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Dimension approximation in smooth dynamical systems

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Abstract. For a non-conformal repeller Λ of a $C^{1+\alpha}$ map f preserving an ergodic measure μ of positive entropy, this paper shows that the Lyapunov dimension of μ can be approximated gradually by the Carathéodory singular dimension of a sequence of horseshoes. For a $C^{1+\alpha}$ diffeomorphism f preserving a hyperbolic ergodic measure μ of positive entropy, if (f, μ) has only two Lyapunov exponents $\lambda_u(\mu) > 0 > \lambda_s(\mu)$, then the Hausdorff or lower box or upper box dimension of μ can be approximated by the corresponding dimension of the horseshoes $\{\Lambda_n\}$. The same statement holds true if f is a C^1 diffeomorphism with a dominated Oseledet's splitting with respect to μ.

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1. *Introduction*

In smooth dynamical systems, a fundamental approximation result asserts that a $C^{1+\alpha}$ diffeomorphism f which preserves a hyperbolic ergodic measure μ of positive entropy can be approximated gradually by compact invariant locally maximal hyperbolic sets–horseshoes $\{\Lambda_n\}$, in the sense that dynamical quantities on the horseshoes such as the topological entropy and pressure, Lyapunov exponents, and averages of continuous functions are approaching those of the measure μ.

This type of result is widely referenced to the landmark work by Katok [20] or Katok and Hasselblatt (see [21]). Misiurewicz and Szlenk [28] earlier proved a related result for continuous and for piecewise monotone maps of the interval. Przytycki and Urbański [34] obtained corresponding properties for holomorphic maps in the case of a measure with only positive Lyapunov exponent. A related setting of dyadic diophantine approximations was established by Persson and Schmeling in [32]. For a general $C^{1+\alpha}$ diffeomorphism f preserving a hyperbolic ergodic measure μ with positive entropy, assume that μ has ℓ different Lyapunov exponents $\{\lambda_j\}_{j=1}^{\ell}$. On each approaching horseshoe Λ_n, Avila, Crovisier, and Wilkinson [2] obtained a continuous splitting

$$T_{\Lambda_n} M = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell$$

and showed that the exponential growth of $D_x f^n|_{E_i}$ is roughly λ_i for each $i = 1, 2, \ldots, \ell$. A corresponding statement for $C^{1+\alpha}$ non-conformal transformations (that is, non-invertible maps) was shown in [11]. See [12], [16, 17], and [44] for other results related to Katok's approximation construction of $C^{1+\alpha}$ maps.

A natural question is how large is that part of the dynamics described by these horseshoes. So, it is interesting to estimate the Hausdorff dimension of the stable and/or unstable Cantor sets of a horseshoe. If μ is a Sinai–Ruelle–Bowen (SRB) measure (that is, a measure with a particular absolute continuity property on unstable manifolds; see [6] for precise definitions), it was shown in [36] that μ can be approximated by ergodic measures supported on horseshoes with arbitrarily large unstable dimensions, which generalized Mendoza's result in [26] for diffeomorphisms in a higher dimensional manifold. The approach in [36] was based on Markov towers that can be described by horseshoes with infinitely many branches and variable return times. However, there is an essential mistake in the proof of the key proposition [36, Proposition 5.1]. The authors in [42] proved the same result by a different method. They used the u-Gibbs property of the conditional measure of the equilibrium measure and the properties of the uniformly hyperbolic dynamical systems. Furthermore, in [42], the authors proved that the Hausdorff dimension of μ can be approximated gradually by the Hausdorff dimension of the horseshoes $\{\Lambda_n\}$ provided that the stable direction is one dimension. See also [24, 25, 27, 29, 37, 38] that represent works close to this topic.

In this work, our main task is to compare the dimension of the horseshoes $\{\Lambda_n\}$ and the given hyperbolic ergodic measure μ of a C^r $(r \geq 1)$ diffeomorphism in a more general setting where μ may not be an SRB measure. For a non-conformal repeller Λ of a $C^{1+\alpha}$ map, using the approximation result in [11], we show that the Lyapunov dimension (see equation (3.1) for the definition) of an f-invariant ergodic measure μ supported on Λ can be approximated gradually by the Carathéodory singular dimension (see equation (3.6) for the definition) of the horseshoes $\{\Lambda_n\}$. For a $C^{1+\alpha}$ diffeomorphism f preserving a hyperbolic ergodic measure μ of positive entropy, if (f, μ) has only two Lyapunov exponents $\lambda_u(\mu) > 0 > \lambda_s(\mu)$, then the Hausdorff or lower box or upper box dimension of μ can be approximated by the corresponding dimension of the horseshoes $\{\Lambda_n\}$. The same statement holds true if f is a C^1 diffeomorphism with a dominated Oseledec's splitting with respect to μ.

We arrange the paper as follows. In §2, we give some basic notions and properties about topological and measure theoretic pressures, and dimensions of sets and measures.

Statements of our main results will be given in §3. In §4, we will give the detailed proofs of the main results.

2. *Definitions and preliminaries*

In this section, we recall the definitions of topological pressure and various dimensions of subsets and/or of invariant measures.

2.1. *Topological and measure theoretic pressures.*

Let $f : X \to X$ be a continuous transformation on a compact metric space X equipped with metric d. A subset $F \subset X$ is called an (n, ϵ)-separated set with respect to f if for any two different points $x, y \in F$, we have $d_n(x, y) := \max_{0 \le k \le n-1} d(f^k(x), f^k(y)) > \epsilon$. A sequence of continuous functions $\Phi = \{\phi_n\}_{n \ge 1}$ is called *sub-additive* if

$$\phi_{m+n} \le \phi_n + \phi_m \circ f^n \quad \text{for all } n, m \in \mathbb{N}.$$

Furthermore, a sequence of continuous functions $\Psi = \{\psi_n\}_{n \ge 1}$ is called *super-additive* if $-\Psi = \{-\psi_n\}_{n \ge 1}$ is sub-additive.

2.1.1. *Topological pressure defined via separated sets.*

Given a sub-additive potential $\Phi = \{\phi_n\}_{n \ge 1}$ on X, put

$$P_n(f, \Phi, \epsilon) = \sup \left\{ \sum_{x \in F} e^{\phi_n(x)} | F \text{ is an } (n, \epsilon)\text{-separated subset of } X \right\}.$$

Definition 2.1. We call the quantity

$$P_{\text{top}}(f, \Phi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(f, \Phi, \epsilon) \tag{2.1}$$

the *sub-additive topological pressure* of (f, Φ).

Remark 2.2. If $\Phi = \{\varphi_n\}_{n \ge 1}$ is *additive* in the sense that $\varphi_n(x) = \varphi(x) + \varphi(fx) + \cdots + \varphi(f^{n-1}x) \triangleq S_n\varphi(x)$ for some continuous function $\varphi : X \to \mathbb{R}$, we simply denote the topological pressure $P_{\text{top}}(f, \Phi)$ as $P_{\text{top}}(f, \varphi)$.

Let $\mathcal{M}_f(X)$ denote the space of all f-invariant measures on X. For $\mu \in \mathcal{M}_f(X)$, let $h_\mu(f)$ denote the metric entropy of f with respect to μ (see Walters' book [39] for details of metric entropy), and let

$$\mathcal{L}_*(\Phi, \mu) = \lim_{n \to \infty} \frac{1}{n} \int \phi_n \, d\mu.$$

The existence of the above limit follows from a sub-additive argument. In [9], the authors proved the following variational principle.

THEOREM 2.3. *Let $f : X \to X$ be a continuous transformation on a compact metric space X, and $\Phi = \{\phi_n\}_{n \ge 1}$ a sub-additive potential on X, then we have*

$$P_{\text{top}}(f, \Phi) = \sup\{h_\mu(f) + \mathcal{L}_*(\Phi, \mu) : \mu \in \mathcal{M}_f(X),\ \mathcal{L}_*(\Phi, \mu) \ne -\infty\}.$$

Here we take the convention that $\sup \emptyset = -\infty$.

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Although it is unknown whether the variational principle holds for super-additive topological pressure, Cao, Pesin, and Zhao gave an alternative definition via variational principle in [11]. Given a sequence of super-additive continuous potentials $\Psi = \{\psi_n\}_{n\geq 1}$ on a compact dynamical system (X, f), the super-additive topological pressure of Ψ is defined as

$$P_{\mathrm{var}}(f, \Psi) := \sup\{h_\mu(f) + \mathcal{L}_*(\Psi, \mu) : \mu \in \mathcal{M}_f(X)\},$$

where

$$\mathcal{L}_*(\Psi, \mu) = \lim_{n\to\infty} \frac{1}{n} \int \psi_n \, d\mu = \sup_{n\geq 1} \frac{1}{n} \int \psi_n \, d\mu.$$

The second equality is due to the standard sub-additive argument.

2.1.2. *Measure theoretic pressure.* We first follow the approach in [33] to give the definitions of topological pressures on arbitrary subsets. Given a sub-additive potential $\Phi = \{\phi_n\}_{n\geq 1}$ on X, a subset $Z \subset X$, and $\alpha \in \mathbb{R}$, let

$$M(Z, \Phi, \alpha, N, \epsilon) = \inf \bigg\{ \sum_i \exp\Big(-\alpha n_i + \sup_{y\in B_{n_i}(x_i,\epsilon)} \phi_{n_i}(y)\Big) : \bigcup_i B_{n_i}(x_i, \epsilon) \supset Z,\ x_i \in X \text{ and } n_i \geq N \text{ for all } i \bigg\}.$$

Since $M(Z, \Phi, \alpha, N, \epsilon)$ is monotonically increasing with N, let

$$m(Z, \Phi, \alpha, \epsilon) := \lim_{N\to\infty} M(Z, \Phi, \alpha, N, \epsilon). \tag{2.2}$$

We denote the jump-up point of $m(Z, \Phi, \alpha, \epsilon)$ by

$$P_Z(f, \Phi, \epsilon) = \inf\{\alpha : m(Z, \Phi, \alpha, \epsilon) = 0\} = \sup\{\alpha : m(Z, \Phi, \alpha, \epsilon) = +\infty\}.$$

Definition 2.4. We call the quantity

$$P_Z(f, \Phi) = \liminf_{\epsilon\to 0} P_Z(f, \Phi, \epsilon)$$

the *topological pressure* of (f, Φ) on the set Z (see [14] for the weighted version of this quantity).

Similarly, for $\alpha \in \mathbb{R}$ and $Z \subset X$, define

$$R(Z, \Phi, \alpha, N, \epsilon) = \inf \bigg\{ \sum_i \exp\Big(-\alpha N + \sup_{y\in B_N(x_i,\epsilon)} \phi_N(y)\Big) : \bigcup_i B_N(x_i, \epsilon) \supset Z,\ x_i \in X \bigg\}.$$

We set

$$\underline{r}(Z, \Phi, \alpha, \epsilon) = \liminf_{N\to\infty} R(Z, \Phi, \alpha, N, \epsilon),$$
$$\overline{r}(Z, \Phi, \alpha, \epsilon) = \limsup_{N\to\infty} R(Z, \Phi, \alpha, N, \epsilon)$$

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and define the jump-up points of $\underline{r}(Z, \Phi, \alpha, \epsilon)$ and $\overline{r}(Z, \Phi, \alpha, \epsilon)$ as

$$\underline{CP}_Z(f, \Phi, \epsilon) = \inf\{\alpha : \underline{r}(Z, \Phi, \alpha, \epsilon) = 0\} = \sup\{\alpha : \underline{r}(Z, \Phi, \alpha, \epsilon) = +\infty\},$$
$$\overline{CP}_Z(f, \Phi, \epsilon) = \inf\{\alpha : \overline{r}(Z, \Phi, \alpha, \epsilon) = 0\} = \sup\{\alpha : \overline{r}(Z, \Phi, \alpha, \epsilon) = +\infty\},$$

respectively.

Definition 2.5. We call the quantities

$$\underline{CP}_Z(f, \Phi) = \liminf_{\epsilon\to 0} \underline{CP}_Z(f, \Phi, \epsilon) \text{ and } \overline{CP}_Z(f, \Phi) = \liminf_{\epsilon\to 0} \overline{CP}_Z(f, \Phi, \epsilon)$$

the *lower* and *upper topological pressures* of (f, Φ) on the set Z, respectively.

Given an f-invariant measure μ, let

$$P_\mu(f, \Phi, \epsilon) = \inf\{P_Z(f, \Phi, \epsilon) \colon \mu(Z) = 1\}$$

and then we call the quantity

$$P_\mu(f, \Phi) := \liminf_{\epsilon\to 0} P_\mu(f, \Phi, \epsilon)$$

the *measure theoretic pressure* of (f, Φ) with respect to μ. Let further

$$\underline{CP}_\mu(f, \Phi, \epsilon) = \lim_{\delta\to 0} \inf\{\underline{CP}_Z(f, \Phi, \epsilon) \colon \mu(Z) \geq 1 - \delta\},$$
$$\overline{CP}_\mu(f, \Phi, \epsilon) = \lim_{\delta\to 0} \inf\{\overline{CP}_Z(f, \Phi, \epsilon) \colon \mu(Z) \geq 1 - \delta\}.$$

We call the quantities

$$\underline{CP}_\mu(f, \Phi) = \liminf_{\epsilon\to 0} \underline{CP}_\mu(f, \Phi, \epsilon), \quad \overline{CP}_\mu(f, \Phi) = \liminf_{\epsilon\to 0} \overline{CP}_\mu(f, \Phi, \epsilon)$$

the *lower and upper measure theoretic pressures* of (f, Φ) with respect to μ, respectively. It is proved in [**10**, Theorem A] that

$$P_\mu(f, \Phi) = \underline{CP}_\mu(f, \Phi) = \overline{CP}_\mu(f, \Phi) = h_\mu(f) + \mathcal{L}_*(\Phi, \mu) \tag{2.3}$$

for any f-invariant ergodic measure μ with $\mathcal{L}_*(\Phi, \mu) \neq -\infty$.

Remark 2.6. In fact, one can show that

$$\mathcal{P}_\mu(f, \Phi) = \inf\{\mathcal{P}_Z(f, \Phi) : \mu(Z) = 1\},$$

where $\mathcal{P}$ denotes either P or $\underline{CP}$ or $\overline{CP}$, see [**46**] for a proof.

2.2. *Dimensions of sets and measures.* Now we recall the definitions of Hausdorff and box dimensions of subsets and measures. Given a subset $Z \subset X$, for any $s \geq 0$, let

$$\mathcal{H}^s_\delta(Z) = \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : \{U_i\}_{i\geq 1} \text{ is a cover of } Z \text{ with } \operatorname{diam} U_i \leq \delta, \text{ for all } i \geq 1\right\}$$

and

$$\mathcal{H}^s(Z) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(Z).$$

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The above limit exists, although the limit may be infinity. We call $\mathcal{H}^s(Z)$ the s-dimensional Hausdorff measure of Z.

Definition 2.7. The following jump-up value of $\mathcal{H}^s(Z)$

$$\dim_H Z = \inf\{s : \mathcal{H}^s(Z) = 0\} = \sup\{s : \mathcal{H}^s(Z) = \infty\}$$

is called the *Hausdorff dimension* of Z. The *lower and upper box dimension* of Z are defined respectively by

$$\underline{\dim}_B Z = \liminf_{\delta\to 0} \frac{\log N(Z,\delta)}{-\log\delta} \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\delta\to 0} \frac{\log N(Z,\delta)}{-\log\delta},$$

where $N(Z,\delta)$ denotes the least number of balls of radius δ that are needed to cover the set Z. If $\underline{\dim}_B Z = \overline{\dim}_B Z$, we will denote the common value by $\dim_B Z$ and call it the *box dimension* of Z.

The following two results are well known in the field of fractal geometry, e.g. see Falconer's book [13] for proofs.

LEMMA 2.8. *Let X and Y be metric spaces. For any $r \in (0,1)$, $\Phi : X \to Y$ is an onto, (C,r)-Hölder continuous map for some $C > 0$. Then*

$$\dim_H Y \le r^{-1}\dim_H X, \quad \underline{\dim}_B Y \le r^{-1}\underline{\dim}_B X \quad \textit{and} \quad \overline{\dim}_B Y \le r^{-1}\overline{\dim}_B X.$$

COROLLARY 2.9. *Let X and Y be metric spaces, and let $\Phi : X \to Y$ be an onto, Lipschitz continuous map. Then*

$$\dim_H Y \le \dim_H X, \quad \underline{\dim}_B Y \le \underline{\dim}_B X \quad \textit{and} \quad \overline{\dim}_B Y \le \overline{\dim}_B X.$$

Given a Borel probability measure μ on X, the quantity

$$\begin{aligned}\dim_H \mu &= \inf\{\dim_H Z : Z \subset X \text{ and } \mu(Z) = 1\}\\ &= \lim_{\delta\to 0} \inf\{\dim_H Z : Z \subset X \text{ and } \mu(Z) \ge 1-\delta\}\end{aligned}$$

is called the *Hausdorff dimension of the measure* μ. Similarly, we call the two quantities

$$\underline{\dim}_B \mu = \lim_{\delta\to 0} \inf\{\underline{\dim}_B Z : Z \subset X \text{ and } \mu(Z) \ge 1-\delta\}$$

and

$$\overline{\dim}_B \mu = \lim_{\delta\to 0} \inf\{\overline{\dim}_B Z : Z \subset X \text{ and } \mu(Z) \ge 1-\delta\}$$

the *lower box dimension* and *upper box dimension* of μ, respectively.

If μ is a finite measure on X and there exists $d \ge 0$ such that

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d$$

for μ-almost every $x \in X$, then

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d.$$

This criterion was established by Young in [45].

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3. *Statements of main results*

In this section, we will give the statements of the main results in this paper, and the proof will be postponed to the next section.

3.1. *Dimension approximation for uniformly expanding systems.*

Let $f : M \to M$ be a smooth map of an m_0-dimensional compact smooth Riemannian manifold M, and Λ a compact f-invariant subset of M. Let $\mathcal{M}_f(\Lambda)$ and $\mathcal{E}_f(\Lambda)$ denote respectively the set of all f-invariant measures and ergodic measures on Λ.

3.1.1. *Definitions of repeller and Lyapunov dimension.*

We call Λ a *repeller* for f or f is *expanding* on Λ if:

(1) there exists an open neighborhood U of Λ such that $\Lambda = \{x \in U : f^n(x) \in U$ for all $n \geq 0\}$;

(2) there is $\kappa > 1$ such that

$$\|D_x f(v)\| \geq \kappa \|v\| \quad \text{for all } x \in \Lambda \text{ and } v \in T_x M,$$

where $\|\cdot\|$ is the norm induced by the Riemannian metric on M, and $D_x f : T_x M \to T_{f(x)} M$ is the differential operator.

Given an f-invariant ergodic measure μ supported on the repeller Λ, let $\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_{m_0}(\mu)$ and $h_\mu(f)$ denote the Lyapunov exponents and the measure theoretic entropy of (f, μ), respectively. We refer the reader to [6, 39] for detailed descriptions of Lyapunov exponents and the measure theoretic entropy. We further define the *Lyapunov dimension* of μ as follows:

$$\dim_{\mathrm{L}} \mu := \begin{cases} \ell + \frac{h_\mu(f) - \lambda_{m_0}(\mu) - \cdots - \lambda_{m_0-\ell+1}(\mu)}{\lambda_{m_0-\ell}(\mu)}, & h_\mu(f) \geq \lambda_{m_0}(\mu), \\ \frac{h_\mu(f)}{\lambda_{m_0}(\mu)}, & 0 \leq h_\mu(f) < \lambda_{m_0}(\mu), \end{cases} \tag{3.1}$$

where $\ell = \max\{i : \lambda_{m_0}(\mu) + \cdots + \lambda_{m_0-i+1}(\mu) \leq h_\mu(f)\}$.

The original definition of Lyapunov dimension in [1, 19, 22] is defined only for hyperbolic systems as follows: assume that ν is an ergodic measure of a smooth diffeomorphism f with Lyapunov exponents $\lambda_1 \geq \cdots \geq \lambda_u > 0 \geq \lambda_{u+1} \geq \cdots \geq \lambda_{m_0}$, then the the Lyapunov dimension is

$$\text{Lya dim } \nu = \ell + \frac{\lambda_1 + \cdots + \lambda_u + \cdots + \lambda_\ell}{|\lambda_{\ell+1}|},$$

where $\ell = \max\{i : \lambda_1 + \cdots + \lambda_i \geq 0\}$. Assume further that ν is an SRB measure, then $h_\nu(f) = \lambda_1 + \cdots + \lambda_u$. As a consequence,

$$\text{Lya dim } \nu = \ell + \frac{h_\nu(f) + \lambda_{u+1} + \cdots + \lambda_\ell}{|\lambda_{\ell+1}|}$$

and $\ell = \max\{i : -\lambda_{u+1} - \cdots - \lambda_i \geq h_\nu(f)\}$. Hence, the definition in equation (3.1) is a reasonable substitute. For a C^1 expanding map f, Feng and Simon [15] defined the Lyapunov dimension of an ergodic measure as the zero of the measure theoretic pressure $P_\mu(f, \Phi_f(t)) = 0$ (see equation (3.4)). In this paper, we will prove that the unique solution of the equation $P_\mu(f, \Phi_f(t)) = 0$ is indeed our definition of Lyapunov dimension

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(see Theorem A). Furthermore, this paper shows that the Lyapunov dimension of an ergodic measure defined in equation (3.1) is equal to its Carathéodory singular dimension (see Proposition 3.2), so the Carathéodory singular dimension (see §3.1.3 for the detailed definition) can be regarded as a geometric explanation of the Lyapunov dimension.

3.1.2. *Singular valued potentials.* Let Λ be a repeller of a smooth map $f : M \to M$. Given $x \in \Lambda$ and $n \geq 1$, consider the differentiable operator $D_x f^n : T_x M \to T_{f^n(x)} M$ and denote the singular values of $D_x f^n$ (square roots of the eigenvalues of $(D_x f^n)^* D_x f^n$) in the decreasing order by

$$\alpha_1(x, f^n) \geq \alpha_2(x, f^n) \geq \cdots \geq \alpha_{m_0}(x, f^n). \tag{3.2}$$

For $t \in [0, m_0]$, set

$$\varphi^t(x, f^n) := \sum_{i=m_0-[t]+1}^{m_0} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{m_0-[t]}(x, f^n). \tag{3.3}$$

Since f is smooth, the functions $x \mapsto \alpha_i(x, f^n)$, $x \mapsto \varphi^t(x, f^n)$ are continuous for any $n \geq 1$. It is easy to see that for all $n, \ell \in \mathbb{N}$,

$$\varphi^t(x, f^{n+\ell}) \geq \varphi^t(x, f^n) + \varphi^t(f^n(x), f^\ell).$$

It follows that the sequence of functions

$$\Phi_f(t) := \{-\varphi^t(\cdot, f^n)\}_{n\geq 1} \tag{3.4}$$

is sub-additive, which is called the *sub-additive singular valued potentials*.

3.1.3. *Carathéodory singular dimension.* We recall the definition of Carathéodory singular dimension of a repeller which is introduced in [11].

Let $\Phi_f(t) = \{-\varphi^t(\cdot, f^n)\}_{n\geq 1}$. Given a subset $Z \subseteq \Lambda$, for each small number $r > 0$, let

$$m(Z, t, r) := \lim_{N\to\infty} \inf \left\{ \sum_i \exp\left(\sup_{y\in B_{n_i}(x_i,r)} -\varphi^t(y, f^{n_i})\right)\right\},$$

where the infimum is taken over all collections $\{B_{n_i}(x_i, r)\}$ of Bowen's balls with $x_i \in \Lambda$, $n_i \geq N$ that cover Z. It is easy to see that there is a jump-up value

$$\dim_{C,r} Z := \inf\{t : m(Z, t, r) = 0\} = \sup\{t : m(Z, t, r) = +\infty\}. \tag{3.5}$$

The quantity

$$\dim_C Z := \liminf_{r\to 0} \dim_{C,r} Z \tag{3.6}$$

is called the *Carathéodory singular dimension of Z*. Particularly, the Carathéodory singular dimension of the repeller Λ is independent of the parameter r for small values of $r > 0$ (see [11, Theorem 4.1]).

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For each f-invariant measure μ supported on Λ, let

$$\dim_{C,r}\mu := \inf\{\dim_{C,r} Z : \mu(Z) = 1\},$$

and the quantity

$$\dim_C \mu := \liminf_{r\to 0} \dim_{C,r}\mu$$

is called the *Carathéodory singular dimension* of the measure μ.

3.1.4. *Approximation of Carathéodory singular dimension of repellers.* Given a repeller Λ of a $C^{1+\alpha}$ map f, the following result shows that the zero of the measure theoretic pressure function is exactly the Lyapunov dimension of an ergodic measure $\mu \in \mathcal{E}_f(\Lambda)$, and the Lyapunov dimension of an ergodic measure of positive entropy can be approximated by the Carathéodory singular dimension of a sequence of invariant sets. Recall that $\Phi_f(t) := \{-\varphi^t(\cdot, f^n)\}_{n\geq 1}$ is the sub-additive singular valued potentials with respect to f (see the definition in equation (3.4)).

The following result gives a measure theoretic version of Bowen's equation, that is, the unique zero of the measure theoretic pressure is exactly the Lyapunov dimension of an ergodic measure.

PROPOSITION 3.1. *Let $f : M \to M$ be a C^1 map of an m_0-dimensional compact smooth Riemannian manifold M, and Λ a repeller of f. For every f-invariant ergodic measure μ supported on Λ, we have that*

$$\dim_{\mathrm{L}}\mu = s_\mu,$$

where s_μ is the unique root of the equation $P_\mu(f, \Phi_f(t)) = 0$.

For an ergodic measure supported on a repeller with positive entropy, one can find a sequence of compact invariant sets whose Carathéodory singular dimension gradually approaches the Lyapunov dimension of the measure.

THEOREM A. *Let $f : M \to M$ be a $C^{1+\alpha}$ map of an m_0-dimensional compact smooth Riemannian manifold M and Λ a repeller of f, and let μ be an f-invariant ergodic measure on Λ with $h_\mu(f) > 0$. For any $\varepsilon > 0$, there exists an f-invariant compact subset $\Lambda_\varepsilon \subset \Lambda$ such that $\dim_C \Lambda_\varepsilon \to \dim_{\mathrm{L}}\mu$ as ε approaches zero.*

Some comments on the previous theorem are in order. First, the map of higher smoothness $C^{1+\alpha}$ is crucial as it allows us to use some powerful results of Pesin theory. Second, if f is a local diffeomorphism preserving an ergodic expanding measure μ of positive entropy, that is, (f, μ) has only positive Lyapunov exponent, in this case, one can also obtain an approximation result as in [11] so that we can obtain the second statement in the previous theorem in this setting. In [37], for a C^2 interval map f with finitely many non-degenerate critical points, the author proved that the Hausdorff dimension of an expanding measure μ can be approximated gradually by the Hausdorff dimension of a sequence of repellers.

For each f-invariant ergodic measure μ supported on Λ, the following result shows that the Carathéodory singular dimension of μ is exactly its Lyapunov dimension.

PROPOSITION 3.2. *Let $f : M \to M$ be a C^1 map of an m_0-dimensional compact smooth Riemannian manifold M, and Λ a repeller for f. Then the following statements hold:*

(1) *for each subset $Z \subset \Lambda$, we have that*

$$\dim_C Z = t_Z,$$

where t_Z is the unique root of the equation $P_Z(f, \Phi_f(t)) = 0$;

(2) *for each f-invariant ergodic measure μ supported on Λ, we have that*

$$\dim_C \mu = \dim_{\mathrm{L}} \mu.$$

3.2. *Dimension approximation in non-uniformly hyperbolic systems.* In this section, we first recall an approximation result in non-uniformly hyperbolic systems that is proved by Avila, Crovisier, and Wilkinson [2], then we give the statement of our dimension approximation result in non-uniformly hyperbolic systems.

3.2.1. *Lyapunov exponents and holonomy maps.* Let $f : M \to M$ be a diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M. By Oseledec's multiplicative ergodic theorem (see [30]), there exists a total measure set $\mathcal{O} \subset M$ such that for each $x \in \mathcal{O}$ and each invariant measure μ, there exist positive integers $d_1(x), d_2(x), \ldots, d_{p(x)}(x)$, numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{p(x)}(x)$, and a splitting

$$T_x M = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{p(x)}(x),$$

which satisfy that:

(1) $D_x f E_i(x) = E_i(f(x))$ for each i and $\sum_{i=1}^{p(x)} d_i(x) = m_0$;

(2) for each $0 \neq v \in E_i(x)$, we have that

$$\lambda_i(x) = \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n(v)\|.$$

Here we call the numbers $\{\lambda_i(x)\}_{i=1}^{p(x)}$ the Lyapunov exponents of (f, μ). In the case that μ is an f-invariant ergodic measure, the numbers $p(x)$, $\{d_i(x)\}$, and $\{\lambda_i(x)\}$ are constants almost everywhere. We denote them simply by p, $\{d_i\}_{i=1}^p$, and $\{\lambda_i\}_{i=1}^p$.

A compact invariant subset $\Lambda \subset M$ is called a *hyperbolic set* if there exists a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$, and constants $C > 0$, $0 < \lambda < 1$ such that for every $x \in \Lambda$:

(1) $D_x f(E^s(x)) = E^s(f(x))$, $D_x f(E^u(x)) = E^u(f(x))$;

(2) for all $n \geq 0$, $\|D_x f^n(v)\| \leq C\lambda^n \|v\|$ if $v \in E^s(x)$, and $\|D_x f^{-n}(v)\| \leq C\lambda^n \|v\|$ if $v \in E^u(x)$.

Given a point $x \in \Lambda$, for each small $\beta > 0$, the *local stable and unstable manifolds* are defined as follows:

$$W^s_\beta(f, x) = \{y \in M : d(f^n(x), f^n(y)) \leq \beta \text{ for all } n \geq 0\},$$
$$W^u_\beta(f, x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \beta \text{ for all } n \geq 0\}.$$

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The global stable and unstable sets of $x \in \Lambda$ are given as follows:

$$W^s(f,x) = \bigcup_{n\geq 0} f^{-n}(W^s_\beta(f, f^n(x))), \quad W^u(f,x) = \bigcup_{n\geq 0} f^{n}(W^u_\beta(f, f^{-n}(x))).$$

A hyperbolic set is called *locally maximal* if there exists a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. Recall that a *horseshoe* for a diffeomorphism f is a transitive, locally maximal hyperbolic set that is totally disconnected and not finite.

Let W^u and W^s be the unstable and stable foliations of a hyperbolic dynamical system (f, Λ). For $x, y \in \Lambda$ with x close to y, let $W^u_\beta(f,x)$ and $W^s_\beta(f,x)$ be the local stable foliations of x and y. Define the map $h : W^s_\beta(f,x) \to W^s_\beta(f,y)$ sending z to $h(z)$ by sliding along the leaves of W^u. The map h is called the holonomy map of W^u. The map h is Lipschitz continuous if

$$d_y(h(z_1), h(z_2)) \leq L d_x(z_1, z_2),$$

where $z_1, z_2 \in W^s_\beta(f,x)$ and d_x, d_y are natural path metrics on $W^s_\beta(f,x)$, $W^u_\beta(f,y)$ with respect to a fixed Riemannian structure on M. The constant L is the Lipschitz constant, and it is independent of the choice of W^s. The map h is α-Hölder continuous if

$$d_y(h(z_1), h(z_2)) \leq H d_x(z_1, z_2)^\alpha,$$

where H is the Hölder constant. Similarly, we can define the holonomy map of W^s.

3.2.2. *Approximation of Lyapunov exponents and entropy.* For a $C^{1+\alpha}$ diffeomorphism $f : M \to M$, Katok [20] showed that an f-invariant ergodic hyperbolic measure (a measure has no zero Lyapunov exponents) with positive metric entropy can be approximated by horseshoes. However, Katok's result does not explicitly mention a control of the Oseledets splitting over the horseshoes. Recently, Avila, Crovisier, and Wilkinson [2] showed that there is a dominated splitting over the horseshoes, with approximately the same Lyapunov exponents on each sub-bundle of the splitting.

Recall that Df-invariant splitting on a compact f-invariant subset Λ:

$$T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell, \quad (\ell \geq 2),$$

is a *dominated splitting* if there exists $N \geq 1$ such that for every $x \in \Lambda$, any unit vectors $v, w \in T_x M$:

$$v \in E_i(x), w \in E_j(x) \quad \text{with } i < j \Longrightarrow \|D_x f^N(v)\| \geq 2\|D_x f^N(w)\|.$$

We write $E_1 \succeq E_2 \succeq \cdots \succeq E_\ell$. Furthermore, if there are numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$, constants $C > 0$, and $0 < \varepsilon < \min_{1\leq i<\ell}(\lambda_i - \lambda_{i+1})/100$ such that for every $x \in \Lambda$, $n \in \mathbb{N}$, $1 \leq j \leq \ell$ and each unit vector $u \in E_j(x)$, it holds that

$$C^{-1} e^{n(\lambda_j - \varepsilon)} \leq \|D_x f^n(u)\| \leq C e^{n(\lambda_j+\varepsilon)},$$

then we say that

$$T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell, \quad (\ell \geq 2)$$

is a $\{\lambda_j\}_{1\leq j\leq\ell}$-*dominated splitting*.

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For the reader's convenience, we recall Avila, Crovisier, and Wilkinson's approximation results in the following, see [2] for more details.

THEOREM 3.3. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism, and μ an f-invariant ergodic hyperbolic measure with $h_\mu(f) > 0$. For each $\varepsilon > 0$ and a weak-$*$ neighborhood $\mathcal{V}$ of μ in the space of f-invariant probability measures on M, there exists a compact set $\Lambda_\varepsilon^* \subset M$ and a positive integer N such that the following properties hold:*

(1) *Λ_ε^* is a locally maximal hyperbolic set and topologically mixing with respect to f^N;*
(2) *$h_\mu(f) - \varepsilon < h_{top}(f, \Lambda_\varepsilon) < h_\mu(f) + \varepsilon$, where $\Lambda_\varepsilon = \Lambda_\varepsilon^* \cup f(\Lambda_\varepsilon^*) \cup \cdots f^{N-1}(\Lambda_\varepsilon^*)$;*
(3) *Λ_ε is ε-close to the support of μ in the Hausdorff distance;*
(4) *each invariant probability measure supported on the horseshoe Λ_ε lies in $\mathcal{V}$;*
(5) *if $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$ are the distinct Lyapunov exponents of (f, μ), with multiplicities $d_1, d_2, \ldots, d_\ell$, then there exists a $\{\lambda_j\}_{1 \le j < \ell}$-dominated splitting $T_{\Lambda_\varepsilon} M = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell$ with $\dim E_i = d_i$ for each i, and for each $x \in \Lambda_\varepsilon$, $k \ge 1$ and each vector $v \in E_i(x)$*

$$e^{(\lambda_i - \varepsilon)kN} \le \|D_x f^{kN}(v)\| \le e^{(\lambda_i + \varepsilon)kN} \quad \textit{for all } i = 1, 2, \ldots, \ell.$$

Remark 3.4. In the second statement, the original result does not give the inequality of the right-hand side. However, only a slight modification can give the upper bound of the topological entropy of f on the horseshoe.

Remark 3.5. By the estimation in the fifth statement of the above theorem, one can further show that there exists a constant $C > 0$ such that

$$C^{-1} e^{(\lambda_i - \varepsilon)n} \le \|D_x f^n(v)\| \le C e^{(\lambda_i + \varepsilon)n} \quad \text{for all } i = 1, 2, \ldots, \ell$$

for all $x \in \Lambda_\varepsilon$, $v \in E_i(x)$, and $n \ge 1$. Without loss of generality, one can assume that $C = 1$ by considering an equivalent norm.

3.2.3. *Statements of results.* Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold M, and let μ be a hyperbolic ergodic f-invariant probability measure with positive entropy. Suppose that (f, μ) has only two Lyapunov exponents $\lambda_u(\mu) > 0 > \lambda_s(\mu)$. Ledrappier, Young [23] and Barreira, Pesin, Schmeling [7] proved that

$$\mathrm{Dim}\mu = \frac{h_\mu(f)}{\lambda_u(\mu)} - \frac{h_\mu(f)}{\lambda_s(\mu)}, \tag{3.7}$$

where Dim denotes either $\dim_H$ or $\underline{\dim}_B$ or $\overline{\dim}_B$. Our strategy used to prove the dimension approximation in this setting is as follows. It follows from Theorem 3.3 that $h_\mu(f)$ can be approximated by the topological entropies of a sequence of horseshoes $\{\Lambda_\varepsilon\}_{\varepsilon > 0}$. Using well-established properties of dimension theory in uniform hyperbolic systems, one can show that

$$\mathrm{Dim}(\Lambda_\varepsilon \cap W_\beta^i(f, x)) \approx \frac{h_\mu(f)}{|\lambda_i(\mu)|}$$

for $i = u, s$ and every $x \in \Lambda$. Burns and Wilkinson [8] prove that the holonomy maps of the stable and unstable foliations for (f, Λ_ε) are Lipschitz continuous. Consequently, one

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can show that

$$\begin{aligned}\dim_H(\Lambda_\varepsilon\cap W^u_\beta(f,x))&+\dim_H(\Lambda_\varepsilon\cap W^s_\beta(f,x))\\&\le \mathrm{Dim}\Lambda_\varepsilon\\&\le \overline{\dim}_B(\Lambda_\varepsilon\cap W^u_\beta(f,x))+\overline{\dim}_B(\Lambda_\varepsilon\cap W^s_\beta(f,x))\end{aligned}$$

for every $x\in\Lambda_\varepsilon$. Hence, $\mathrm{Dim}\mu$ is approximately equal to $\mathrm{Dim}\Lambda_\varepsilon$. The detailed proofs will be given in the next section.

THEOREM B. *Let $f: M\to M$ be a $C^{1+\alpha}$ diffeomorphism, and μ be an f-invariant ergodic hyperbolic measure with $h_\mu(f)>0$. Assume that (f,μ) has only two Lyapunov exponents $\lambda_u(\mu)>0>\lambda_s(\mu)$. For each $\varepsilon>0$, there exists a horseshoe Λ_ε such that*

$$|\mathrm{Dim}\ \Lambda_\varepsilon-\mathrm{Dim}\ \mu|<\varepsilon,$$

where Dim *denotes either* $\dim_H$ *or* $\underline{\dim}_B$ *or* $\overline{\dim}_B$.

In [41], the authors relaxed the smoothness of Theorem 3.3 to C^1 under the additional condition that Oseledec's splitting $E^u\oplus E^s$ of (f,μ) is dominated. In this setting, one does not have Lipschitz continuity of the holonomy map in general. However, using Palis and Viana's method [31], one can show that for every $\gamma\in(0,1)$, there is some $D_\gamma>0$ such that the holonomy maps of the stable and unstable foliations for the hyperbolic dynamical system (f,Λ_ε) (see Lemma 4.2) are (D_γ,γ)-Hölder continuous. Since γ is arbitrary, using the ideas in [43], one can prove the following theorem.

THEOREM C. *Let $f: M\to M$ be a C^1 diffeomorphism, and let μ be an f-invariant ergodic hyperbolic measure with $h_\mu(f)>0$. Assume that (f,μ) has only two Lyapunov exponents $\lambda_u(\mu)>0>\lambda_s(\mu)$ and the corresponding Oseledec's splitting $E^u\oplus E^s$ is dominated. For each $\varepsilon>0$, there exists a horseshoe Λ_ε such that*

$$|\mathrm{Dim}\Lambda_\varepsilon-\mathrm{Dim}\mu|<\varepsilon,$$

where Dim *denotes either* $\dim_H$ *or* $\underline{\dim}_B$ *or* $\overline{\dim}_B$.

4. *Proofs*

In this section, we provide the proof of the main results presented in the previous section.

4.1. *Proof of Proposition 3.1.* Given an f-invariant ergodic measure μ, let $P(t):=P_\mu(f|_\Lambda,\Phi_f(t))$, it is easy to see that the function $t\mapsto P(t)$ is continuous and strictly decreasing on the interval $[0,m_0]$. It follows from equation (2.3) that $P(0)=h_\mu(f)\ge 0$, and $P(m_0)\le 0$ by Margulis-Ruelle's inequality. Consequently, there exists a unique root s_μ of the equation $P_\mu(f|_\Lambda,\Phi_f(t))=0$.

If $h_\mu(f)=0$, it is easy to see that $h_\mu(f)=s_\mu=0$. Hence, $\dim_L\mu=s_\mu$.

If $0<h_\mu(f)<\lambda_{m_0}(\mu)$, then $P(0)>0$ and $P(1)<0$. This implies that $s_\mu\in(0,1)$ and $0=P(s_\mu)=h_\mu(f)-s_\mu\lambda_{m_0}(\mu)$. As a consequence, we have that

$$s_\mu=\dim_L\mu=\frac{h_\mu(f)}{\lambda_{m_0}(\mu)}.$$

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If $h_\mu(f) \geq \lambda_{m_0}(\mu)$, note that

$$\begin{aligned} 0 &= h_\mu(f) + \mathcal{L}_*(\Phi_f(s_\mu), \mu) \\ &= h_\mu(f) - \sum_{i=m_0-[s_\mu]+1}^{m_0} \lambda_i(\mu) - (s_\mu - [s_\mu])\lambda_{m_0-[s_\mu]}(\mu). \end{aligned}$$

Hence,

$$s_\mu = [s_\mu] + \frac{h_\mu(f) - \sum_{i=m_0-[s_\mu]+1}^{m_0} \lambda_i(\mu)}{\lambda_{m_0-[s_\mu]}(\mu)}.$$

However, since $t \mapsto P(t)$ is strictly decreasing in t, we have that

$$[s_\mu] = \max\{i : \lambda_{m_0}(\mu) + \cdots + \lambda_{m_0-i+1}(\mu) \leq h_\mu(f)\}.$$

This yields that

$$s_\mu = \dim_{\mathrm{L}} \mu.$$

This completes the proof of the proposition.

4.2. *Proof of Theorem A*. By [11, Theorem 5.1], for each f-invariant ergodic measure μ with positive entropy and for each $\varepsilon > 0$, there exists an f-invariant compact subset $\Lambda_\varepsilon \subset \Lambda$ such that the following statements hold:

(i) $h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) \geq h_\mu(f) - \varepsilon$;

(ii) there is a continuous invariant splitting $T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_\ell(x)$ over Λ_ε and a constant $C > 0$ so that

$$C^{-1}\exp(n(\lambda_j(\mu) - \varepsilon)) \leq \|D_x f^n(u)\| \leq C\exp(n(\lambda_j(\mu) + \varepsilon))$$

for any unit vector $u \in E_j(x)$, where $\lambda_1(\mu) < \cdots < \lambda_\ell(\mu)$ are distinct Lyapunov exponents of f with respect to the measure μ.

By modifying the arguments in [11, Theorem 5.1], one may improve the estimate in statement (i) as follows:

(i)′ $h_\mu(f) + \varepsilon \geq h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) \geq h_\mu(f) - \varepsilon$.

Since Λ_ε is a repeller of f, one can choose an f-invariant ergodic measure μ_ε on Λ_ε so that $h_{\mu_\varepsilon}(f) = h_{\mathrm{top}}(f|_{\Lambda_\varepsilon})$ yields that

$$\begin{aligned} P_{\mathrm{top}}(f|_{\Lambda_\varepsilon}, \Phi_f(t)) &\geq h_{\mu_\varepsilon}(f) + \mathcal{L}_*(\Phi_f(t), \mu_\varepsilon) \\ &\geq h_\mu(f) + \mathcal{L}_*(\Phi_f(t), \mu) - (t+1)\varepsilon \qquad \text{(by statement (ii))} \\ &\geq P_\mu(f|_\Lambda, \Phi_f(t)) - (m_0+1)\varepsilon. \end{aligned}$$

However, since f is expanding, by the variational principle, there exists an f-invariant ergodic measure $\widetilde{\mu}_\varepsilon$ on Λ_ε so that

$$\begin{aligned} P_{\mathrm{top}}(f|_{\Lambda_\varepsilon}, \Phi_f(t)) &= h_{\widetilde{\mu}_\varepsilon}(f) + \mathcal{L}_*(\Phi_f(t), \widetilde{\mu}_\varepsilon) \\ &\leq h_\mu(f) + \mathcal{L}_*(\Phi_f(t), \mu) + (t+1)\varepsilon \\ &\leq P_\mu(f|_\Lambda, \Phi_f(t)) + (m_0+1)\varepsilon. \end{aligned}$$

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Hence,

$$|P_{\mathrm{top}}(f|_{\Lambda_\varepsilon}, \Phi_f(t)) - P_\mu(f|_\Lambda, \Phi_f(t))| \leq (m_0+1)\varepsilon.$$

By [11, Theorem 4.1], the Carathéodory singular dimension $\dim_C \Lambda_\varepsilon$ of Λ_ε is given by the unique root of the following equation:

$$P_{\mathrm{top}}(f|_{\Lambda_\varepsilon}, \Phi_f(t)) = 0.$$

This, together with Proposition 3.1, yields that

$$\begin{aligned} K|\dim_C \Lambda_\varepsilon - \dim_{\mathrm{L}} \mu| &\leq |P_\mu(f|_\Lambda, \Phi_f(\dim_C \Lambda_\varepsilon)) - P_\mu(f|_\Lambda, \Phi_f(\dim_{\mathrm{L}} \mu))| \\ &= |P_\mu(f|_\Lambda, \Phi_f(\dim_C \Lambda_\varepsilon)) - P_{\mathrm{top}}(f|_{\Lambda_\varepsilon}, \Phi_f(\dim_C \Lambda_\varepsilon))| \\ &\leq (m_0+1)\varepsilon, \end{aligned}$$

where $K = \min_{x\in\Lambda} \log m(D_x f)$ and $m(\cdot)$ denotes the minimum norm of an operator. Consequently, we have that $\dim_C \Lambda_\varepsilon \to \dim_{\mathrm{L}} \mu$ as ε approaches zero.

4.3. *Proof of Proposition 3.2.* Given a subset $Z \subset \Lambda$, since $P_Z(f|_\Lambda, \Phi_f(t))$ is continuous and strictly decreasing in t, let t_Z denote the unique root of the equation $P_Z(f|_\Lambda, \Phi_f(t)) = 0$. For every $t < t_Z$, we have that $P_Z(f|_\Lambda, \Phi_f(t)) > 0$. Fix such a number t, and take $\beta > 0$ so that $P_Z(f|_\Lambda, \Phi_f(t)) - \beta > 0$. Since

$$P_Z(f|_\Lambda, \Phi_f(t)) = \liminf_{r\to 0} P_Z(f|_\Lambda, \Phi_f(t), r),$$

there exists $r_0 > 0$ such that for each $0 < r < r_0$, one has

$$P_Z(f|_\Lambda, \Phi_f(t), r) > P_Z(f|_\Lambda, \Phi_f(t)) - \beta.$$

Fix such a small $r > 0$. By the definition of topological pressure on arbitrary subsets, one has

$$m(Z, \Phi_f(t), P_Z(f|_\Lambda, \Phi_f(t)) - \beta, r) = +\infty.$$

Hence, for each $\xi > 0$, there exists $L \in \mathbb{N}$ so that for any $N > L$, we have that

$$\begin{aligned} &\exp(-N(P_Z(f|_\Lambda, \Phi_f(t)) - \beta)) \inf\bigg\{\sum_i \exp\Big(\sup_{y\in B_{n_i}(x_i,r)} -\varphi^t(y, f^{n_i})\Big)\bigg\} \\ &\geq \inf\bigg\{\sum_i \exp(-(P_Z(f|_\Lambda, \Phi_f(t)) - \beta)n_i + \sup_{y\in B_{n_i}(x_i,r)} -\varphi^t(y, f^{n_i}))\bigg\} > \xi, \end{aligned}$$

where the infimum is taken over all collections $\{B_{n_i}(x_i, r)\}$ of Bowen's balls with $n_i \geq N$, which covers Z. This yields that

$$\inf\bigg\{\sum_i \exp\Big(\sup_{y\in B_{n_i}(x_i,r)} -\varphi^t(y, f^{n_i})\Big)\bigg\} > \xi \exp(N(P_Z(f|_\Lambda, \Phi_f(t)) - \beta)).$$

Letting $N \to \infty$, we have that

$$m(Z, t, r) = +\infty.$$

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Hence,

$$\dim_{C,r} Z \geq t$$

for all $0 < r < r_0$. Consequently, since $t < t_Z$ is arbitrary, we have that

$$\dim_C Z \geq t_Z. \tag{4.1}$$

However, for each $t > t_Z$, one has that $P_Z(f|_\Lambda, \Phi_f(t)) < 0$. Fix such a number t, and take $\widetilde{\beta} > 0$ so that $P_Z(f|_\Lambda, \Phi_f(t)) + \widetilde{\beta} < 0$. By the definition of topological pressure on arbitrary subsets, for any $R > 0$, there exists $0 < r < R$ such that

$$P_Z(f|_\Lambda, \Phi_f(t), r) < P_Z(f|_\Lambda, \Phi_f(t)) + \widetilde{\beta}.$$

For such a small $r > 0$, one has

$$m(Z, \Phi_f(t), P_Z(f|_\Lambda, \Phi_f(t)) + \widetilde{\beta}, r) = 0.$$

Hence, for each small $\widetilde{\xi} > 0$, there exists $\widetilde{L} \in \mathbb{N}$ so that for any $N > \widetilde{L}$, we have that

$$\exp(-N(P_Z(f|_\Lambda, \Phi_f(t)) + \widetilde{\beta})) \inf \bigg\{ \sum_i \exp \bigg(\sup_{y \in B_{n_i}(x_i,r)} -\varphi^t(y, f^{n_i}) \bigg) \bigg\}$$
$$\leq \inf \bigg\{ \sum_i \exp \bigg(- (P_Z(f|_\Lambda, \Phi_f(t)) + \widetilde{\beta}) n_i + \sup_{y \in B_{n_i}(x_i,r)} -\varphi^t(y, f^{n_i}) \bigg) \bigg\} \leq \widetilde{\xi},$$

where the infimum is taken over all collections $\{B_{n_i}(x_i, r)\}$ of Bowen's balls with $n_i \geq N$, which covers Z. This yields that

$$\inf \bigg\{ \sum_i \exp \bigg(\sup_{y \in B_{n_i}(x_i,r)} -\varphi^t(y, f^{n_i}) \bigg) \bigg\} \leq \widetilde{\xi} \exp(N(P_Z(f|_\Lambda, \Phi_f(t)) + \widetilde{\beta})).$$

Letting $N \to \infty$, one has

$$m(Z, t, r) = 0.$$

Consequently, for such $r > 0$, one has

$$\dim_{C,r} Z \leq t.$$

Hence, we have that

$$\dim_C Z = \liminf_{r \to 0} \dim_{C,r} Z \leq t_Z. \tag{4.2}$$

It follows from equations (4.1) and (4.2) that

$$\dim_C Z = t_Z.$$

To show the second statement, for a given f-invariant ergodic measure μ supported on Λ, and a subset $Z \subset \Lambda$ with $\mu(Z) = 1$, we have that

$$P_Z(f|_\Lambda, \Phi_f(t)) \geq P_\mu(f|_\Lambda, \Phi_f(t)).$$

By item (1) of Theorem A and the first statement, one has

$$\dim_C Z \geq \dim_L \mu.$$

By the definition of Carathéodory singular dimension of arbitrary subsets, one has

$$\dim_{C,r} Z \geq \dim_L \mu$$

for all sufficiently small $r > 0$. Consequently, we have that

$$\dim_C \mu = \liminf_{r\to 0} \dim_{C,r} \mu = \liminf_{r\to 0} \inf\{\dim_{C,r} Z : \mu(Z) = 1\} \geq \dim_L \mu.$$

To prove that $\dim_C \mu = \dim_L \mu$, we assume that $\dim_C \mu > \widetilde{t} > \dim_L \mu$. By the first statement in Theorem A, we have that

$$P_\mu(f|_\Lambda, \Phi(\widetilde{t})) < 0.$$

By the definition of measure theoretic pressure, for each $n \in \mathbb{N}$, there exists $0 < r_n < 1/n$ so that

$$\inf\{P_Z(f|_\Lambda, \Phi(\widetilde{t}), r_n) : \mu(Z) = 1\} < 0.$$

Hence, there exists a subset $Z_n \subset \Lambda$ with $\mu(Z_n) = 1$ so that

$$P_{Z_n}(f|_\Lambda, \Phi(\widetilde{t}), r_n) < 0.$$

Put $\widetilde{Z} := \bigcap_{n\geq 1} Z_n$, then $\mu(\widetilde{Z}) = 1$ and

$$\begin{aligned} P_{\widetilde{Z}}(f|_\Lambda, \Phi(\widetilde{t})) &= \liminf_{r\to 0} P_{\widetilde{Z}}(f|_\Lambda, \Phi(\widetilde{t}), r) \\ &\leq \liminf_{n\to\infty} P_{Z_n}(f|_\Lambda, \Phi(\widetilde{t}), r_n) \leq 0. \end{aligned}$$

It follows from the first statement and the definition of Carathéodory singular dimension of μ that

$$\dim_C \mu = \liminf_{r\to 0} \dim_{C,r} \mu \leq \liminf_{r\to 0} \dim_{C,r} \widetilde{Z} = \dim_C \widetilde{Z} \leq \widetilde{t},$$

which yields a contraction. Hence, we have that $\dim_C \mu = \dim_L \mu$.

4.4. *Proof of Theorem B.* Ledrappier, Young [23] and Barreira, Pesin, Schmeling [7] proved that

$$\mathrm{Dim}\mu = \frac{h_\mu(f)}{\lambda_u(\mu)} - \frac{h_\mu(f)}{\lambda_s(\mu)},$$

where Dim denotes either $\dim_H$ or $\underline{\dim}_B$ or $\overline{\dim}_B$. Fix a small number $\varepsilon > 0$. By Theorem 3.3, there exists a horseshoe Λ_ε such that:

(i) $|h_{\mathrm{top}}(f, \Lambda_\varepsilon) - h_\mu(f)| < \varepsilon$;

(ii) there exists a dominated splitting $T_{\Lambda_\varepsilon} M = E^u \oplus E^s$ with $\dim E^i = d_i$ $(i = u, s)$, and for each $x \in \Lambda_\varepsilon$, every $n \geq 1$, and each vector $v \in E^i(x) (i = s, u)$,

$$e^{(\lambda_i(\mu)-\varepsilon)n} < \|D_x f^n(v)\| < e^{(\lambda_i(\mu)+\varepsilon)n}.$$

The above estimation can be proven in similar fashion as Remark 3.2.

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Fixed any $k \in \mathbb{N}$ and denote $F = f^{2^k}$. Since Λ_ε is a locally maximal hyperbolic set for f, Λ_ε is also a locally maximal hyperbolic set for F. Notice that

$$W^u_\beta(F, x) \cap \Lambda_\varepsilon = W^u_\beta(f, x) \cap \Lambda_\varepsilon \quad \text{and} \quad W^s_\beta(F, x) \cap \Lambda_\varepsilon = W^s_\beta(f, x) \cap \Lambda_\varepsilon.$$

Let $\|\cdot\|$ and $m(\cdot)$ denote the maximal and minimal norm of an operator. For every $x \in \Lambda_\varepsilon$, Barreira [4] proved that

$$\underline{t}^k_u \leq \dim_H(\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \overline{\dim}_B(\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \bar{t}^k_u,$$

where $\underline{t}^k_u, \bar{t}^k_u$ are the unique solutions of

$$P_{\text{top}}(F, -t \log \|D_x F|_{E^u(x)}\|) = 0 \quad \text{and} \quad P_{\text{top}}(F, -t \log m(D_x F|_{E^u(x)})) = 0,$$

respectively. Using the same arguments as in the proof of [3, Theorems 6.2 and 6.3], one can prove that the sequences $\{\underline{t}^k_u\}$ and $\{\bar{t}^k_u\}$ are monotone. Furthermore, setting

$$\underline{t}_u := \lim_{k\to\infty} \underline{t}^k_u \quad \text{and} \quad \bar{t}_u := \lim_{k\to\infty} \bar{t}^k_u,$$

one can show that $\underline{t}_u, \bar{t}_u$ are the unique solutions of the following equations:

$$P_{\text{var}}(f, -t\{\log \|D_x f^n|_{E^u}\|\}) = 0, \quad P_{\text{top}}(f, -t\{\log m(D_x f^n|_{E^u})\}) = 0,$$

respectively.

Consequently, we have that

$$\underline{t}_u \leq \dim_H(\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \underline{\dim}_B(\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \overline{\dim}_B(\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \bar{t}_u$$

and

$$\underline{t}_u = \sup\left\{\frac{h_\nu(f)}{\lim_{n\to\infty}\frac{1}{n}\int \log \|D_x f^n|_{E^u}\|\, d\nu} : \nu \in \mathcal{M}_f(\Lambda_\varepsilon)\right\},$$
$$\bar{t}_u = \sup\left\{\frac{h_\nu(f)}{\lim_{n\to\infty}\frac{1}{n}\int \log m(D_x f^n|_{E^u})\, d\nu} : \nu \in \mathcal{M}_f(\Lambda_\varepsilon)\right\}.$$

Combining with items (i) and (ii), one has

$$\frac{h_\mu(f) - \varepsilon}{\lambda_u(\mu) + \varepsilon} \leq \mathrm{Dim}(\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \frac{h_\mu(f) + \varepsilon}{\lambda_u(\mu) - \varepsilon} \tag{4.3}$$

for every $x \in \Lambda_\varepsilon$, where Dim denotes either $\dim_H$ or $\underline{\dim}_B$ or $\overline{\dim}_B$. One can show in a similar fashion that

$$-\frac{h_\mu(f) - \varepsilon}{\lambda_s(\mu) - \varepsilon} \leq \mathrm{Dim}(\Lambda_\varepsilon \cap W^s_\beta(f, x)) \leq -\frac{h_\mu(f) + \varepsilon}{\lambda_s(\mu) + \varepsilon} \tag{4.4}$$

for every $x \in \Lambda_\varepsilon$.

LEMMA 4.1. *The holonomy maps of the stable and unstable foliations for (f, Λ_ε) are Lipschitz continuous.*

Proof. Fix a positive integer N, put $F := f^N$ and $\Lambda = \Lambda_\varepsilon$. Since Λ is a locally maximal hyperbolic set for f, so is Λ for F. Notice that

$$W^u_\beta(F, x) \cap \Lambda = W^u_\beta(f, x) \cap \Lambda \quad \text{and} \quad W^s_\beta(F, x) \cap \Lambda = W^s_\beta(f, x) \cap \Lambda. \tag{4.5}$$

Let

$$a_F = \|DF^{-1}|_{E^u}\|, \quad b_F = \|DF|_{E^s}\|, \quad c_F = \|DF|_{E^u}\|, \quad d_F = \|DF^{-1}|_{E^s}\|.$$

It follows from item (ii) that

$$1 < \frac{\|D_x F|_{E^i(x)}\|}{m(D_x F|_{E^i(x)})} < e^{2\varepsilon N} \quad \text{for every } x \in \Lambda \text{ and } i \in \{s, u\}.$$

Hence,

$$a_F b_F c_F = \frac{\|DF|_{E^s}\| \cdot \|DF|_{E^u}\|}{m(DF|_{E^u})} < e^{(\lambda_s(\mu)+3\varepsilon)N} < 1,$$

provided that $\varepsilon > 0$ is sufficiently small such that $\lambda_s(\mu) + 3\varepsilon < 0$. By [8, Theorem 0.2], we have that the holonomy map of the stable foliation for (F, Λ) is C^1. Similarly, note that

$$a_F b_F d_F = \frac{\|DF|_{E^s}\|}{m(DF||_{E^s})m(DF||_{E^u})} < e^{(-\lambda_u(\mu)+3\varepsilon)N} < 1$$

provided that $\varepsilon > 0$ is sufficiently small such that $\lambda_u(\mu) - 3\varepsilon > 0$. It follows from [8, Theorem 0.2] that the holonomy map of the unstable foliation for (F, Λ) is C^1. Combining with equation (4.5), one has the holonomy maps of the stable and unstable foliations for (f, Λ) are Lipschitz continuous. □

By Lemma 4.1 and the fact f is topologically mixing on Λ_ε, one has $\dim_H(\Lambda_\varepsilon \cap W^u_\beta(f, x))$, $\underline{\dim}_B(\Lambda_\varepsilon \cap W^u_\beta(f, x))$, and $\overline{\dim}_B(\Lambda_\varepsilon \cap W^u_\beta(f, x))$ are independent of β and x (see the proof of [5, Theorem 4.3.2] for more details). Let

$$A_{\varepsilon,x} = (\Lambda_\varepsilon \cap W^u_\beta(f, x)) \times (\Lambda_\varepsilon \cap W^s_\beta(f, x)).$$

By the properties of dimension (e.g. see [13, 33]), one has

$$\begin{aligned} &\dim_H(\Lambda_\varepsilon \cap W^u_\beta(f, x)) + \dim_H(\Lambda_\varepsilon \cap W^s_\beta(f, x)) \\ &\quad \le \dim_H A_{\varepsilon,x} \\ &\quad \le \underline{\dim}_B A_{\varepsilon,x} \\ &\quad \le \overline{\dim}_B A_{\varepsilon,x} \\ &\quad \le \overline{\dim}_B(\Lambda_\varepsilon \cap W^u_\beta(f, x)) + \overline{\dim}_B(\Lambda_\varepsilon \cap W^s_\beta(f, x)). \end{aligned} \tag{4.6}$$

Let $\Phi : A_{\varepsilon,x} \to \Lambda_\varepsilon$ be given by

$$\Phi(y, z) = W^s_\beta(f, y) \cap W^u_\beta(f, z).$$

It is easy to see Φ is a homeomorphism onto a neighborhood V_x of x in Λ_ε. It follows from Lemma 4.1 that Φ and Φ^{-1} are Lipschitz continuous (see [5, Theorem 4.3.2] for detailed proofs). It follows from Corollary 2.9 that

$$\operatorname{Dim} V_x = \operatorname{Dim} A_{\varepsilon,x},$$

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where Dim denotes either $\dim_H$ or $\underline{\dim}_B$ or $\overline{\dim}_B$. Since $\{V_x : x \in \Lambda_\varepsilon\}$ is an open cover of Λ_ε, one can choose a finite open cover $\{V_{x_1}, V_{x_2}, \ldots, V_{x_k}\}$ of Λ_ε. It follows from equation (4.6) that

$$\begin{aligned}
&\dim_H (\Lambda_\varepsilon \cap W^u_\beta(f, x)) + \dim_H (\Lambda_\varepsilon \cap W^s_\beta(f, x)) \\
&\quad \le \dim_H \Lambda_\varepsilon = \max_{1\le i\le k} \dim_H V_{x_i} \\
&\quad \le \overline{\dim}_B \Lambda_\varepsilon = \max_{1\le i\le k} \overline{\dim}_B V_{x_i} \\
&\quad \le \overline{\dim}_B (\Lambda_\varepsilon \cap W^u_\beta(f, x)) + \overline{\dim}_B (\Lambda_\varepsilon \cap W^s_\beta(f, x)),
\end{aligned}$$

for every $x \in \Lambda_\varepsilon$. Combining equations (4.3) and (4.4), we obtain

$$\lim_{\varepsilon\to 0} \mathrm{Dim}\, \Lambda_\varepsilon = \frac{h_\mu(f)}{\lambda_u(\mu)} - \frac{h_\mu(f)}{\lambda_s(\mu)} = \mathrm{Dim}\, \mu,$$

where Dim denotes either $\dim_H$ or $\underline{\dim}_B$ or $\overline{\dim}_B$. This completes the proof of Theorem B.

4.5. *Proof of Theorem C.* For every pair (f, μ) satisfying the assumptions, Wang and Cao [**40**, Corollary 1] proved that

$$\dim_H \mu = \frac{h_\mu(f)}{\lambda_u(\mu)} - \frac{h_\mu(f)}{\lambda_s(\mu)}.$$

Fix a small number $\varepsilon > 0$. Wang, Cao, and Zou [**41**, Theorem 1.1] proved that there exists a horseshoe Λ_ε such that:

(i) $|h_{\mathrm{top}}(f, \Lambda_\varepsilon) - h_\mu(f)| < \varepsilon$;

(ii) there exists a dominated splitting $T_{\Lambda_\varepsilon} M = E^u \oplus E^s$ with $\dim E^i = d_i$ $(i = u, s)$, and for each $x \in \Lambda_\varepsilon$, every $n \ge 1$, and each vector $v \in E^i(x) (i = s, u)$,

$$e^{(\lambda_i(\mu)-\varepsilon)n} < \|D_x f^n(v)\| < e^{(\lambda_i(\mu)+\varepsilon)n}.$$

The above estimation can be obtained by item (v) of [**41**, Theorem 1.1] and Remark 3.2.

Fix a positive integer $k \in \mathbb{N}$, denote $F = f^{2^k}$. Since Λ_ε is a locally maximal hyperbolic set for *f*, so is Λ_ε for *F*. Notice that

$$W^u_\beta(F, x) \cap \Lambda_\varepsilon = W^u_\beta(f, x) \cap \Lambda_\varepsilon \quad \text{and} \quad W^s_\beta(F, x) \cap \Lambda_\varepsilon = W^s_\beta(f, x) \cap \Lambda_\varepsilon.$$

For every $x \in \Lambda_\varepsilon$, it follows from [**43**, Lemmas 3.5 and 3.6] that

$$\underline{t}^k_u \le \dim_H (\Lambda_\varepsilon \cap W^u_\beta(f, x)) \le \overline{\dim}_B (\Lambda_\varepsilon \cap W^u_\beta(f, x)) \le \overline{t}^k_u,$$

where $\underline{t}^k_u, \overline{t}^k_u$ are the unique roots of

$$P_{\mathrm{top}}(F, -t \log \|D_x F|_{E^u(x)}\|) = 0, \qquad P_{\mathrm{top}}(F, -t \log m(D_x F|_{E^u(x)})) = 0,$$

respectively. Using the same arguments as in the proof of [**3**, Theorems 6.2 and 6.3], one can prove that the sequences $\{\underline{t}^k_u\}$ and $\{\overline{t}^k_u\}$ are monotone. Set

$$\underline{t}_u := \lim_{k\to\infty} \underline{t}^k_u \quad \text{and} \quad \overline{t}_u := \lim_{k\to\infty} \overline{t}^k_u,$$

where $\underline{t}_u, \overline{t}_u$ are the unique solutions of the following equations:

$$P_{\text{var}}(f, -t\{\log \|D_x f^n|_{E^u(x)}\|\}) = 0, \quad P_{\text{top}}(f, -t\{\log m(D_x f^n|_{E^u(x)})\}) = 0,$$

respectively. Hence, we have that

$$\underline{t}_u \leq \dim_H (\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \underline{\dim}_B (\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \overline{\dim}_B (\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \overline{t}_u.$$

Since

$$\underline{t}_u = \sup \left\{ \frac{h_\nu(f)}{\lim_{n\to\infty} \frac{1}{n} \int \log \|D_x f^n|_{E^u(x)}\| \, d\nu} : \nu \in \mathcal{M}_f(\Lambda_\varepsilon) \right\}$$

and

$$\overline{t}_u = \sup \left\{ \frac{h_\nu(f)}{\lim_{n\to\infty} \frac{1}{n} \int \log m(D_x f^n|_{E^u(x)}) \, d\nu} : \nu \in \mathcal{M}_f(\Lambda_\varepsilon) \right\},$$

using items (i) and (ii), one can show that

$$\frac{h_\mu(f) - \varepsilon}{\lambda_u(\mu) + \varepsilon} \leq \text{Dim} \, (\Lambda_\varepsilon \cap W^u_\beta(f, x)) \leq \frac{h_\mu(f) + \varepsilon}{\lambda_u(\mu) - \varepsilon} \tag{4.7}$$

for every $x \in \Lambda_\varepsilon$, where Dim denotes either $\dim_H$ or $\underline{\dim}_B$ or $\overline{\dim}_B$. Similarly, we obtain that

$$-\frac{h_\mu(f) - \varepsilon}{\lambda_s(\mu) - \varepsilon} \leq \text{Dim} \, (\Lambda_\varepsilon \cap W^s_\beta(f, x)) \leq -\frac{h_\mu(f) + \varepsilon}{\lambda_s(\mu) + \varepsilon} \tag{4.8}$$

for every $x \in \Lambda_\varepsilon$.

LEMMA 4.2. *Let Λ be a locally maximal hyperbolic set of a C^1 diffeomorphism such that f is topologically mixing on Λ. Assume that the diffeomorphism $f|_\Lambda$ possesses a $\{\lambda_u(\mu), \lambda_s(\mu)\}$-dominated splitting $T_\Lambda M = E^u \oplus E^s$ with $E^u \succeq E^s$ and $\lambda_u(\mu) > 0 > \lambda_s(\mu)$. Then for every $\gamma \in (0, 1)$, there exists $D_\gamma > 0$ such that the holonomy maps of the stable and unstable foliations for f are (D_γ, γ)-Hölder continuous.*

Proof. Since the diffeomorphism $f|_\Lambda$ possesses a $\{\lambda_u(\mu), \lambda_s(\mu)\}$-dominated splitting $T_\Lambda M = E^u \oplus E^s$ with $E^u \succeq E^s$ and $\lambda_u(\mu) > 0 > \lambda_s(\mu)$, there exists a constant $C > 0$ such that for every $x \in \Lambda$, $n \in \mathbb{N}$ and each unit vector $v \in E^i(x)$ $(i = u, s)$,

$$C^{-1} e^{n(\lambda_i(\mu) - \varepsilon)} \leq \|D_x f^n(v)\| \leq C e^{n(\lambda_i(\mu) + \varepsilon)}.$$

As illustrated in Remark 3.2, for simplicity, we assume that $C = 1$ in the rest of the proof. Fix a positive integer N and put $F := f^N$. This implies that for every $x \in \Lambda$,

$$1 \leq \frac{\|D_x F|_{E^u}\|}{m(D_x F|_{E^u})} \leq e^{2N\varepsilon}, \quad 1 \leq \frac{\|D_x F|_{E^s}\|}{m(D_x F|_{E^s})} \leq e^{2N\varepsilon}.$$

Notice that Λ is also a locally maximal hyperbolic set for F and

$$W^u_\beta(F, x) \cap \Lambda = W^u_\beta(f, x) \cap \Lambda \quad \text{and} \quad W^s_\beta(F, x) \cap \Lambda = W^s_\beta(f, x) \cap \Lambda$$

for every $x \in \Lambda$.

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Let π^s and π^u be the holonomy maps of stable and unstable foliations for f, that is, for any $x \in \Lambda$, $x' \in W^s_\beta(f, x)$, and $x'' \in W^u_\beta(f, x)$ close to x,

$$\pi^s : W^u_\beta(f, x) \cap \Lambda \to W^u_\beta(f, x') \cap \Lambda \text{ with } \pi^s(y) = W^s_\beta(f, y) \cap W^u_\beta(f, x')$$

and

$$\pi^u : W^s_\beta(f, x) \cap \Lambda \to W^s_\beta(f, x'') \cap \Lambda \text{ with } \pi^u(z) = W^u_\beta(f, z) \cap W^s_\beta(f, x'').$$

Therefore, π^s is also a map from $W^u_\beta(F, x) \cap \Lambda$ to $W^u_\beta(F, x') \cap \Lambda$ and π^u is also a map from $W^s_\beta(F, x) \cap \Lambda$ to $W^s_\beta(F, x'') \cap \Lambda$.

Let $U \subset M$ be an open subset such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, and $\mathcal{U} \subset \mathrm{Diff}^1(M)$ be a neighborhood of f such that for each $g \in \mathcal{U}$, $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is a locally maximal hyperbolic set for g and there is a homeomorphism $h_g : \Lambda \to \Lambda_g$ that satisfies $g \circ h_g = h_g \circ f$, with h_g C^0-close to identity if g is C^1-close to f. For $g \in \mathcal{U}$, let $T_{\Lambda_g} M = E^u_g \oplus E^s_g$ denote the hyperbolic splitting over Λ_g. For $i \in \{u, s\}$, $\{W^i_\beta(g, z) : z \in \Lambda_g\}$ is continuous on g in the following sense: there is $\{\theta^i_{g,x} : x \in \Lambda\}$, where $\theta^i_{g,x} : W^i_\beta(f, x) \to W^i_\beta(g, h_g(x))$ is a C^1 diffeomorphism with $\theta^i_{g,x}(x) = h_g(x)$, such that if g is C^1-close to f then, for all $x \in \Lambda$, $\theta^i_{g,x}$ is uniformly C^1-close to the inclusion of $W^i_\beta(f, x)$ in M.

For any $\gamma \in (0, 1)$, let $\mathcal{U}^F_\gamma$ be a small C^1 neighborhood of F (recall $F = f^N$). Taking $G \in \mathcal{U}^F_\gamma \cap \mathrm{Diff}^2(M)$ such that for every $x \in \Lambda_G$ (here Λ_G is a locally maximal hyperbolic set for G), $n \in \mathbb{N}$, and $i = u, s$,

$$e^{nN(\lambda_i(\mu)-2\varepsilon)} \le \|D_x G^n|_{E^i(x)}\| \le e^{nN(\lambda_i(\mu)+2\varepsilon)}. \tag{4.9}$$

CLAIM 4.3. *The following properties hold:*

(a) $h_G|_{W^u_\beta(F,x)\cap\Lambda}$ *and* $(h_G|_{W^u_\beta(F,x)\cap\Lambda})^{-1}$ *are* (C_γ, γ)*-Hölder continuous for some* $C_\gamma > 0$*;*

(b) *the stable and unstable foliations*

$$\{W^s(G, z) : z \in \Lambda_G\}, \quad \{W^u(G, z) : z \in \Lambda_G\}$$

are C^1 *and invariant for G. Thus the holonomy maps*

$$\pi^s_G : W^u_\beta(G, h_G(x)) \cap \Lambda_G \to W^u_\beta(G, h_G(x')) \cap \Lambda_G \quad \textit{with}$$
$$\pi^s_G(y) = W^s_\beta(G, y) \cap W^u_\beta(G, h_G(x')),$$

and

$$\pi^u_G : W^s_\beta(G, h_G(x)) \cap \Lambda_G \to W^s_\beta(G, h_G(x'')) \cap \Lambda_G \quad \textit{with}$$
$$\pi^u_G(z) = W^u_\beta(G, z) \cap W^s_\beta(G, h_G(x''))$$

are Lipschitz continuous.

Proof. (a) See [43, Claim 3.1].

(b) Since G satisfies equation (4.9), we conclude

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$$\frac{\|DG|_{E^u}\| \cdot \|DG|_{E^s}\|}{m(DG|_{E^u})} \leq e^{4N\varepsilon} e^{N(\lambda_s(\mu)+2\varepsilon)}$$
$$= e^{N(\lambda_s(\mu)+6\varepsilon)}$$
$$\leq 1,$$

provided that $\lambda_s(\mu) + 6\varepsilon < 0$. By [**18**, Theorem 6.3], the stable foliation is C^1. A similar argument shows that the unstable foliation is also C^1. Then the corresponding maps are uniformly C^1 (see [**35**, pp. 540–541] for more details), which implies the desired result. □

We proceed to prove Lemma 4.2. For any $y \in W^u_\beta(F, x) \cap \Lambda$,

$$\begin{aligned} h_G(\pi^s(y)) &= h_G(W^s_\beta(F, y) \cap W^u_\beta(F, x')) \\ &= W^s_\beta(G, h_G(y)) \cap W^u_\beta(G, h_G(x')) \\ &= \pi^s_G(h_G(y)). \end{aligned}$$

For the above γ, by Claim 4.3, there exists $D_\gamma > 0$ such that

$$\pi^s = h_G^{-1} \circ \pi^s_G \circ h_G$$

is (D_γ, γ)-Hölder continuous. Using the same arguments, one can prove $(\pi^s)^{-1}$, π^u, and $(\pi^u)^{-1}$ are also (D_γ, γ)-Hölder continuous. □

By Lemma 4.2 and the fact that f is topologically mixing on Λ_ε, one has $\dim_H (\Lambda_\varepsilon \cap W^u_\beta(f, x))$, $\underline{\dim}_B(\Lambda_\varepsilon \cap W^u_\beta(f, x))$, and $\overline{\dim}_B(\Lambda_\varepsilon \cap W^u_\beta(f, x))$ are independent of β and x (see the proof of [**43**, Lemma 3.4] for more details). Let

$$A_{\varepsilon,x} = (\Lambda_\varepsilon \cap W^u_\beta(f, x)) \times (\Lambda_\varepsilon \cap W^s_\beta(f, x))$$

be a product space. By the properties of dimension (see [**33**, Theorem 6.5] for details), one has

$$\begin{aligned} &\dim_H (\Lambda_\varepsilon \cap W^u_\beta(f, x)) + \dim_H (\Lambda_\varepsilon \cap W^s_\beta(f, x)) \\ &\quad\leq \dim_H A_{\varepsilon,x} \\ &\quad\leq \underline{\dim}_B A_{\varepsilon,x} \\ &\quad\leq \overline{\dim}_B A_{\varepsilon,x} \\ &\quad\leq \overline{\dim}_B (\Lambda_\varepsilon \cap W^u_\beta(f, x)) + \overline{\dim}_B (\Lambda_\varepsilon \cap W^s_\beta(f, x)). \end{aligned} \tag{4.10}$$

Let $\Phi : A_{\varepsilon,x} \to \Lambda_\varepsilon$ be given by

$$\Phi(y, z) = W^s_\beta(f, y) \cap W^u_\beta(f, z).$$

It is easy to see that Φ is a homeomorphism onto a neighborhood V_x of x in Λ_ε. For any $\gamma \in (0, 1)$, by Lemma 4.2, there is $E_\gamma > 0$ such that Φ and Φ^{-1} are (E_γ, γ)-Hölder continuous (see Step 2 in the proof of [**43**, Theorem A] for more details). By Lemma 2.8 and the arbitrariness of γ, one has

$$\operatorname{Dim} V_x = \operatorname{Dim} A_{\varepsilon,x},$$

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where Dim denotes either $\dim_H$ or $\underline{\dim}_B$ or $\overline{\dim}_B$. Since $\{V_x : x \in \Lambda_\varepsilon\}$ is an open cover of Λ_ε, one can choose a finite open cover $\{V_{x_1}, V_{x_2}, \ldots, V_{x_k}\}$ of Λ_ε. It follows from equation (4.10) that

$$\begin{aligned}
&\dim_H (\Lambda_\varepsilon \cap W^u_\beta(f,x)) + \dim_H (\Lambda_\varepsilon \cap W^s_\beta(f,x)) \\
&\quad \le \dim_H \Lambda_\varepsilon = \max_{1\le i\le k} \dim_H V_{x_i} \\
&\quad \le \overline{\dim}_B \Lambda_\varepsilon = \max_{1\le i\le k} \overline{\dim}_B V_{x_i} \\
&\quad \le \overline{\dim}_B (\Lambda_\varepsilon \cap W^u_\beta(f,x)) + \overline{\dim}_B (\Lambda_\varepsilon \cap W^s_\beta(f,x)),
\end{aligned}$$

for every $x \in \Lambda_\varepsilon$. This together with equations (4.7) and (4.8) yields that

$$\lim_{\varepsilon\to 0} \mathrm{Dim}\, \Lambda_\varepsilon = \frac{h_\mu(f)}{\lambda_u(\mu)} - \frac{h_\mu(f)}{\lambda_s(\mu)} = \mathrm{Dim}\, \mu,$$

where Dim denotes either $\dim_H$ or $\underline{\dim}_B$ or $\overline{\dim}_B$. This completes the proof of Theorem C.

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