \uparrow a \rightarrow x \in \sigma \uparrow b$										
-	$w \notin \sigma \uparrow a$										
-	$w \in \sigma \uparrow b$										
(r, s) -	$\neg \exists xy. w = \langle xy \rangle \wedge w \in \sigma \wedge x \in a$										
2	$\exists xy. w = \langle xy \rangle \wedge w \in \sigma \wedge x \in b$										
3	$w \neq \langle rs \rangle$										
4	$w \notin \sigma$										
5	$r \notin a$										
(2)	<table style="display: inline-table; border: none;"> <tr> <td style="padding-right: 10px;">$w = \langle rs \rangle$ (3)</td> <td style="padding-right: 10px;">$w \in \sigma$ (4)</td> <td style="padding-right: 10px;">6</td> <td style="padding-right: 10px;">$r \in b$</td> </tr> <tr> <td></td> <td></td> <td style="padding-right: 10px;">(1)</td> <td> <table style="display: inline-table; border: none;"> <tr> <td style="padding-right: 10px;">$r \in a$ (5)</td> <td>$r \notin b$ (6)</td> </tr> </table> </td> </tr> </table>	$w = \langle rs \rangle$ (3)	$w \in \sigma$ (4)	6	$r \in b$			(1)	<table style="display: inline-table; border: none;"> <tr> <td style="padding-right: 10px;">$r \in a$ (5)</td> <td>$r \notin b$ (6)</td> </tr> </table>	$r \in a$ (5)	$r \notin b$ (6)
$w = \langle rs \rangle$ (3)	$w \in \sigma$ (4)	6	$r \in b$								
		(1)	<table style="display: inline-table; border: none;"> <tr> <td style="padding-right: 10px;">$r \in a$ (5)</td> <td>$r \notin b$ (6)</td> </tr> </table>	$r \in a$ (5)	$r \notin b$ (6)						
$r \in a$ (5)	$r \notin b$ (6)										

$\uparrow * 3 \quad \langle pq \rangle \in \sigma \wedge p \in a \rightarrow \langle pq \rangle \in \sigma \uparrow a$

$* \uparrow * 4 \quad \sigma \in \text{Un} \rightarrow \sigma \uparrow a \in \text{Un}$

-	$* \uparrow * 4$			
1	$\sigma \notin \text{Un}$			
2	$\sigma \uparrow a \in \text{Un} \quad \text{Cut Im} * 1$			
-	$\neg. \sigma \in \text{Un} \wedge \sigma \uparrow a \subseteq a \rightarrow \sigma \uparrow a \in \text{Un}$			
	<table style="display: inline-table; border: none;"> <tr> <td style="padding-right: 10px;">$\sigma \in \text{Un}$ (1)</td> <td style="padding-right: 10px;">$\sigma \uparrow a \subseteq a$ * $\uparrow * 1$</td> <td style="padding-right: 10px;">$\sigma \uparrow a \notin \text{Un}$ (2)</td> </tr> </table>	$\sigma \in \text{Un}$ (1)	$\sigma \uparrow a \subseteq a$ * $\uparrow * 1$	$\sigma \uparrow a \notin \text{Un}$ (2)
$\sigma \in \text{Un}$ (1)	$\sigma \uparrow a \subseteq a$ * $\uparrow * 1$	$\sigma \uparrow a \notin \text{Un}$ (2)		

$* \uparrow * 5 \quad \sigma^{-1} \in \text{Un} \rightarrow (\sigma \uparrow a)^{-1} \in \text{Un}$

-	$* \uparrow * 5$
-	$\sigma^{-1} \notin \text{Un}$
-	$(\sigma \uparrow a)^{-1} \in \text{Un}$
1	$\neg \forall xyz. \langle xy \rangle \in \sigma^{-1} \wedge \langle xz \rangle \in \sigma^{-1} \rightarrow y = z$
(r, s, t) -	$\forall xyz. \langle xy \rangle \in (\sigma \uparrow a)^{-1} \wedge \langle xz \rangle \in (\sigma \uparrow a)^{-1} \rightarrow y = z$
-	$\langle rs \rangle \in (\sigma \uparrow a)^{-1} \wedge \langle rt \rangle \in (\sigma \uparrow a)^{-1} \rightarrow s = t$
-	$\langle rs \rangle \notin (\sigma \uparrow a)^{-1}$
-	$\langle rt \rangle \in (\sigma \uparrow a)^{-1}$
2	$s = t$

$$\begin{array}{c}
 \text{---} \\
 (r, s) \text{---} \quad \neg \exists xy. \langle rs \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma \uparrow a \\
 (r, t) \text{---} \quad \neg \exists xy. \langle rt \rangle = \langle xy \rangle \wedge \langle yx \rangle \in \sigma \uparrow a \\
 \text{---} \\
 \quad \langle sr \rangle \notin \sigma \uparrow a \\
 \quad \langle tr \rangle \notin \sigma \uparrow a \\
 \text{---} \\
 \quad \neg. \langle sr \rangle \in \sigma \wedge s \in a \\
 \quad \neg. \langle tr \rangle \in \sigma \wedge t \in a \\
 \text{---} \\
 3 \quad \langle sr \rangle \notin \sigma \\
 4 \quad \langle tr \rangle \notin \sigma \\
 * \text{Spf.} \quad s \in a \\
 * \text{Spf.} \quad t \in a \\
 \text{---} \\
 (1) \text{---} \quad \neg. \langle rs \rangle \in \sigma^{-1} \wedge \langle rt \rangle \in \sigma^{-1} \rightarrow s = t \\
 \text{---} \quad \langle rs \rangle \in \sigma^{-1} \quad \text{---} \quad \langle rt \rangle \in \sigma^{-1} \quad s \neq t \\
 \quad \langle sr \rangle \in \sigma \quad \quad \quad \langle tr \rangle \in \sigma \\
 \quad (3, =, I) \quad \quad \quad (4, =, I)
 \end{array}$$

*↑*6 $\sigma \in \text{Un}_2 \rightarrow \sigma \uparrow a \in \text{Un}_2$

$$\begin{array}{c}
 \text{---} \\
 \quad * \uparrow * 6 \\
 \text{---} \\
 \quad \sigma \notin \text{Un}_2 \\
 1 \quad \sigma \uparrow a \in \text{Un}_2 \\
 \text{---} \\
 \quad \neg. \sigma \in \text{Un} \wedge \sigma^{-1} \in \text{Un} \\
 \text{---} \\
 2 \quad \sigma \notin \text{Un} \\
 3 \quad \sigma^{-1} \notin \text{Un} \\
 (1) \text{---} \\
 \quad \sigma \uparrow a \in \text{Un} \wedge (\sigma \uparrow a)^{-1} \in \text{Un} \\
 \text{---} \\
 4 \quad \sigma \uparrow a \in \text{Un} \quad \text{Cut } * \uparrow * 4 \quad 4 \quad (\sigma \uparrow a)^{-1} \in \text{Un} \quad \text{Cut } * \uparrow * 5 \\
 \text{---} \quad \neg. \sigma \in \text{Un} \rightarrow \sigma \uparrow a \in \text{Un} \quad \text{---} \quad \neg. \sigma^{-1} \in \text{Un} \rightarrow (\sigma \uparrow a)^{-1} \in \text{Un} \\
 \quad \sigma \in \text{Un} \quad \sigma \uparrow a \notin \text{Un} \quad \quad \quad \sigma^{-1} \in \text{Un} \quad (\sigma \uparrow a)^{-1} \notin \text{Un} \\
 \quad (2) \quad (4) \quad \quad \quad (3) \quad (4)
 \end{array}$$

↑*7 $b \in a \rightarrow (\sigma \uparrow a) \uparrow b = \sigma \uparrow b$

$$\begin{array}{c}
 \text{---} \\
 \quad \uparrow * 7 \\
 1 \quad \neg \forall x. x \in b \rightarrow x \in a \\
 \quad (\sigma \uparrow a) \uparrow b = \sigma \uparrow b \\
 \text{---} \\
 \quad \neg \forall x. x \in (\sigma \uparrow a) \uparrow b \equiv x \in \sigma \uparrow b \\
 (w) \text{---} \\
 \quad w \in (\sigma \uparrow a) \uparrow b \equiv w \in \sigma \uparrow b \\
 \text{---} \\
 \quad w \in (\sigma \uparrow a) \uparrow b \quad \quad \quad w \in \sigma \uparrow b \\
 \quad w \notin \sigma \uparrow b \quad \quad \quad w \notin (\sigma \uparrow a) \uparrow b \\
 \text{---} \\
 2 \quad \exists xy. w = \langle xy \rangle \wedge w \in \sigma \uparrow a \wedge x \in b \quad 2 \quad \exists xy. w = \langle xy \rangle \wedge w \in \sigma \wedge x \in b \\
 (r, s) \text{---} \neg \exists xy. w = \langle xy \rangle \wedge w \in \sigma \wedge x \in b \quad (r, s) \text{---} \neg \exists xy. w = \langle xy \rangle \wedge w \in \sigma \uparrow a \wedge x \in b \\
 \text{---} \quad \neg. w = \langle rs \rangle \wedge w \in \sigma \wedge r \in b \quad \text{---} \quad \neg. w = \langle rs \rangle \wedge w \in \sigma \uparrow a \wedge r \in b \\
 3 \quad w \neq \langle rs \rangle \quad \quad \quad 3 \quad w \neq \langle rs \rangle \\
 4 \quad w \notin \sigma \quad \quad \quad 4 \quad w \notin \sigma \uparrow a \\
 5 \quad r \notin b \quad \quad \quad 5 \quad r \notin b
 \end{array}$$

$\frac{\begin{array}{l} \text{(2)} \\ - \\ w = \langle rs \rangle \wedge w \in \sigma \wedge a \wedge r \in b \\ \hline w = \langle rs \rangle \quad \text{(3)} \\ - \\ \exists xy. w = \langle xy \rangle \wedge w \in \sigma \wedge x \in a \\ \hline - \\ w = \langle rs \rangle \wedge w \in \sigma \wedge r \in a \\ \hline w = \langle rs \rangle \quad w \in \sigma \quad \text{(3)} \quad \text{(4)} \quad \text{(1)} \\ \hline - \\ \neg. r \in b \rightarrow r \in a \\ \hline r \in b \quad r \in a \\ \text{(5)} \quad \text{(6)} \end{array}}{\text{---}}$	$\frac{\begin{array}{l} - \\ \neg \exists xy. w = \langle xy \rangle \wedge w \in \sigma \wedge x \in a \\ \hline \text{(r}_1, s_1) \\ - \\ \neg. w = \langle r_1 s_1 \rangle \wedge w \in \sigma \wedge r_1 \in a \\ \hline \text{Spf.} \quad w \neq \langle r_1 s_1 \rangle \\ w \in \sigma \\ \text{(5)} \\ \text{Spf.} \quad r_1 \notin a \\ \hline \text{(2)} \\ - \\ w = \langle rs \rangle \wedge w \in \sigma \wedge r \in b \\ \hline w = \langle rs \rangle \quad w \in \sigma \quad r \in b \\ \text{(3)} \quad \text{(5)} \quad \text{(4)} \end{array}}{\text{---}}$
---	--

†*8 $p \in a \rightarrow p^{\sigma a} = p^\sigma$

$\frac{\begin{array}{l} - \\ \text{†*8} \\ 1 \\ p \in a \\ p^{\sigma a} = p^\sigma \\ \hline \text{(w)} \\ - \\ \forall x. x \in p^{\sigma a} \equiv x \in p^\sigma \\ \hline - \\ w \in p^{\sigma a} \equiv w \in p^\sigma \\ \hline - \\ w \in p^{\sigma a} \\ - \\ w \notin p^\sigma \\ \hline 2 \\ \exists x. \langle px \rangle \in \sigma \wedge a \wedge w \in x \\ \hline \text{(q)} \\ \neg \exists x. \langle px \rangle \in \sigma \wedge w \in x \\ \hline - \\ \neg. \langle pq \rangle \in \sigma \wedge w \in q \\ \hline 3 \\ \langle pq \rangle \notin \sigma \\ \hline 4 \\ w \notin q \\ \hline \text{(2)} \\ - \\ \langle pq \rangle \in \sigma \wedge a \wedge w \in q \\ \hline - \\ \langle pq \rangle \in \sigma \wedge a \\ \hline - \\ \langle pq \rangle \in \sigma \wedge p \in a \\ \hline \langle pq \rangle \in \sigma \quad p \in a \\ \text{(3)} \quad \text{(1)} \end{array}}{\text{---}}$	$\frac{\begin{array}{l} - \\ \text{†*8} \\ 1 \\ p \in a \\ p^{\sigma a} = p^\sigma \\ \hline \text{(w)} \\ - \\ \forall x. x \in p^{\sigma a} \equiv x \in p^\sigma \\ \hline - \\ w \in p^{\sigma a} \equiv w \in p^\sigma \\ \hline - \\ w \in p^\sigma \\ - \\ w \notin p^{\sigma a} \\ \hline 2 \\ \exists x. \langle px \rangle \in \sigma \wedge w \in x \\ \hline \text{(q)} \\ \neg \exists x. \langle px \rangle \in \sigma \wedge a \wedge w \in x \\ \hline - \\ \neg. \langle pq \rangle \in \sigma \wedge a \wedge w \in q \\ \hline 3 \\ \langle pq \rangle \notin \sigma \wedge a \\ \hline w \notin q \\ \hline - \\ \langle pq \rangle \in \sigma \wedge p \in a \\ \hline 4 \\ \langle pq \rangle \in \sigma \\ \hline \text{(2)} \\ - \\ \langle pq \rangle \in \sigma \wedge w \in q \\ \hline \langle pq \rangle \in \sigma \quad w \in q \\ \text{(4)} \quad \text{(3)} \end{array}}{\text{---}}$
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†*9 $b \subseteq a \rightarrow b^{\sigma a} = b^\sigma$

$\frac{\begin{array}{l} - \\ \text{†*9} \\ 1 \\ b \subseteq a \\ b^{\sigma a} = b^\sigma \\ \hline - \\ \forall x. x \in b^{\sigma a} \equiv x \in b^\sigma \\ \hline - \\ s \in b^{\sigma a} \equiv s \in b^\sigma \\ \hline - \\ s \in b^{\sigma a} \\ - \\ s \notin b^\sigma \\ \hline 2 \\ \exists x. x \in b \wedge \langle xs \rangle \in \sigma \wedge a \\ \hline \text{(r)} \\ \neg \exists x. x \in b \wedge \langle xs \rangle \in \sigma \\ \hline - \\ \neg. r \in b \wedge \langle rs \rangle \in \sigma \end{array}}{\text{---}}$	$\frac{\begin{array}{l} - \\ \text{†*9} \\ 1 \\ b \subseteq a \\ b^{\sigma a} = b^\sigma \\ \hline - \\ \forall x. x \in b^{\sigma a} \equiv x \in b^\sigma \\ \hline - \\ s \in b^{\sigma a} \equiv s \in b^\sigma \\ \hline - \\ s \in b^\sigma \\ - \\ s \notin b^{\sigma a} \\ \hline 2 \\ \exists x. x \in b \wedge \langle xs \rangle \in \sigma \\ \hline \text{(r)} \\ \neg \exists x. x \in b \wedge \langle xs \rangle \in \sigma \wedge a \\ \hline - \\ \neg. r \in b \wedge \langle rs \rangle \in \sigma \wedge a \end{array}}{\text{---}}$
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$ \begin{array}{l} \frac{3 \quad r \notin b}{4 \quad \langle rs \rangle \notin \sigma} \\ \frac{(2) \quad r \in b \wedge \langle rs \rangle \in \sigma \uparrow a}{r \in b \quad - \quad \langle rs \rangle \in \sigma \uparrow a} \\ \frac{(3) \quad - \quad \langle rs \rangle \in \sigma \wedge r \in a}{\langle rs \rangle \in \sigma} \quad \frac{5 \quad r \in a}{(1) \quad - \quad \neg \forall x. x \in b \rightarrow x \in a} \\ \frac{(1) \quad - \quad \neg \forall x. x \in b \rightarrow x \in a}{- \quad \neg \forall x. r \in b \rightarrow r \in a} \\ \frac{r \in b \quad r \notin a}{(3) \quad (5)} \end{array} $	$ \begin{array}{l} \frac{3 \quad r \notin b}{- \quad \langle rs \rangle \notin \sigma \uparrow a} \\ \frac{- \quad \neg \forall x. \langle rs \rangle \in \sigma \wedge r \in a}{4 \quad \langle rs \rangle \notin \sigma} \\ \frac{4 \quad \langle rs \rangle \notin \sigma}{\text{Spf.} \quad r \notin a} \\ \frac{(2) \quad - \quad r \in b \wedge \langle rs \rangle \in \sigma}{r \in b \quad \langle rs \rangle \in \sigma} \\ \frac{(3) \quad (4)}{} \end{array} $
---	---

⊢ * 10 $\sigma \in \text{Un} \wedge a \subseteq D_\sigma \rightarrow D_{\sigma \uparrow a} = a$

$ \begin{array}{l} - \quad \vdash * 10 \\ \frac{1 \quad \sigma \notin \text{Un}}{2 \quad a \notin D_\sigma} \\ \frac{2 \quad a \notin D_\sigma}{- \quad D_{\sigma \uparrow a} = a} \\ \frac{(r) \quad - \quad \forall x. x \in D_{\sigma \uparrow a} \equiv x \in a}{- \quad r \in D_{\sigma \uparrow a} \equiv r \in a} \\ \frac{- \quad r \in D_{\sigma \uparrow a} \quad r \notin a}{3 \quad \exists x. \langle rx \rangle \in \sigma \uparrow a} \\ \frac{- \quad \langle rr^\sigma \rangle \in \sigma \uparrow a}{- \quad \langle rr^\sigma \rangle \in \sigma \wedge r \in a} \\ \frac{4 \quad \langle rr^\sigma \rangle \in \sigma \quad r \in a}{(2) \quad - \quad \neg \forall x. x \in a \rightarrow x \in D_\sigma} \\ \frac{- \quad \neg \forall x. x \in a \rightarrow x \in D_\sigma}{- \quad \neg \forall x. r \in a \rightarrow r \in D_\sigma} \\ \frac{r \in a \quad 5 \quad r \notin D_\sigma \quad \text{Cut Im*3}}{(3) \quad - \quad \neg \forall x. \sigma \in \text{Un} \wedge r \in D_\sigma \rightarrow \langle rr^\sigma \rangle \in \sigma} \\ \frac{\sigma \in \text{Un} \quad r \in D_\sigma \quad \langle rr^\sigma \rangle \notin \sigma}{(1) \quad (5) \quad (4)} \end{array} $	$ \begin{array}{l} \frac{3 \quad r \in a}{- \quad r \notin D_{\sigma \uparrow a}} \\ \frac{(s) \quad - \quad \neg \exists x. \langle rx \rangle \in \sigma \uparrow a}{- \quad \langle rs \rangle \notin \sigma \uparrow a} \\ \frac{- \quad \langle rs \rangle \notin \sigma \uparrow a}{- \quad \neg \forall x. \langle rs \rangle \in \sigma \wedge r \in a} \\ \frac{\text{Spf.} \quad \langle rs \rangle \notin \sigma}{r \notin a} \\ \frac{r \notin a}{(3)} \end{array} $
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⊢ * 11 $\sigma \in \text{Un} \wedge a \subseteq D_\sigma \rightarrow W_{\sigma \uparrow a} = a^\sigma$

$ \begin{array}{l} - \quad \vdash * 11 \\ \frac{1 \quad \sigma \notin \text{Un}}{2 \quad a \notin D_\sigma} \\ \frac{2 \quad a \notin D_\sigma}{3 \quad W_{\sigma \uparrow a} = a^\sigma \quad \text{Cut Im*9}} \\ \frac{3 \quad W_{\sigma \uparrow a} = a^\sigma}{4 \quad W_{\sigma \uparrow a} \neq (D_{\sigma \uparrow a})^{\sigma \uparrow a} \quad \text{Cut } \vdash * 10} \\ \frac{- \quad \neg \forall x. \sigma \in \text{Un} \wedge a \subseteq D_\sigma \rightarrow D_{\sigma \uparrow a} = a}{\sigma \in \text{Un} \quad a \subseteq D_\sigma} \\ \frac{(1) \quad (2) \quad 5 \quad D_{\sigma \uparrow a} \neq a \quad \text{Cut } \vdash * 9}{- \quad \neg \forall x. a \subseteq a \rightarrow a^{\sigma \uparrow a} = a^\sigma} \\ \frac{a \subseteq a \quad a^{\sigma \uparrow a} \neq a^\sigma}{= * 1 \quad (3, 4, 5, =, 1)} \end{array} $	
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⊢*12 $\sigma \in \text{Un}_2 \wedge a \in D_\sigma \rightarrow \sigma \upharpoonright a \in \text{Map}_2^{a, a^2}$

$$\begin{array}{c}
 - \\
 \hline
 \text{1} \quad \sigma \notin \text{Un}_2 \\
 \text{2} \quad a \notin D_\sigma \\
 - \\
 \sigma \upharpoonright a \in \text{Map}_2^{a, a^2} \\
 \hline
 - \\
 \sigma \upharpoonright a \in \text{Un}_2 \wedge D_{\sigma \upharpoonright a} = a \wedge W_{\sigma \upharpoonright a} = a^2 \\
 \sigma \upharpoonright a \in \text{Un}_2 \quad D_{\sigma \upharpoonright a} = a \quad W_{\sigma \upharpoonright a} = a^2 \\
 \text{(1)} \quad \text{(1, 2)} \quad \text{(1, 2)} \\
 \text{Cut } * \upharpoonright * 6 \quad \text{Cut } \upharpoonright * 10 \quad \text{Cut } \upharpoonright * 11
 \end{array}$$

⊢*13 $\sigma \in \text{Un} \wedge \tau \in \text{Un} \wedge a \in D_\sigma \wedge a \in D_\tau \wedge \forall x [x^\sigma = x^\tau] \rightarrow \sigma \upharpoonright a = \tau \upharpoonright a$

$$\begin{array}{c}
 - \\
 \hline
 \text{1} \quad \sigma \notin \text{Un} \quad \text{3} \quad a \notin D_\sigma \quad \text{5} \quad \neg \forall x. x^\sigma = x^\tau \\
 \text{2} \quad \tau \notin \text{Un} \quad \text{4} \quad a \notin D_\tau \quad \text{6} \quad \sigma \upharpoonright a = \tau \upharpoonright a \\
 \text{(6)} \\
 - \\
 \forall x. x \in \sigma \upharpoonright a \equiv x \in \tau \upharpoonright a \\
 \text{(w)} \\
 - \\
 w \in \sigma \upharpoonright a \equiv w \in \tau \upharpoonright a \\
 \hline
 - \\
 w \notin \sigma \upharpoonright a \quad w \notin \tau \upharpoonright a \\
 - \\
 w \in \tau \upharpoonright a \quad w \in \sigma \upharpoonright a \\
 \text{(r, s)} \quad - \quad \neg \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \wedge x \in a \quad \text{Symmetric to the left.} \\
 \text{7} \quad \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \tau \wedge x \in a \quad \text{(3) is used here.} \\
 - \\
 \neg. w = \langle rs \rangle \wedge \langle rs \rangle \in \sigma \wedge r \in a \\
 \text{8} \quad w \neq \langle rs \rangle \\
 \text{9} \quad \langle rs \rangle \notin \sigma \\
 \text{10} \quad r \notin a \\
 \text{(7)} \\
 - \\
 w = \langle rs \rangle \wedge \langle rs \rangle \in \tau \wedge r \in a \\
 w = \langle rs \rangle \quad \text{11} \quad \langle rs \rangle \in \tau \quad r \in a \\
 \text{(8)} \quad \text{(5)} \quad \text{(10)} \\
 - \quad \neg. r \in a \rightarrow r^\sigma = r^\tau \\
 \text{(10)} \quad \text{12} \quad r^\sigma \neq r^\tau \quad \text{Cut Im*2} \\
 - \quad \neg. \sigma \in \text{Un} \wedge \langle rs \rangle \in \sigma \rightarrow r^\sigma = s \\
 \sigma \in \text{Un} \quad \langle rs \rangle \in \sigma \quad \text{13} \quad r^\sigma \neq s \quad \text{Cut Im*3} \\
 \text{(1)} \quad \text{(9)} \\
 - \quad \neg. \tau \in \text{Un} \wedge r \in D_\tau \rightarrow \langle rr^\tau \rangle \in \tau \\
 \tau \in \text{Un} \quad r \in D_\tau \quad \langle rr^\tau \rangle \notin \tau \\
 \text{(2)} \quad \text{(4, 10)} \quad \text{(11, 12, 13, =, I)} \\
 \text{Cut } = * 9
 \end{array}$$

⊢*14 $\sigma \circ (\tau \upharpoonright a) = (\sigma \circ \tau) \upharpoonright a$

$$\begin{array}{c}
 - \\
 \hline
 \text{(w)} \quad \forall x. x \in \sigma \circ (\tau \upharpoonright a) \equiv x \in (\sigma \circ \tau) \upharpoonright a \\
 - \\
 w \in \sigma \circ (\tau \upharpoonright a) \equiv w \in (\sigma \circ \tau) \upharpoonright a
 \end{array}$$

$\frac{- \quad w \in \sigma \circ (\tau \uparrow a) \quad - \quad w \notin (\sigma \circ \tau) \uparrow a}{1 \quad \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \tau \uparrow a \wedge \langle yz \rangle \in \sigma}$ $(r, t) \quad - \quad \neg \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \circ \tau \wedge x \in a$ $- \quad \neg \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \circ \tau \wedge x \in a$ <hr/> $2 \quad w \neq \langle rt \rangle$ $- \quad \langle rt \rangle \notin \sigma \circ \tau$ $3 \quad r \notin a$ <hr/> $(r, s, t) \quad - \quad \neg \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \tau \wedge \langle yz \rangle \in \sigma$ $4 \quad \langle rs \rangle \notin \tau$ $5 \quad \langle st \rangle \notin \sigma$ <hr/> $(1) \quad - \quad w = \langle rt \rangle \wedge \langle rs \rangle \in \tau \uparrow a \wedge \langle st \rangle \in \sigma$ <hr/> $w = \langle rt \rangle \quad - \quad \langle rs \rangle \in \tau \uparrow a \quad \langle st \rangle \in \sigma$ $(2) \quad - \quad \langle rs \rangle \in \tau \wedge r \in a \quad (5, =, 1)$ <hr/> $\langle rs \rangle \in \tau \quad r \in a$ $(4, =, 1) \quad (3)$	$\frac{- \quad w \in (\sigma \circ \tau) \uparrow a \quad - \quad w \notin \sigma \circ (\tau \uparrow a)}{1 \quad \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \circ \tau \wedge x \in a}$ $(r, s, t) \quad - \quad \neg \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \tau \uparrow a \wedge \langle yz \rangle \in \sigma$ $- \quad \neg \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \tau \uparrow a \wedge \langle yz \rangle \in \sigma$ <hr/> $2 \quad w \neq \langle rt \rangle$ $- \quad \langle rs \rangle \notin \tau \uparrow a$ $3 \quad \langle st \rangle \notin \sigma$ <hr/> $(r, s) \quad - \quad \neg \exists xy. \langle rs \rangle = \langle xy \rangle \wedge \langle xy \rangle \in \tau \wedge x \in a$ $4 \quad \langle rs \rangle \notin \tau$ $5 \quad r \notin a$ <hr/> $(1) \quad - \quad w = \langle rt \rangle \wedge \langle rt \rangle \in \sigma \circ \tau \wedge r \in a$ <hr/> $w = \langle rt \rangle \quad - \quad \langle rt \rangle \in \sigma \circ \tau \quad r \in a$ $(2) \quad - \quad \langle rs \rangle \in \tau \wedge \langle st \rangle \in \sigma \quad (5, =, 1)$ <hr/> $\langle rs \rangle \in \tau \quad \langle st \rangle \in \sigma$ $(4, =, 1) \quad (3, =, 1)$
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$$\uparrow * 15 \quad \tau \in \text{Un} \rightarrow (\sigma \uparrow a) \circ \tau = (\sigma \circ \tau) \uparrow a^{\tau^{-1}}$$

$\frac{- \quad \uparrow * 15}{0 \quad \tau \in \text{Un}}$ $- \quad (\sigma \uparrow a) \circ \tau = (\sigma \circ \tau) \uparrow a^{\tau^{-1}}$ <hr/> $(w) \quad - \quad \forall x. x \in (\sigma \uparrow a) \circ \tau \equiv x \in (\sigma \circ \tau) \uparrow a^{\tau^{-1}}$ <hr/> $- \quad w \in (\sigma \uparrow a) \circ \tau \equiv w \in (\sigma \circ \tau) \uparrow a^{\tau^{-1}}$ <hr/> $- \quad w \in (\sigma \uparrow a) \circ \tau \quad - \quad w \in (\sigma \circ \tau) \uparrow a^{\tau^{-1}}$ $- \quad w \notin (\sigma \circ \tau) \uparrow a^{\tau^{-1}} \quad - \quad w \notin (\sigma \uparrow a) \circ \tau$ <hr/> $(*) \quad (**) \quad$	$(*)$ $1 \quad \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \tau \wedge \langle yz \rangle \in \sigma \uparrow a$ $(r, t) \quad - \quad \neg \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \circ \tau \wedge x \in a^{\tau^{-1}}$ $- \quad \neg \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \circ \tau \wedge x \in a^{\tau^{-1}}$ <hr/> $2 \quad w \neq \langle rt \rangle$ $- \quad \langle rt \rangle \notin \sigma \circ \tau$ $3 \quad r \notin a^{\tau^{-1}}$ <hr/> $(r, s, t) \quad - \quad \neg \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \tau \wedge \langle yz \rangle \in \sigma$ $4 \quad \langle rs \rangle \notin \tau$ $5 \quad \langle st \rangle \notin \sigma$ <hr/> $(1) \quad - \quad w = \langle rt \rangle \wedge \langle rs \rangle \in \tau \wedge \langle st \rangle \in \sigma \uparrow a$ <hr/> $w = \langle rt \rangle \quad \langle rs \rangle \in \tau \quad - \quad \langle st \rangle \in \sigma \uparrow a$ $(2) \quad (4, =, 1) \quad - \quad \exists xy. \langle st \rangle = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \wedge x \in a$
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	$\frac{\langle st \rangle \in \sigma \quad s \in a}{(5) \quad (2) \quad (6)}$	
	$\frac{- \quad \neg \exists x. x \in a \wedge \langle xr \rangle \in \tau^{-1}}{(S_1) \quad (7)}$	
	$\frac{- \quad \neg \exists s_1 \in a \wedge \langle s_1 r \rangle \in \tau^{-1}}{(8)}$	
	$\frac{s_1 \notin a}{(9)}$	
	$\frac{\langle s_1 r \rangle \in \tau^{-1}}{(10)}$	
	$\frac{\langle rs_1 \rangle \in \tau}{(11)}$	
	$\frac{\neg \exists xyz. \langle xy \rangle \in \tau \wedge \langle xz \rangle \in \tau \rightarrow y = z}{(12)}$	
	$\frac{\neg \exists \langle rs \rangle \in \tau \wedge \langle rs_1 \rangle \in \tau \rightarrow s = s_1}{(13)}$	
	$\frac{\langle rs \rangle \in \tau \quad \langle rs_1 \rangle \in \tau \quad s \neq s_1}{(4, =, I) \quad (8) \quad (6, \tau, =, I)}$	
	$(**)$	
	$\frac{\exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \circ \tau \wedge x \in a^{-1}}{(1)}$	
$\frac{(r, s, t) \quad \neg \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \tau \wedge \langle yz \rangle \in \sigma \uparrow a}{(14)}$		
	$\frac{- \quad \neg \exists w = \langle rt \rangle \wedge \langle rs \rangle \in \tau \wedge \langle st \rangle \in \sigma \uparrow a}{(15)}$	
	$\frac{w \neq \langle rt \rangle}{(16)}$	
	$\frac{\langle rs \rangle \notin \tau}{(17)}$	
	$\frac{\langle st \rangle \notin \sigma \uparrow a}{(18)}$	
$\frac{(s, t) \quad \neg \exists xy. \langle st \rangle = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \wedge x \in a}{(19)}$		
	$\frac{\langle st \rangle \notin \sigma}{(20)}$	
	$\frac{s \notin a}{(21)}$	
	$\frac{- \quad w = \langle rt \rangle \wedge \langle rt \rangle \in \sigma \circ \tau \wedge r \in a^{-1}}{(22)}$	
$\frac{w = \langle rt \rangle}{(2)}$	$\frac{\langle rt \rangle \in \sigma \circ \tau}{(23)}$	$\frac{r \in a^{-1}}{(24)}$
	$\frac{- \quad \exists xyz. \langle rt \rangle = \langle xz \rangle \wedge \langle xy \rangle \in \tau \wedge \langle yz \rangle \in \sigma}{(25)}$	$\frac{- \quad \exists x. x \in a \wedge \langle xr \rangle \in \tau^{-1}}{(26)}$
	$\frac{- \quad \langle rt \rangle = \langle rt \rangle \wedge \langle rs \rangle \in \tau \wedge \langle st \rangle \in \sigma}{(27)}$	$\frac{- \quad s \in a \wedge \langle sr \rangle \in \tau^{-1}}{(28)}$
	$\frac{\langle rs \rangle \in \tau \quad \langle st \rangle \in \sigma}{(3) \quad (4, =, I)}$	$\frac{s \in a \quad \langle sr \rangle \in \tau^{-1}}{(5) \quad (3, =, I)}$
		$\frac{\langle rs \rangle \in \tau}{(3, =, I)}$

†*16 $(\sigma \cap V^2) \uparrow a = \sigma \uparrow a$

	$\frac{- \quad \uparrow * 16}{(w) \quad \neg \forall x. x \in (\sigma \cap V^2) \uparrow a \equiv x \in \sigma \uparrow a}$
	$\frac{- \quad w \in (\sigma \cap V^2) \uparrow a \equiv w \in \sigma \uparrow a}{(29)}$
$\frac{- \quad w \in (\sigma \cap V^2) \uparrow a}{(30)}$	$\frac{- \quad w \in \sigma \uparrow a}{(31)}$
$\frac{- \quad w \notin \sigma \uparrow a}{(32)}$	$\frac{- \quad w \notin (\sigma \cap V^2) \uparrow a}{(33)}$
$\frac{1 \quad \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \cap V^2 \wedge x \in a}{(r, s) \quad \neg \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \wedge x \in a}$	$\frac{1 \quad \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \wedge x \in a}{(r, s) \quad \neg \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \cap V^2 \wedge x \in a}$
$\frac{2 \quad w = \langle rs \rangle}{(34)}$	$\frac{2 \quad w \neq \langle rs \rangle}{(35)}$
$\frac{3 \quad \langle rs \rangle \notin \sigma}{(36)}$	$\frac{- \quad \langle rs \rangle \notin \sigma \cap V^2}{(37)}$
$\frac{4 \quad r \neq a}{(38)}$	$\frac{3 \quad r \neq a}{(39)}$

$$\begin{array}{c}
 \frac{(1) \quad \frac{w = \langle rs \rangle \quad - \quad \langle rs \rangle \in \sigma \cap V^2 \quad r \in a}{\langle rs \rangle \in \sigma} \quad - \quad \frac{\langle rs \rangle \in V^2 \quad r \in a}{- \exists xy. \langle rs \rangle = \langle xy \rangle}}{\langle rs \rangle = \langle rs \rangle} \\
 \frac{(3) \quad \frac{\langle rs \rangle \in \sigma \quad - \quad \langle rs \rangle \in V^2 \quad r \in a}{- \exists xy. \langle rs \rangle = \langle xy \rangle}}{\langle rs \rangle = \langle rs \rangle} \\
 \frac{(4) \quad \frac{1 \quad \text{Spf.} \quad \frac{\langle rs \rangle \notin \sigma}{\langle rs \rangle \notin V^2}}{(1) \quad \frac{w = \langle rs \rangle \quad \langle rs \rangle \in \sigma \quad r \in a}{(2) \quad (4) \quad (3)}}
 \end{array}$$

⊢* 17 $\sigma \in \text{Un}_2 \wedge a \subseteq D_\sigma \rightarrow \sigma \upharpoonright a \in \text{Map}_2^{a, a^\sigma}$

$$\begin{array}{c}
 - \quad \frac{\text{⊢* 17}}{\frac{1 \quad \sigma \notin \text{Un} \quad 2 \quad \sigma^{-1} \notin \text{Un} \quad 3 \quad a \notin D_\sigma}{- \quad \sigma \upharpoonright a \in \text{Map}_2^{a, a^\sigma}} \\
 - \quad \frac{\sigma \upharpoonright a \in \text{Un}_2 \wedge D_{\sigma \upharpoonright a} = a \wedge W_{\sigma \upharpoonright a} = a^\sigma}{\frac{\sigma \upharpoonright a \in \text{Un}_2 \quad D_{\sigma \upharpoonright a} = a \quad W_{\sigma \upharpoonright a} = a^\sigma}{\text{Cut } * \upharpoonright * 6 \quad \text{Cut } \upharpoonright * 10 \quad \text{Cut } \upharpoonright * 11}}
 \end{array}$$

Identical Operator (ι)

ι *1 $\langle aa \rangle \in \iota$

$$\begin{array}{c}
 - \quad \frac{\iota * 1}{- \quad \frac{\exists x. \langle aa \rangle = \langle xx \rangle}{\langle aa \rangle = \langle aa \rangle}} \\
 \frac{(=)}{\langle aa \rangle = \langle aa \rangle}
 \end{array}$$

ι *2 $a' = a$

$$\begin{array}{c}
 - \quad \frac{\iota * 2}{(w) \quad \frac{\forall x. x \in a' \equiv x \in a}{w \in a' \equiv w \in a}} \\
 \frac{1 \quad \frac{w \in a'}{w \notin a} \quad 1 \quad \frac{w \in a}{w \notin a'}}{\frac{- \quad \exists x. \langle ax \rangle \in \iota \wedge w \in x}{- \quad \langle aa \rangle \in \iota \wedge w \in a} \quad (b) \quad \frac{- \quad \neg \exists x. \langle ax \rangle \in \iota \wedge w \in x}{- \quad \neg \langle ab \rangle \in \iota \wedge w \in b}}{\frac{\langle aa \rangle \in \iota \quad w \in a}{\iota * 1} \quad (1) \quad \frac{- \quad \langle ab \rangle \in \iota}{2 \quad w \notin b} \quad (r) \quad \frac{- \quad \neg \exists x. \langle ab \rangle = \langle xx \rangle}{\langle ab \rangle \neq \langle rr \rangle}}
 \end{array}$$

ι *3 $\iota \in \text{Un}$

$$\begin{array}{c}
 - \quad \frac{\iota * 3}{(r, s, t) \quad \frac{\forall xyz. \langle xy \rangle \in \iota \wedge \langle xz \rangle \in \iota \rightarrow y = z}{- \quad \langle rs \rangle \in \iota \wedge \langle rt \rangle \in \iota \rightarrow s = t}}
 \end{array}$$

$$\begin{array}{c}
 \hline
 - \quad \langle rs \rangle \notin \iota \\
 - \quad \langle rt \rangle \notin \iota \\
 1 \quad s = t \\
 \hline
 (a) \quad \neg \exists x. \langle rs \rangle = \langle xx \rangle \\
 (b) \quad \neg \exists x. \langle rt \rangle = \langle xx \rangle \\
 \hline
 \langle rs \rangle \neq \langle aa \rangle \\
 \langle rt \rangle \neq \langle bb \rangle \\
 (1, =)
 \end{array}$$

- $\iota^*4 \quad \iota^{-1} \in \text{Un} \quad (\text{Similar to } \iota^*3)$
- $\iota^*5 \quad \iota \in \text{Un}_2 \quad (\text{From } \iota^*3 \text{ and } \iota^*4)$
- $*\iota^*6 \quad \iota \uparrow a \in \text{Un}_2 \quad (\text{From } *\iota^*6 \text{ and } \iota^*5)$
- $\iota^*7 \quad D_\iota = V$

$$\begin{array}{c}
 - \quad \iota^*7 \\
 (r) \quad \forall x. x \in D_\iota \\
 - \quad r \in D_\iota \\
 - \quad \exists x. \langle rx \rangle \in \iota \\
 - \quad \langle rr \rangle \in \iota \\
 - \quad \exists x. \langle rr \rangle = \langle xx \rangle \\
 \hline
 \langle rr \rangle = \langle rr \rangle \\
 (=)
 \end{array}$$

- $\iota^*8 \quad W_\iota = V$
- $\iota^*9 \quad D_{\iota \uparrow a} = a$

$$\begin{array}{c}
 - \quad \iota^*9 \\
 (r) \quad \forall x. x \in D_{\iota \uparrow a} \equiv x \in a \\
 - \quad r \in D_{\iota \uparrow a} \equiv r \in a \\
 \hline
 - \quad r \in D_{\iota \uparrow a} \quad 1 \quad r \in a \\
 1 \quad r \notin a \quad - \quad r \notin D_{\iota \uparrow a} \\
 \hline
 - \quad \exists x. \langle rx \rangle \in \iota \uparrow a \quad (s) \quad \neg \exists x. \langle rx \rangle \in \iota \uparrow a \\
 - \quad \langle rr \rangle \in \iota \uparrow a \quad - \quad \langle rs \rangle \notin \iota \uparrow a \\
 \hline
 - \quad \exists x. \langle rr \rangle = \langle xx \rangle \wedge x \in a \quad (t) \quad \neg \exists x. \langle rs \rangle = \langle xx \rangle \wedge x \in a \\
 - \quad \langle rr \rangle = \langle rr \rangle \wedge r \in a \quad - \quad \neg \langle rs \rangle = \langle tt \rangle \wedge t \in a \\
 \hline
 \langle rr \rangle = \langle rr \rangle \quad r \in a \quad \langle rs \rangle \neq \langle tt \rangle \\
 (=) \quad (1) \quad (1, =, 1) \\
 t \notin a
 \end{array}$$

- $\iota^*10 \quad W_{\iota \uparrow a} = a \quad (\text{Similar to } \iota^*9)$
- $\iota^*11 \quad \iota \uparrow a \in \text{Map}_2^{a, a} \quad (\text{From } *\iota^*6, \iota^*9 \text{ and } \iota^*10)$

$\iota^*12 \quad \sigma^{-1} \in \text{Un} \rightarrow \sigma^{-1} \circ \sigma = \iota \upharpoonright D_\sigma$

<div style="text-align: center; margin-bottom: 10px;">ι^*12</div> <div style="text-align: center; margin-bottom: 10px;">1 $\sigma^{-1} \notin \text{Un}$</div> <div style="text-align: center; margin-bottom: 10px;">- $\sigma^{-1} \circ \sigma = \iota \upharpoonright D_\sigma$</div> <div style="text-align: center; margin-bottom: 10px;">(w) - $\forall x. x \in \sigma^{-1} \circ \sigma \equiv x \in \iota \upharpoonright D_\sigma$</div> <div style="text-align: center; margin-bottom: 10px;">- $w \in \sigma^{-1} \circ \sigma \equiv w \in \iota \upharpoonright D_\sigma$</div> <hr/> <div style="display: flex; justify-content: space-between;"> <div style="width: 45%;"> <div style="text-align: center; margin-bottom: 10px;">2 $w \in \sigma^{-1} \circ \sigma$</div> <div style="text-align: center; margin-bottom: 10px;">- $w \notin \iota \upharpoonright D_\sigma$</div> <div style="text-align: center; margin-bottom: 10px;">(r, r₁) - $\neg \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \iota \wedge x \in D_\sigma$</div> <div style="text-align: center; margin-bottom: 10px;">- $\neg \exists x. w = \langle rr_1 \rangle \wedge \langle rr_1 \rangle \in \iota \wedge r \in D_\sigma$</div> <div style="text-align: center; margin-bottom: 10px;">3 $w \neq \langle rr_1 \rangle$</div> <div style="text-align: center; margin-bottom: 10px;">- $\langle rr_1 \rangle \notin \iota$</div> <div style="text-align: center; margin-bottom: 10px;">- $r \notin D_\sigma$</div> <hr/> <div style="text-align: center; margin-bottom: 10px;">(r₂) - $\neg \exists x. \langle rr_1 \rangle = \langle xx \rangle$</div> <div style="text-align: center; margin-bottom: 10px;">(5) - $\neg \exists x. \langle rx \rangle \in \sigma$</div> <div style="text-align: center; margin-bottom: 10px;">4 $\langle rr_1 \rangle \neq \langle rr_2 \rangle$</div> <div style="text-align: center; margin-bottom: 10px;">5 $\langle rs \rangle \notin \sigma$</div> <div style="text-align: center; margin-bottom: 10px;">(2) - $\exists yxz. w = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \sigma^{-1}$</div> <div style="text-align: center; margin-bottom: 10px;">- $w = \langle rr_1 \rangle \wedge \langle rs \rangle \in \sigma \wedge \langle sr_1 \rangle \in \sigma^{-1}$</div> <div style="text-align: center; margin-bottom: 10px;">- $w = \langle rr_1 \rangle \quad \langle rs \rangle \in \sigma \quad - \quad \langle sr_1 \rangle \in \sigma^{-1}$</div> <div style="text-align: center; margin-bottom: 10px;">(3) (5) (4, 5, =, 1)</div> </div> <div style="width: 45%;"> <div style="text-align: center; margin-bottom: 10px;">2 $w \in \iota \upharpoonright D_\sigma$</div> <div style="text-align: center; margin-bottom: 10px;">- $w \notin \sigma^{-1} \circ \sigma$</div> <div style="text-align: center; margin-bottom: 10px;">(r, s, t) - $\neg \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \sigma \wedge \langle yz \rangle \in \sigma^{-1}$</div> <div style="text-align: center; margin-bottom: 10px;">- $\neg \exists x. w = \langle rt \rangle \wedge \langle rs \rangle \in \sigma \wedge \langle st \rangle \in \sigma^{-1}$</div> <div style="text-align: center; margin-bottom: 10px;">3 $w \neq \langle rt \rangle$</div> <div style="text-align: center; margin-bottom: 10px;">4 $\langle rs \rangle \notin \sigma$</div> <div style="text-align: center; margin-bottom: 10px;">5 $\langle st \rangle \notin \sigma^{-1}$</div> <hr/> <div style="text-align: center; margin-bottom: 10px;">(2) - $\exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \iota \wedge x \in D_\sigma$</div> <div style="text-align: center; margin-bottom: 10px;">- $w = \langle rt \rangle \wedge \langle rt \rangle \in \iota \wedge r \in D_\sigma$</div> <div style="text-align: center; margin-bottom: 10px;">(3) $w = \langle rt \rangle \quad - \quad \langle rt \rangle \in \iota \quad - \quad r \in D_\sigma$</div> <div style="text-align: center; margin-bottom: 10px;">- $\exists x. \langle rt \rangle = \langle xx \rangle \quad - \quad \exists x. \langle rx \rangle \in \sigma$</div> <div style="text-align: center; margin-bottom: 10px;">6 $\langle rt \rangle = \langle rr \rangle \quad \langle rs \rangle \in \sigma$</div> <div style="text-align: center; margin-bottom: 10px;">(*) (4)</div> </div> </div> <div style="text-align: center; margin-top: 10px;">(*)</div> <div style="text-align: center; margin-bottom: 10px;">(1) - $\neg \forall xyz. \langle xy \rangle \in \sigma^{-1} \wedge \langle xz \rangle \in \sigma^{-1} \rightarrow y = z$</div> <div style="text-align: center; margin-bottom: 10px;">- $\neg \exists st. \langle st \rangle \in \sigma^{-1} \wedge \langle sr \rangle \in \sigma^{-1} \rightarrow t = r$</div> <div style="text-align: center; margin-bottom: 10px;">(5, =, 1) $\langle st \rangle \in \sigma^{-1} \quad - \quad \langle sr \rangle \in \sigma^{-1} \quad \neg \quad t \neq r$</div> <div style="text-align: center; margin-bottom: 10px;">(6, =)</div> <div style="text-align: center; margin-bottom: 10px;">(4) $\langle rs \rangle \in \sigma$</div>	<div style="text-align: center; margin-bottom: 10px;">ι^*13</div> <div style="text-align: center; margin-bottom: 10px;">$\sigma \in \text{Un} \rightarrow \sigma \circ \sigma^{-1} = \iota \upharpoonright D_{\sigma^{-1}}$</div> <div style="text-align: center; margin-bottom: 10px;">(Similar to ι^*12)</div> <div style="text-align: center; margin-bottom: 10px;">ι^*14</div> <div style="text-align: center; margin-bottom: 10px;">$\sigma \in \text{Un}_2 \rightarrow \sigma^{-1} \circ \sigma = \iota \upharpoonright D_\sigma \wedge \sigma \circ \sigma^{-1} = \iota \upharpoonright D_{\sigma^{-1}}$</div> <div style="text-align: center; margin-bottom: 10px;">(From ι^*12 and ι^*13)</div> <div style="text-align: center; margin-bottom: 10px;">ι^*15</div> <div style="text-align: center; margin-bottom: 10px;">$\sigma \circ \iota = \sigma \cap V^2$</div> <div style="text-align: center; margin-bottom: 10px;">ι^*15</div> <div style="text-align: center; margin-bottom: 10px;">(w) - $\forall x. x \in \sigma \circ \iota \equiv x \in \sigma \cap V^2$</div> <div style="text-align: center; margin-bottom: 10px;">- $w \in \sigma \circ \iota \equiv w \in \sigma \cap V^2$</div> <hr/> <div style="display: flex; justify-content: space-between;"> <div style="width: 45%;"> <div style="text-align: center; margin-bottom: 10px;">1 $w \in \sigma \circ \iota$</div> <div style="text-align: center; margin-bottom: 10px;">- $w \notin \sigma \cap V^2$</div> <div style="text-align: center; margin-bottom: 10px;">- $\neg \exists x. w \in \sigma \wedge w \in V^2$</div> </div> <div style="width: 45%;"> <div style="text-align: center; margin-bottom: 10px;">1 $w \in \sigma \cap V^2$</div> <div style="text-align: center; margin-bottom: 10px;">- $w \notin \sigma \circ \iota$</div> <div style="text-align: center; margin-bottom: 10px;">(r, s, t) - $\neg \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \iota \wedge \langle yz \rangle \in \sigma$</div> </div> </div>
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- $\iota^*13 \quad \sigma \in \text{Un} \rightarrow \sigma \circ \sigma^{-1} = \iota \upharpoonright D_{\sigma^{-1}} \quad (\text{Similar to } \iota^*12)$
- $\iota^*14 \quad \sigma \in \text{Un}_2 \rightarrow \sigma^{-1} \circ \sigma = \iota \upharpoonright D_\sigma \wedge \sigma \circ \sigma^{-1} = \iota \upharpoonright D_{\sigma^{-1}} \quad (\text{From } \iota^*12 \text{ and } \iota^*13)$
- $\iota^*15 \quad \sigma \circ \iota = \sigma \cap V^2$

<div style="text-align: center; margin-bottom: 10px;">ι^*15</div> <div style="text-align: center; margin-bottom: 10px;">(w) - $\forall x. x \in \sigma \circ \iota \equiv x \in \sigma \cap V^2$</div> <div style="text-align: center; margin-bottom: 10px;">- $w \in \sigma \circ \iota \equiv w \in \sigma \cap V^2$</div> <hr/> <div style="display: flex; justify-content: space-between;"> <div style="width: 45%;"> <div style="text-align: center; margin-bottom: 10px;">1 $w \in \sigma \circ \iota$</div> <div style="text-align: center; margin-bottom: 10px;">- $w \notin \sigma \cap V^2$</div> <div style="text-align: center; margin-bottom: 10px;">- $\neg \exists x. w \in \sigma \wedge w \in V^2$</div> </div> <div style="width: 45%;"> <div style="text-align: center; margin-bottom: 10px;">1 $w \in \sigma \cap V^2$</div> <div style="text-align: center; margin-bottom: 10px;">- $w \notin \sigma \circ \iota$</div> <div style="text-align: center; margin-bottom: 10px;">(r, s, t) - $\neg \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \iota \wedge \langle yz \rangle \in \sigma$</div> </div> </div>	
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$ \begin{array}{c} \frac{2 \quad w \notin \sigma}{- \quad w \notin V^2} \\ \frac{(r, s) \quad - \quad \neg \exists xy. w = \langle xy \rangle}{3 \quad w \neq \langle rs \rangle} \\ \frac{(1) \quad - \quad \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \iota \wedge \langle yz \rangle \in \sigma}{- \quad w = \langle rs \rangle \wedge \langle rr \rangle \in \iota \wedge \langle rs \rangle \in \sigma} \\ \frac{w = \langle rs \rangle \quad \langle rr \rangle \in \iota \quad \langle rs \rangle \in \sigma}{(3) \quad \iota * 1 \quad (2, 3, =, I)} \end{array} $	$ \begin{array}{c} - \quad \neg. w = \langle rt \rangle \wedge \langle rs \rangle \in \iota \wedge \langle st \rangle \in \sigma \\ \frac{2 \quad w \neq \langle rt \rangle}{- \quad \langle rs \rangle \notin \iota} \\ \frac{3 \quad \langle st \rangle \notin \sigma}{- \quad \neg \exists x. \langle rs \rangle = \langle xx \rangle} \\ \frac{(r) \quad - \quad \neg \exists x. \langle rs \rangle = \langle xx \rangle}{(1) \quad \frac{4 \quad \langle rs \rangle \neq \langle rr \rangle}{- \quad w \in \sigma \wedge w \in V^2}} \\ \frac{w \in \sigma \quad - \quad w \in V^2}{(2, 3, 4, =, I) \quad - \quad \exists xy. w = \langle xy \rangle} \\ \frac{- \quad \exists xy. w = \langle xy \rangle}{(2) \quad w = \langle rt \rangle} \end{array} $
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$\iota * 16 \quad \sigma \circ \iota = \iota \circ \sigma = \sigma \cap V^2$

We can prove the latter equality similarly as $\iota * 15$.

$\iota * 17 \quad \sigma \circ (\iota \uparrow a) = \sigma \uparrow a$

$ \begin{array}{c} - \quad \iota * 17 \\ \frac{(w) \quad - \quad \forall x. x \in \sigma \circ (\iota \uparrow a) \equiv x \in \sigma \uparrow a}{- \quad w \in \sigma \circ (\iota \uparrow a) \equiv w \in \sigma \uparrow a} \\ \frac{- \quad w \in \sigma \circ (\iota \uparrow a)}{- \quad w \in \sigma \uparrow a} \\ \frac{1 \quad \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \iota \uparrow a \wedge \langle yz \rangle \in \sigma}{(r, s) \quad - \quad \neg \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \wedge x \in a} \\ - \quad \neg. w = \langle rs \rangle \wedge \langle rs \rangle \in \sigma \wedge r \in a \\ \frac{2 \quad w \neq \langle rs \rangle}{3 \quad \langle rs \rangle \notin \sigma} \\ \frac{4 \quad r \notin a}{(1) \quad - \quad w = \langle rs \rangle \wedge \langle rr \rangle \in \iota \uparrow a \wedge \langle rs \rangle \in \sigma} \\ \frac{w = \langle rs \rangle \quad - \quad \langle rr \rangle \in \iota \uparrow a \quad \langle rs \rangle \in \sigma}{(2) \quad - \quad \langle rr \rangle \in \iota \wedge r \in a} \\ \frac{\langle rr \rangle \in \iota \quad r \in a}{\iota * 1 \quad (4)} \end{array} $	$ \begin{array}{c} - \quad w \in \sigma \uparrow a \\ - \quad w \notin \sigma \circ (\iota \uparrow a) \\ \frac{1 \quad \exists xy. w = \langle xy \rangle \wedge \langle xy \rangle \in \sigma \wedge x \in a}{(r, s, t) \quad - \quad \neg \exists xyz. w = \langle xz \rangle \wedge \langle xy \rangle \in \iota \uparrow a \wedge \langle yz \rangle \in \sigma} \\ - \quad \neg. w = \langle rt \rangle \wedge \langle rs \rangle \in \iota \uparrow a \wedge \langle st \rangle \in \sigma \\ \frac{2 \quad w \neq \langle rt \rangle}{- \quad \langle rs \rangle \notin \iota \uparrow a} \\ \frac{3 \quad \langle st \rangle \notin \sigma}{- \quad \neg \exists xy. \langle rs \rangle = \langle xy \rangle \wedge \langle xy \rangle \in \iota \wedge x \in a} \\ \frac{(r, s) \quad - \quad \neg \exists xy. \langle rs \rangle = \langle xy \rangle \wedge \langle xy \rangle \in \iota \wedge x \in a}{- \quad \langle rs \rangle \notin \iota} \\ \frac{4 \quad s \notin a}{(r) \quad - \quad \neg \exists x = \langle rs \rangle = \langle xx \rangle} \\ \frac{(1) \quad - \quad \neg \exists x = \langle rs \rangle = \langle xx \rangle}{5 \quad \langle rs \rangle \neq \langle rr \rangle} \\ \frac{- \quad w = \langle rt \rangle \wedge \langle rt \rangle \in \sigma \wedge r \in a}{(2) \quad w = \langle rt \rangle \quad \langle rt \rangle \in \sigma \quad r \in a} \\ \frac{(3, 5, =, I) \quad (4, 5, =, I)}{(2) \quad (3, 5, =, I) \quad (4, 5, =, I)} \end{array} $
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$\iota * 18 \quad (\iota \circ \sigma) \uparrow a = \sigma \uparrow a$

$$\begin{array}{c}
 1 \quad \iota * 18 \quad \text{Cut } \iota * 16 \\
 2 \quad \iota \circ \sigma \neq \sigma \cap V^2 \quad \text{Cut } \uparrow * 16 \\
 \hline
 (\sigma \cap V^2) \uparrow a \neq \sigma \uparrow a \\
 (1, 2, =, I)
 \end{array}$$

$\iota * 19 \quad (\sigma \circ \iota) \uparrow a = \sigma \uparrow a$

(Added in proof) (i) The principle of arranging the formulas in this Part (III) is as follows. The main symbols used are classified into seven groups: (i) $=, \subseteq$; (ii) $\{a, b\}, \langle ab \rangle$; (iii) $Un, Un_2, D_\sigma, W_\sigma, a^\sigma, A^\sigma$; (iv) σ^{-1} ; (v) $\sigma \circ \tau$; (vi) \uparrow ; and (vii) ι . To these groups correspond respectively the seven Sections: $=, El, Im, -1, \circ, \uparrow$, and ι . If, among the symbols listed above and contained in a formula A to be proved, x is the symbol which belongs to the rightmost group, say y , then A is proved in the Section corresponding to the group y . Owing to this principle it happens that in the proof of $-1*12$ are used some cuts of which the cut formulas are proved after $-1*12$.

(ii) Most proofs are analytic (see the last page of Part (IV)). For instance, the proofs for $-1*12, \iota*18, -1*14, \uparrow*10$, etc. are not analytic: note that in the proofs of $\uparrow*10$ is found the dependent variable $\langle rr^? \rangle$ and in that of $-1*14$ the variables $a^{\sigma^{-1}(\sigma^{-1})^{-1}}$ and $W_{\sigma^{-1}}$, which do not belong to the closure of the variables in the formulas to be proved. Some formulas with non-analytic proofs may be proved analytically.

(iii) All the proofs are given in the reduced form (see §20, Part (II)), or in the normal form except some practical change. The weakly irreducible proofs are those for $*El*6, *El*7$; and for $*\uparrow*1, *\uparrow*4, *\uparrow*5, *\uparrow*6$ and $*\iota*6$. Thus, the proofs are given without roundabout way and without superfluity as far as we can.

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