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# Remarks on Littlewood–Paley Analysis

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*Abstract.* Littlewood–Paley analysis is generalized in this article. We show that the compactness of the Fourier support imposed on the analyzing function can be removed. We also prove that the Littlewood–Paley decomposition of tempered distributions converges under a topology stronger than the weak-star topology, namely, the inductive limit topology. Finally, we construct a multiparameter Littlewood–Paley analysis and obtain the corresponding "renormalization" for the convergence of this multiparameter Littlewood–Paley analysis.

## 1 Introduction

Littlewood–Paley analysis was initially observed by Peetre [17, pp. 51–54]. It is of fundamental importance to function theory. The formulation of Littlewood–Paley analysis involves a function $\varphi$ that satisfies

$$\varphi \in \mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

$$\operatorname{supp} \hat{\varphi} \subset \{\xi : 1/2 \leq |\xi| \leq 2\}, \tag{1.2}$$

$$\sum_{i=-\infty}^{\infty} \hat{\varphi}(2^i \xi) = 1, \quad \text{if } \xi \neq 0, \tag{1.3}$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions (rapidly decaying smooth functions) and $\mathcal{S}'(\mathbb{R}^n)$ is its dual space, which is the space of tempered distributions (Schwartz distributions). Let $\hat{\varphi}$ denote the Fourier transform of $\varphi$. We call $\operatorname{supp}\hat{\varphi}$ the Fourier support of $\varphi$. Let $\langle \cdot\,,\,\cdot \rangle$ be the pairing between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$.

We call $\varphi$ the *analyzing function* if it satisfies (1.3).

For any $i \in \mathbb{Z}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, define $\varphi_i(x) = 2^{ni}\varphi(2^i x)$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let $f * \varphi$ denote the convolution of $f$ and $\varphi$.

The following proposition is the main result of Littlewood–Paley analysis. It guarantees that any tempered distribution $f$ can be represented as a series of smooth functions and, with a sequence of "floating polynomials", the series converges to $f$ in the weak-star topology of $\mathcal{S}'(\mathbb{R}^n)$.

***Proposition 1.1*** *If $f$ is a tempered distribution and $\varphi$ satisfies* (1.1)–(1.3)*, then there exist an integer $N$ and a sequence $P_q(x)$ of polynomials of degrees less than or equal to $N$ such that*

$$f = \lim_{q\to+\infty} \Big\{ \sum_{j=-q}^{0} \varphi_j * f - P_q \Big\} + \sum_{j=1}^{\infty} \varphi_j * f. \tag{1.4}$$

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There are many proofs for the above result. One of the essential components for proving (1.4) is condition (1.2). For instance, Meyer proved the above proposition by using Bernstein's inequality [16]. Bownik and Ho [1] generalized (1.4) by replacing the dilation $2I_{n\times n}$ ($I_{n\times n}$ is the identity matrix in $\mathbb{R}^n$) by an "expansive matrix dilation". The idea behind proving the Littlewood–Paley analysis in [1] is based on the compactness of the Fourier support of the analyzing function. Notice that in [1, 16], the authors prove the convergence of the Littlewood–Paley analysis in $\mathcal{S}'(\mathbb{R}^n)$ (1.4) under the weak-star topology. Recall that the weak-star topology of $\mathcal{S}'(\mathbb{R}^n)$ is the finest topology in $\mathcal{S}'(\mathbb{R}^n)$ such that all mappings $f \to \langle f, \psi\rangle$, $\psi \in \mathcal{S}(\mathbb{R}^n)$, are continuous.

In this article, we present a new proof of the "classical" Littlewood–Paley analysis, on one hand. On the other hand, our results generalize the Littlewood–Paley analysis. We show that (1.2) can be removed and (1.1) can be relaxed. Note that wavelets provide many examples of functions satisfying (1.3), but not (1.1) and (1.2). Moreover, we prove that the expansion (1.4) converges in a topology stronger than the weak-star topology, namely, the inductive limit topology. We recall the definition of the inductive limit topology for $\mathcal{S}'(\mathbb{R}^n)$ in Section 2 and briefly discuss some properties of this topology. In Theorem 3.1 we use a sufficiently smooth analyzing function that only satisfies (1.3) to construct a Littlewood–Paley analysis on the distribution space $\mathcal{S}_\alpha(\mathbb{R}^n)^*$ (the definition of $\mathcal{S}_\alpha(\mathbb{R}^n)^*$ is given in Section 2), while $\alpha$ depends on the smoothness of the analyzing function.

For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (1.2), it is obvious that all moments of $\varphi$ are zero. More precisely, $\varphi$ satisfies

$$\int_{\mathbb{R}^n} x^\lambda \varphi(x)\,dx = 0, \quad \lambda \in \mathbb{N}^n. \tag{1.5}$$

The vanishing moment condition is a remarkable feature satisfied by the analyzing functions used in [1, 16]. One of the novelties in this article is that our idea of proving the convergence of the Littlewood–Paley analysis does not require any "extra" vanishing moment condition.

We clarify the meaning of the "extra" vanishing moment condition. If $\hat{\varphi}$ is continuous at $\xi = 0$ and $\varphi$ satisfies (1.3), then the convergence of the series (1.3) enables us to conclude that for any fixed $\xi \neq 0$, $\lim_{i\to-\infty} \hat{\varphi}(2^i\xi) = 0$. The continuity of $\hat{\varphi}$ at $\xi = 0$ forces $\hat{\varphi}(0) = 0$. Thus, $\varphi$ satisfies the zeroth-order vanishing moment,

$$\int_{\mathbb{R}^n} \varphi(x)\,dx = 0. \tag{1.6}$$

Therefore, the zeroth-order vanishing moment condition (1.6) comes with the identity (1.3). Hence, the "extra" vanishing moment condition is condition (1.5) with $|\lambda| \neq 0$. In our main result, we do not impose any extra vanishing moment conditions on our analyzing function, $\varphi$. Theorem 3.1, and its supporting theorem, Theorem 2.3, only rely on the zeroth-order vanishing moment condition (1.6).

Furthermore, by using our idea, we obtain the other main result of this article, Theorem 5.4. In this theorem, we construct the Littlewood–Paley analysis under a multiparameter setting. That is, the Euclidean space $\mathbb{R}^n$ is endowed with the dilations $(x_1, \dots, x_n) \mapsto (2^{i_1}x_1, \dots, 2^{i_n}x_n)$, $(i_1, \dots, i_n) \in \mathbb{Z}^n$.

Similar to the one parameter setting, the multiparameter Littlewood–Paley analysis provides an important tool on the study of multiparameter function spaces. Thus, there is significant interest in a detailed investigation of the multiparameter Littlewood–Paley analysis. One of the first studies on the multiparameter function spaces was accomplished by Gundy and Stein [12]. They introduced and studied the Hardy spaces on product domains. Chang and Fefferman [2–9] obtained the atomic decomposition for the Hardy spaces on product domains and studied the corresponding function spaces of bounded mean oscillation on product domains.

It is easy to see that the multiparameter Littlewood–Paley analysis cannot be obtained by iterating the one parameter results if the analyzing function is not separable. Here a function $\varphi$ on $\mathbb{R}^2$ is separable if $\varphi(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)$. The non-separable analyzing function arises naturally on the study of function spaces. We provide an example of non-separable analyzing function at the end of this article. Furthermore, a further obstacle is found in the polynomials $P_q$ in (1.4). In the multiparameter setting, it should be replaced by a family of distributions where their Fourier supports are subsets of $\{(\xi_1, \dots, \xi_n) \in \mathbb{R}^n : \prod_{j=1}^{n} \xi_j = 0\}$. Thus, the "renormalization" is not a family of polynomials. Our method overcomes these difficulties by considering the convergence on smooth functions instead of distributions. In addition, our method explicitly constructs the "renormalization" in Theorem 5.4. In order to accomplish this, the analyzing function for the multiparameter Littlewood–Paley analysis requires some extra smoothness compared to the one parameter analyzing function. Moreover, the multiparameter Littlewood–Paley analysis also converges in the inductive limit topology generated by the Schwartz functions.

This paper is organized as follows. We present some definitions and the supporting result Theorem 2.3 of Theorem 3.1 in Section 2. The statement and proof of Theorem 3.1 is given in Section 3. In Section 4, we provide the proof for Theorem 2.3. The multiparameter Littlewood–Paley analysis is presented and constructed in Section 5.

## 2 Preliminary Results

Let $\partial^\gamma \varphi$, $\gamma \in \mathbb{N}^n$ denote the $\gamma$-th partial derivative of $\varphi$.

***Definition 2.1*** Given a fixed $\epsilon > 0$, for any $\alpha > 0$, we denote by $\mathcal{C}_\alpha(\mathbb{R}^n)$ the class of functions $\varphi(x)$ satisfying

$$\|\varphi\|_{\alpha*} = \sup_{0 \le |\gamma| \le [\alpha]} \sup_{x \in \mathbb{R}^n} |(1 + |x|)^{\alpha+n+\epsilon} \partial^\gamma \varphi(x)| < \infty,$$

where $\gamma \in \mathbb{N}^n$ and $[\alpha]$ is the integer part of $\alpha$, and

$$\|\varphi\|_{\alpha**} = \sup_{|\gamma|=[\alpha]} \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y \\ |z| \le |x-y|}} (1 + |x - z|)^{\alpha+n+\epsilon} \frac{|\partial^\gamma \varphi(x) - \partial^\gamma \varphi(y)|}{|x - y|^{\alpha-[\alpha]}} < \infty$$

if $\alpha > [\alpha]$. Let the norm on $\mathcal{C}_\alpha(\mathbb{R}^n)$ be $\|\varphi\|_{\mathcal{C}_\alpha} = \max\{\|\varphi\|_{\alpha*}, \|\varphi\|_{\alpha**}\}$. Let $\mathcal{S}_\alpha(\mathbb{R}^n)$ consist of functions satisfying

$$\|\varphi\|_{\alpha\sharp} = \sup_{0\le|\gamma|\le[\alpha]} \sup_{x\in\mathbb{R}^n} |(1+|x|)^{\alpha+n}\partial^\gamma\varphi(x)| < \infty,$$

where $\gamma \in \mathbb{N}^n$ and

$$\|\varphi\|_{\alpha\sharp\sharp} = \sup_{|\gamma|=[\alpha]} \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y\\ |z|\le|x-y|}} (1+|x-z|)^{\alpha+n} \frac{|\partial^\gamma\varphi(x)-\partial^\gamma\varphi(y)|}{|x-y|^{\alpha-[\alpha]}} < \infty$$

if $|\gamma| = [\alpha]$. Let the norm on $\mathcal{S}_\alpha(\mathbb{R}^n)$ be $\|\varphi\|_{\mathcal{S}_\alpha} = \max\{\|\varphi\|_{\alpha\sharp}, \|\varphi\|_{\alpha\sharp\sharp}\}$.

The function spaces $\mathcal{C}_\alpha(\mathbb{R}^n)$ and $\mathcal{S}_\alpha(\mathbb{R}^n)$ are Banach spaces. We have the continuous embedding $\mathcal{C}_\alpha(\mathbb{R}^n) \hookrightarrow \mathcal{C}_\beta(\mathbb{R}^n)$ if $\alpha \ge \beta$.

Let $\mathcal{S}_\alpha(\mathbb{R}^n)^*$ and $\mathcal{C}_\alpha(\mathbb{R}^n)^*$ denote the dual space of $\mathcal{S}_\alpha(\mathbb{R}^n)$ and $\mathcal{C}_\alpha(\mathbb{R}^n)$, respectively. It is easy to see that we have

$$\mathcal{S}'(\mathbb{R}^n) = \bigcup_{\alpha>0} \mathcal{S}_\alpha(\mathbb{R}^n)^* = \bigcup_{\alpha>0} \mathcal{C}_\alpha(\mathbb{R}^n)^*. \tag{2.1}$$

We define the *order* of $f \in \mathcal{S}'(\mathbb{R}^n)$, $\omega$, to be the infimum of those $\alpha$ that satisfy $f \in \mathcal{S}_\alpha(\mathbb{R}^n)^*$. Thus, we have $f \in \mathcal{C}_\alpha(\mathbb{R}^n)^*$ if $\alpha > \omega$.

With the decomposition (2.1) we can endow $\mathcal{S}'(\mathbb{R}^n)$ with the inductive limit topology induced by the "inductive system" $\{\mathcal{C}_\alpha(\mathbb{R}^n)^*\}_{\alpha>0}$ because for any fixed $\epsilon > 0$, $\{\mathcal{C}_\alpha(\mathbb{R}^n)^*\}_{\alpha>0}$ are Banach spaces and they satisfy $\mathcal{C}_\beta(\mathbb{R}^n)^* \hookrightarrow \mathcal{C}_\alpha(\mathbb{R}^n)^*$ if $\alpha \ge \beta$. Therefore, we can define the inductive limit topology on $\mathcal{S}'(\mathbb{R}^n)$ to be the finest locally convex topology on $\mathcal{S}'(\mathbb{R}^n)$ such that all inclusion mappings $\mathcal{C}_\alpha(\mathbb{R}^n)^* \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, $\alpha > 0$, are continuous. More precisely, the open base of the inductive limit topology for $\mathcal{S}'(\mathbb{R}^n)$ consists of all convex subsets $U \subset \mathcal{S}'(\mathbb{R}^n)$ such that $U \cap \mathcal{C}_\alpha(\mathbb{R}^n)^*$ is an open set in $\mathcal{C}_\alpha(\mathbb{R}^n)^*$ for any $\alpha > 0$. It is obvious that the inductive limit topology is stronger than the weak-star topology.

Let $\kappa > 0$ and $N > \kappa + n$. We say that $\varphi$ satisfies the *smoothness condition of order* $(\kappa, N)$ if there exists a constant $C > 0$ independent of $x$ and $y$ such that

$$|\partial^\gamma\varphi(x)| \le C\Big(\frac{1}{1+|x|}\Big)^N, \tag{2.2}$$

for $|\gamma| \le [\kappa]$, and

$$|\partial^\gamma\varphi(x) - \partial^\gamma\varphi(y)| \le C|x-y|^{\kappa-[\kappa]} \sup_{|z|\le|x-y|} \Big(\frac{1}{1+|x-z|}\Big)^N \tag{2.3}$$

for $|\gamma| = [\kappa]$.

Furthermore, $\varphi$ is said to satisfy the *vanishing moment condition of order* $\kappa$ if

$$\int_{\mathbb{R}^n} x^\lambda\varphi(x)\,dx = 0 \quad \text{for } \lambda \in \mathbb{N}^n \text{ and } 0 \le |\lambda| \le [\kappa]. \tag{2.4}$$

***Lemma 2.2*** *Let $\kappa > [\kappa] > 0$ and $N > \kappa + n$. Let $\varphi, \psi$ satisfy*

$$|\varphi(x)|, |\psi(x)| \le C\Big(\frac{1}{1+|x|}\Big)^N,$$

*for a constant $C > 0$.*

(i) *Let $i \ge 0$. If $\varphi$ satisfies the vanishing moment condition of order $\kappa$ and $\psi$ satisfies the smoothness condition of order $(\kappa, N)$, then we have a constant $C$, independent of $i$, such that*

$$|(\varphi_i * \psi)(x)| \le C2^{-i\kappa}(1+|x|)^{-N}. \tag{2.5}$$

(ii) *Let $i < 0$. If $\varphi$ satisfies the smoothness condition of order $(\kappa, N)$ and $\psi$ satisfies the vanishing moment condition of order $\kappa$, then there exists a constant $C > 0$, independent of $i$, such that*

$$|(\varphi_i * \psi)(x)| \le C2^{i(n+\kappa)}(1+2^i|x|)^{-N}. \tag{2.6}$$

This is a straightforward result from [10, Lemma B.1]; therefore, we skip the proof.

Later, we will see that the vanishing moments satisfied by $\varphi$ and $\psi$ are important in the proof of Theorem 2.3. Therefore, we would like to clarify the role of vanishing moments in the above proof. Notice that we do not require that both functions $\psi$ and $\varphi_i$ satisfy the vanishing moment conditions for all $i \in \mathbb{Z}$. According to the proof of [10, Lemma B.1], for (2.5), we only need the vanishing moments for $\varphi_i$; and for (2.6), we only need the vanishing moments for $\psi$.

The following result is the main supporting result for our main theorem. On the other hand, Theorem 2.3 has its own independent interest. It shows that the Littlewood–Paley analysis for $\psi \in \mathcal{C}_\alpha(\mathbb{R}^n)$ converges in $\mathcal{C}_\beta(\mathbb{R}^n)$, where $\alpha > \beta + \epsilon$ (the $\epsilon$ in Definition 2.1) and $[\alpha] = [\beta]$, if and only if the moments of $\psi$ up to order $[\alpha]$ are zero. The proof of Theorem 2.3 is given in Section 4.

***Theorem 2.3*** *Let $\alpha$, $\beta$, and $\epsilon$ (the $\epsilon$ in Definition 2.1) satisfy $\alpha > [\alpha]$, $\alpha > \beta + \epsilon$ and $[\alpha] = [\beta]$. Suppose that $\varphi \in \mathcal{C}_\alpha(\mathbb{R}^n)$ satisfies (1.3) and there exists a constant $C > 0$ such that*

$$\sum_{i=-\infty}^{\infty} |\hat{\varphi}(2^i\xi)| < C, \quad \xi \ne 0.$$

*Then for any $\psi \in \mathcal{C}_\alpha(\mathbb{R}^n)$, we have*

$$\psi = \lim_{M,M'\to\infty} \sum_{i=-M'}^{M} (\varphi_i * \psi)$$

*in $\mathcal{C}_\beta(\mathbb{R}^n)$ if $\psi$ satisfies*

$$\int_{\mathbb{R}^n} x^\lambda \psi(x)\, dx = 0 \quad \text{for } \lambda \in \mathbb{N}^n \text{ and } 0 \le |\lambda| \le [\alpha]. \tag{2.7}$$

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*Moreover, we have a constant $C(\alpha,\beta,\epsilon) > 0$, independent of $M$ and $M'$, such that for any $\psi$ satisfying* (2.7),

$$\Big\| \psi - \sum_{i=-M'}^{M} (\varphi_i * \psi) \Big\|_{\mathcal{C}_\beta} \le C(\alpha,\beta,\epsilon)\|\psi\|_{\mathcal{C}_\alpha}(2^{(-M'+1)(\alpha-\beta-\epsilon)} + 2^{(-M-1)(\alpha-\beta)}). \tag{2.8}$$

## 3 Main Result of One parameter Littlewood–Paley Analysis

We now present and prove the main result for this paper. It states that the smoothness of the analyzing function can be relaxed and the compactness assumption of the Fourier support of the analyzing function in the Littlewood–Paley analysis is redundant.

***Theorem 3.1*** *Let $\tau > \omega > 0$ and $\varphi \in \mathcal{S}_\tau(\mathbb{R}^n)$ satisfy*

$$\sum_{i=-\infty}^{\infty} \hat{\varphi}(2^i\xi) = 1 \quad \text{if } \xi \ne 0. \tag{3.1}$$

*and there exists a constant $C > 0$ such that*

$$\sum_{i=-\infty}^{\infty} |\hat{\varphi}(2^i\xi)| < C, \quad \xi \ne 0. \tag{3.2}$$

*Then for any $f \in \mathcal{S}'(\mathbb{R}^n)$ of order $\omega$, there exists a sequence of polynomials $P_{M,M'}$ of degrees less than or equal to $[\omega]$ such that*

$$f = \lim_{M,M'\to\infty} \Big\{ \sum_{i=-M'}^{M} (\varphi_i * f) - P_{M,M'} \Big\} \tag{3.3}$$

*in the inductive limit topology of $\mathcal{S}'(\mathbb{R}^n)$. More specifically,*

$$P_{M,M'}(x) = -\sum_{0\le|\lambda|\le[\omega]} \Big( \langle f,\theta_\lambda\rangle - \sum_{i=-M'}^{M} \langle \varphi_i * f, \theta_\lambda\rangle \Big) x^\lambda \tag{3.4}$$

*where $\theta_\lambda \in \mathcal{S}(\mathbb{R}^n)$, $\lambda \in \mathbb{N}^n$ satisfies*

$$\int x^\gamma \theta_\lambda(x)\,dx = \delta_{\gamma\lambda} = \begin{cases} 1 & \gamma = \lambda, \\ 0 & \gamma \ne \lambda. \end{cases}$$

**Proof** Without loss of generality, we assume that $[\omega] + 1 > \tau$. We consider the family of function spaces $\mathcal{C}_\alpha(\mathbb{R}^n)$ with $\epsilon = (\tau - \omega)/4$. Thus, $\varphi \in \mathcal{C}_\eta(\mathbb{R}^n)$ where $\eta = \tau - \epsilon > \omega + \epsilon$. Let $\beta$ satisfy $\beta > \omega$ and $\eta > \beta + \epsilon$.

For any $\psi \in \mathcal{C}_\eta(\mathbb{R}^n)$, we define

$$\tilde{\psi}(x) = \psi(x) - \sum_{0\le|\lambda|\le[\omega]} c_\lambda \theta_\lambda(x),$$

where $c_\lambda = \langle x^\lambda, \psi\rangle = \int_{\mathbb{R}^n} x^\lambda \psi(x)\,dx$.

It is obvious that $\tilde{\psi}$ satisfies the vanishing moment condition (2.7) in Theorem 2.3 with $\alpha = \omega$. Since $[\tau] = [\eta] = [\beta]$, by applying Theorem 2.3, we find that there is a constant $C > 0$ such that

$$\Big\| \tilde{\psi} - \sum_{i=-M'}^{M} (\varphi_i * \tilde{\psi}) \Big\|_{\mathcal{C}_\beta} \le C \|\psi\|_{\mathcal{C}_\eta} (2^{(-M'+1)(\eta-\beta-\epsilon)} + 2^{(-M-1)(\eta-\beta)}). \tag{3.5}$$

We consider $\|f - \sum_{i=-M'}^{M} (\varphi_i * f) + P_{M,M'}\|_{\mathcal{C}_\eta^*}$, where

$$P_{M,M'}(x) = -\sum_{0\le|\lambda|\le[\omega]} \Big( \langle f, \theta_\lambda\rangle - \sum_{i=-M'}^{M} \langle \varphi_i * f, \theta_\lambda\rangle \Big) x^\lambda.$$

It is trivial that $f - \sum_{i=-M'}^{M} (\varphi_i * f) + P_{M,M'} \in \mathcal{C}_\eta(\mathbb{R}^n)^*$ because $f$ is of order $\omega$, $(\varphi_i * f)$ is a bounded Lebesgue measurable function, and $P_{M,M'} \in \mathcal{C}_{[\omega]}(\mathbb{R}^n)^* \hookrightarrow \mathcal{C}_\eta(\mathbb{R}^n)^*$.

We have

$$\begin{aligned}\Big\| f - \sum_{i=-M'}^{M} (\varphi_i * f) + P_{M,M'} \Big\|_{\mathcal{C}_\eta^*} &= \sup_{\|\psi\|_{\mathcal{C}_\eta}=1} \Big| \Big\langle f - \sum_{i=-M'}^{M} (\varphi_i * f) + P_{M,M'}, \psi \Big\rangle \Big| \\ &= \sup_{\|\psi\|_{\mathcal{C}_\eta}=1} \Big| \langle f, \psi\rangle - \sum_{i=-M'}^{M} \langle (\varphi_i * f), \psi\rangle + \langle P_{M,M'}, \psi\rangle \Big|.\end{aligned}$$

According to the definition of $P_{M,M'}$ (3.4) and the definition of $\tilde{\psi}$, we find that

$$\Big\| f - \sum_{i=-M'}^{M} (\varphi_i * f) + P_{M,M'} \Big\|_{\mathcal{C}_\eta^*} = \sup_{\|\psi\|_{\mathcal{C}_\eta}=1} \Big| \langle f, \tilde{\psi}\rangle - \sum_{i=-M'}^{M} \langle (\varphi_i * f), \tilde{\psi}\rangle \Big|.$$

As $f \in \mathcal{C}_\beta(\mathbb{R}^n)^*$, we are allowed to consider $\langle \cdot\,, \cdot \rangle$ as the pairing between $\mathcal{C}_\beta(\mathbb{R}^n)^*$ and $\mathcal{C}_\beta(\mathbb{R}^n)$. We have

$$\Big\| f - \sum_{i=-M'}^{M} (\varphi_i * f) + P_{M,M'} \Big\|_{\mathcal{C}_\eta^*} \le \sup_{\|\psi\|_{\mathcal{C}_\eta}=1} \|f\|_{\mathcal{C}_\beta^*} \Big\| \tilde{\psi} - \sum_{i=-M'}^{M} (\varphi_i * \tilde{\psi}) \Big\|_{\mathcal{C}_\beta}.$$

By (3.5), we obtain

$$\begin{aligned}&\Big\| f - \sum_{i=-M'}^{M} (\varphi_i * f) + P_{M,M'} \Big\|_{\mathcal{C}_\eta^*} \\ &\qquad \le C \sup_{\|\psi\|_{\mathcal{C}_\eta}=1} \|f\|_{\mathcal{C}_\beta^*} \|\tilde{\psi}\|_{\mathcal{C}_\eta} (2^{(-M'+1)(\eta-\beta-\epsilon)} + 2^{(-M-1)(\eta-\beta)}).\end{aligned}$$

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It is obvious that $\|\tilde{\psi}\|_{\mathcal{C}_\eta} \le C\|\psi\|_{\mathcal{C}_\eta}$ for some constant $C > 0$ independent of $\psi$. Hence,

$$\Big\| f - \sum_{i=-M'}^{M} (\varphi_i * f) + P_{M,M'} \Big\|_{\mathcal{C}_\eta^*} \le C\|f\|_{\mathcal{C}_\beta^*} (2^{(-M'+1)(\eta-\beta-\epsilon)} + 2^{(-M-1)(\eta-\beta)}).$$

Therefore, $\sum_{i=-M'}^{M} (\varphi_i * f) - P_{M,M'}$ converges to $f$ in $\mathcal{C}_\eta(\mathbb{R}^n)^*$. Due to the fact that we have the continuous embedding $\mathcal{C}_\eta(\mathbb{R}^n)^* \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ when $\mathcal{S}'(\mathbb{R}^n)$ is endowed with the inductive limit topology, we obtain our desired result: $\sum_{i=-M'}^{M} (\varphi_i * f) - P_{M,M'}$ converges to $f$ in $\mathcal{S}'(\mathbb{R}^n)$ under the inductive limit topology. ■

Here are some remarks and discussions for the above results. There is a similar result in the Appendix of [11]. Frazier, Jawerth and Weiss prove the convergence of the "continuous Calderón reproducing formula" for tempered distributions modulo a polynomial. In fact, by using the argument for proving Theorem 3.1, [11, Appendix, Theorem 3] can be further generalized. For instance, we can provide an explicit formula for the floating polynomials in the convergence of the continuous Calderón reproducing formula. Moreover, we can show that the continuous Calderón reproducing formula converges under the inductive limit topology while the topology used in [11] is the weak-star topology.

From the proof of Theorem 3.1, we observe that the degree of the floating polynomials is determined by the integral part of the order of the distribution. Hence, it is necessary to introduce the function spaces $\mathcal{C}_\alpha(\mathbb{R}^n)$ with non-integer order. Otherwise, we cannot obtain the best result for the degree of the floating polynomials.

The condition $\tau > \omega$ is almost the best possible condition as the convolution $f * \varphi_i$ is not well defined if $f \in \mathcal{S}'(\mathbb{R}^n)$ is of an order greater than $\tau$.

The degree of the floating polynomials $P_{M,M'}$ is also optimal. This fact can be verified by a special case: if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $0 \notin \operatorname{supp} \hat{\varphi}$ and $f(x) = x^\gamma$, $|\gamma| = [\omega]$, then $f * \varphi_i \equiv 0$. Therefore, we need to have $P_{M,M'}(x) = -x^\gamma$ in (3.3). Notice that the distribution $f(x) = x^\gamma$, $|\gamma| = [\omega]$ is of order $[\omega]$.

Our result includes some interesting examples of analyzing functions that cannot be covered by the "classical" Littlewood–Paley analysis, Proposition 1.1. For instance, if $\varphi$ is a Daubechies wavelet and $\hat{\Phi} = |\hat{\varphi}|^2$, then $\Phi$ satisfies condition (3.1), but it does not belong to $\mathcal{S}(\mathbb{R}^n)$. Therefore, Proposition 1.1 is useless when we take $\Phi$ as the analyzing function. On the other hand, we can use Daubechies wavelets to decompose functions and distributions. Thus, it is natural to expect that we have a Littlewood–Paley type decomposition for distributions by using $\Phi$ as the analyzing function, and Theorem 3.1 provides a positive answer.

## 4 Proof of Theorem 2.3

Since $\varphi \in \mathcal{C}_\alpha(\mathbb{R}^n)$, $\varphi$ is integrable and its Fourier transform is a continuous function. Thus, the zeroth-order moment of $\varphi$ is zero, that is, it satisfies (1.6). Without loss of generality, we assume that $\|\psi\|_{\mathcal{C}_\alpha} = 1$.

Given integers $L > M > 0$ and $L' > M' > 0$, for any $\psi \in \mathcal{C}_\alpha(\mathbb{R}^n)$ satisfying the vanishing moment condition of order $\alpha$ (see (2.4)), we write

$$\begin{aligned}\Psi(x) &= \sum_{i=-L'}^{L} (\varphi_i * \psi)(x) - \sum_{i=-M'}^{M} (\varphi_i * \psi)(x) \\ &= \sum_{i=-L'}^{-M'-1} (\varphi_i * \psi)(x) + \sum_{i=M+1}^{L} (\varphi_i * \psi)(x) = \Psi_1(x) + \Psi_2(x).\end{aligned}$$

By Lemma 2.2, for any $|\gamma| \le [\beta]$, we have a constant $C > 0$ such that

$$\begin{aligned}(4.1)\qquad |(\partial^\gamma \Psi)(x)| &\le \sum_{i=-L'}^{-M'-1} 2^{i|\gamma|} |((\partial^\gamma \varphi)_i * \psi)(x)| + \sum_{i=M+1}^{L} |(\varphi_i * \partial^\gamma \psi)(x)| \\ &\le C\Big( \sum_{i=-L'}^{-M'-1} 2^{i|\gamma|} 2^{i(n+\alpha-|\gamma|)} (1 + 2^i|x|)^{-\beta-n-\epsilon} \\ &\qquad + \sum_{i=M+1}^{L} 2^{-i(\alpha-[\beta])} (1+|x|)^{-\beta-n-\epsilon} \Big).\end{aligned}$$

For $-L' \le i \le -M'-1$, we take the partial derivative $\partial^\gamma$ on $\varphi_i$, since it produces the optimal decay, $2^{i|\gamma|}$, for the summation on $i$ when $i < 0$. After that, we use Lemma 2.2 with $\kappa = \alpha - |\gamma|$ and $N = n + \beta + \epsilon$. We can apply the second part of Lemma 2.2 to conclude our result because we only need the vanishing moments satisfied by $\psi$ when $i \le 0$.

However, we take the partial derivative to the function $\psi$ when $M + 1 \le i \le L$. Notice that $\alpha - [\beta] < 1$ and the zeroth-order moment of $\varphi$ is zero. Therefore, we can apply the first part of Lemma 2.2 with $\kappa = \alpha - [\beta]$ and $N = n+\beta+\epsilon$. Moreover, the $\kappa$ we used in this case is the best possible condition since $\varphi$ only satisfies (1.6).

We find that

$$\begin{aligned}&|(\partial^\gamma \Psi)(x)| \\ &\le C\Big( \sum_{i=-L'}^{-M'-1} 2^{i(n+\alpha)} \Big( \frac{2^{-i}}{2^{-i} + |x|} \Big)^{\beta+n+\epsilon} + \sum_{i=M+1}^{L} 2^{-i(\alpha-[\beta])} (1+|x|)^{-\beta-n-\epsilon} \Big) \\ &\le C\Big( \sum_{i=-L'}^{-M'-1} 2^{i(n+\alpha)-i(\beta+n+\epsilon)} (1+|x|)^{-\beta-n-\epsilon} + \sum_{i=M+1}^{L} 2^{-i(\alpha-[\beta])} (1+|x|)^{-\beta-n-\epsilon} \Big)\end{aligned}$$

as $2^{-i} > 1$ when $-L' \le i \le -M'-1 \le 0$.

Observe that when $i < 0$, we need the decay $2^{i|\gamma|}$ generated by the partial derivative on $\varphi_i$ to balance the growth $2^{-i(\beta+n+\epsilon)}$ given by $(1 + 2^i|x|)^{-\beta-n-\epsilon}$.

Using $\alpha > \beta + \epsilon$, we have

$$(4.2)\qquad |(\partial^\gamma \Psi)(x)| \le C(2^{(-M'-1)(\alpha-\beta-\epsilon)} + 2^{(-M-1)(\alpha-[\beta])})(1+|x|)^{-\beta-n-\epsilon}.$$

For $|\gamma| = [\beta] = [\alpha]$, in order to estimate $|(\partial^\gamma \Psi)(x) - (\partial^\gamma \Psi)(y)|$, we write

$$|(\partial^\gamma \Psi)(x) - (\partial^\gamma \Psi)(y)| \le |(\partial^\gamma \Psi_1)(x) - (\partial^\gamma \Psi_1)(y)| + |(\partial^\gamma \Psi_2)(x) - (\partial^\gamma \Psi_2)(y)|.$$

Moreover, we assume that $|x - y| \le 1$. Otherwise, this is a straightforward consequence of (4.2). We find that there exists a constant $C > 0$ such that

(4.3)
$$\begin{aligned} |(\partial^\gamma \Psi_1)(x) - (\partial^\gamma \Psi_1)(y)| &\le \sum_{i=-L'}^{-M'-1} 2^{i[\beta]} |((\partial^\gamma \varphi)_i * \psi)(x) - ((\partial^\gamma \varphi)_i * \psi)(y)| \\ &\le C \sum_{i=-L'}^{-M'-1} 2^{i(n+\alpha)} |x-y|^{\beta-[\beta]} \left(1 + 2^i |x|\right)^{-\beta-n-\epsilon} \\ &\le C 2^{(-M'-1)(\alpha-\beta-\epsilon)} |x-y|^{\beta-[\beta]} \left(1 + |x|\right)^{-\beta-n-\epsilon}. \end{aligned}$$

Note that in this case we represent $((\partial^\gamma \varphi)_i * \psi)(x) - ((\partial^\gamma \varphi)_i * \psi)(y)$ by

$$\int (\partial^\gamma \varphi)_i(z) [\psi(x-z) - \psi(y-z)]\, dz. \tag{4.4}$$

The second inequality in (4.3) is established by applying the second part of Lemma 2.2 with the facts that $\partial^\gamma \varphi$ satisfies conditions (2.2) and (2.3) with $N = n + \alpha + \epsilon$ and $\kappa = \alpha - [\alpha]$. The zeroth-order moment of the function $\psi(x - \cdot) - \psi(y - \cdot)$ is equal to zero and

$$|\psi(x-z) - \psi(y-z)| \le C|x-y|^{\beta-[\beta]} (1 + |x-z|)^{-\alpha-n-\epsilon}. \tag{4.5}$$

To prove inequality (4.5), we apply the mean value theorem to $\psi$ if $\alpha \ge 1$ or use the Lipschitz condition satisfied by $\psi$ if $0 < \alpha < 1$. Hence, we obtain a constant $C > 0$ such that for $|\gamma| = [\beta] = [\alpha]$ and any fixed $x, y \in \mathbb{R}^n$ satisfying $|x - y| \le 1$,

(4.6)
$$\begin{aligned} |\psi(x-z) - \psi(y-z)| &\le \|\psi\|_{\mathcal{C}_\alpha} |x-y|^{\alpha-[\alpha]} \sup_{|u| \le |x-y|} \left( \frac{1}{1 + |x-z-u|} \right)^{\alpha+n+\epsilon} \\ &\le C|x-y|^{\beta-[\beta]} \left( \frac{1}{1+|x-z|} \right)^{\alpha+n+\epsilon}. \end{aligned}$$

We have the last inequality because $|u| \le |x - y| \le 1$.

Let $l \in \mathbb{N}$ satisfy $2^{-l} \le |x - y| < 2^{-l+1}$. For the estimate of $\partial^\gamma \Psi_2$, we write

$$\begin{aligned} |(\partial^\gamma \Psi_2)(x) - (\partial^\gamma \Psi_2)(y)| \le &\sum_{i=\max(M+1,l)}^{L} |(\varphi_i * \partial^\gamma \psi)(x)| + |(\varphi_i * \partial^\gamma \psi)(y)| \\ &+ \sum_{i=M+1}^{\max(M+1,l)-1} |(\varphi_i * \partial^\gamma \psi)(x) - (\varphi_i * \partial^\gamma \psi)(y)| = \mathrm{I} + \mathrm{II}. \end{aligned}$$

We write $(\varphi_i * \partial^\gamma \psi)(x) - (\varphi_i * \partial^\gamma \psi)(y)$ as

$$\int [\varphi_i(x-z) - \varphi_i(y-z)]\partial^\gamma \psi(z)\, dz. \tag{4.7}$$

Notice that II vanishes when $\max(M+1, l) = M+1$. Therefore, we have $|x-y| \le 2^{-l+1} \le 2^{-i}$ when we deal with II. Similar to (4.6), we find that

$$|\varphi_i(x-z) - \varphi_i(y-z)| \le C2^{i(n+\beta-[\beta])}|x-y|^{\beta-[\beta]}\Big(\frac{1}{1+2^i|x-z|}\Big)^{\alpha+n+\epsilon}.$$

Since $\psi \in \mathcal{C}_\alpha(\mathbb{R}^n)$ and $\varphi(x-z) - \varphi(y-z)$, as a function of $z$, satisfies the zeroth-order vanishing moment condition, and observing that $[\alpha] = [\beta]$, we can use the first part of Lemma 2.2 with $\kappa = \alpha - [\alpha]$, $N = \alpha + n + \epsilon$. We assert that

$$\mathrm{II} \le 2^{(-M-1)(\alpha-\beta)}|x-y|^{\beta-[\beta]}\Big(\frac{1}{1+|x|}\Big)^{\beta+n+\epsilon}.$$

For I, by an estimate similar to the estimate of $\partial^\gamma \Psi_2$ on (4.1), we obtain a constant $C > 0$ such that

$$\begin{aligned}\mathrm{I} &\le \sum_{i=\max(M+1,l)}^{L} 2^{-i(\alpha-[\beta])}\big[(1+|x|)^{-\beta-n-\epsilon} + (1+|y|)^{-\beta-n-\epsilon}\big]\\ &\le C\sum_{i=\max(M+1,l)}^{L} 2^{-i(\alpha-[\beta])} \sup_{|z|\le|x-y|}\frac{1}{(1+|x-z|)^{\beta+n+\epsilon}}.\end{aligned}$$

By the Cauchy–Schwartz inequality, we have

$$\begin{aligned}\sum_{i=\max(M+1,l)}^{L} 2^{-i(\alpha-[\beta])} &\le \Big(\sum_{i=M+1} 2^{-2i(\alpha-\beta)}\Big)^{1/2}\Big(\sum_{i=l} 2^{-2i(\beta-[\beta])}\Big)^{1/2}\\ &\le C2^{-(M+1)(\alpha-\beta)}|x-y|^{\beta-[\beta]}.\end{aligned}$$

Hence, for $I$, we obtain

$$\mathrm{I} \le C2^{-(M+1)(\alpha-\beta)}|x-y|^{\beta-[\beta]} \sup_{|z|\le|x-y|}\frac{1}{(1+|x-z|)^{\beta+n+\epsilon}}.$$

Therefore, by the estimates of I and II, we find that

$$\big|(\partial^\gamma\Psi_2)(x) - (\partial^\gamma\Psi_2)(y)\big| \le 2^{(-M-1)(\alpha-\beta)}|x-y|^{\beta-[\beta]} \sup_{|z|\le|x-y|}\frac{1}{(1+|x-z|)^{\beta+n+\epsilon}}. \tag{4.8}$$

By combining (4.2), (4.3) and (4.8), we assert that

$$\Big\|\sum_{i=-L'}^{L}(\varphi_i * \psi) - \sum_{i=-M'}^{M}(\varphi_i * \psi)\Big\|_{\mathcal{C}_\beta} \le C(2^{(-M'+1)(\alpha-\beta-\epsilon)} + 2^{(-M-1)(\alpha-\beta)}), \tag{4.9}$$

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and hence, $\sum_{i=-M'}^{M}(\varphi_i * \psi)(x)$ is a Cauchy sequence in $\mathcal{C}_\beta(\mathbb{R}^n)$. Thus, the limit

$$\lim_{M,M'\to\infty} \sum_{i=-M'}^{M} (\varphi_i * \psi),$$

exists in $\mathcal{C}_\beta(\mathbb{R}^n)$ and the limit function must be equal to the limit function in $L^2(\mathbb{R}^n)$. Hence, with condition (3.2), the limit function equals to $\psi(x)$.

The estimate (2.8) is obtained by taking $L, L' \to \infty$ in (4.9). ∎

*Remark* 4.1. We see that for $i \le 0$ and $i > 0$, we use the representations (4.4) and (4.7), respectively. With these representations, the smoothness requirement is transferred to the "single function" (that is, the function $(\partial^\gamma \varphi)_i$ in (4.4)) and the vanishing moment is assigned to the "difference function" (the function $\psi(x - z) - \psi(y - z)$ in (4.4)). Without these representations, we need to have an extra smoothness condition on $\varphi$.

Finally, the above proof also shows that inequality (2.8) is still valid without the assumption $[\alpha] = [\beta]$ if $\varphi$ satisfies

$$\int_{\mathbb{R}^n} x^\lambda \varphi(x)\, dx = 0 \quad \text{for } \lambda \in \mathbb{N}^n \text{ and } 0 \le |\lambda| \le [\alpha] - [\beta].$$

## 5 Multiparameter Littlewood–Paley Analysis

Chang and Fefferman studied the Hardy spaces and the function space of bounded mean oscillation under the multiparameter setting (the product domains) [2–9]. Littlewood–Paley analysis in the multiparameter setting (no matter whether it is the "continuous" or the "discrete" version) is a fundamental tool for multiparameter function spaces. Multiparameter Littlewood–Paley analysis (MP Littlewood–Paley analysis) can be easily constructed if we use a separable analyzing function, that is, $\varphi(x_1, x_2) = \varphi^1(x_1)\varphi^2(x_2)$. For non-separable analyzing functions, it is clear that we can construct the corresponding MP Littlewood–Paley analysis by using our ideas in the previous sections. Specifically, we study the convergence of

$$\sum_{I\in\mathbb{Z}^n} \varphi_I * f, \quad f \in \mathcal{S}'(\mathbb{R}^n), \tag{5.1}$$

where $I = (i_1, \dots, i_n) \in \mathbb{Z}^n$ and

$$\varphi_I(x_1, \dots, x_n) = 2^{i_1+\cdots+i_n}\varphi(2^{i_1}x_1, \dots, 2^{i_n}x_n). \tag{5.2}$$

In order to obtain the multiparameter version of Lemma 2.2, we need to introduce the *combined Lipschitz condition* (see (5.5)) to the non-separable analyzing function. The formulation and the use of this condition are presented in Definition 5.1 and Lemma 5.2.

For simplicity, we only consider the multiparameter setting on $\mathbb{R}^2$. The general case, (5.1) and (5.2), follows easily. For the multiparameter setting on $\mathbb{R}^2$, we have

$\varphi_{i,j}(x_1, x_2) = 2^{i+j}\varphi(2^i x_1, 2^j x_2)$, $(i, j) \in \mathbb{Z}^2$. Under this setting, the condition (3.1) is replaced by

$$\sum_{i,j\in\mathbb{Z}} \hat{\varphi}(2^i\xi_1, 2^j\xi_2) = 1, \quad \text{if } \xi_1\xi_2 \neq 0. \tag{5.3}$$

Let $\partial_1$ and $\partial_2$ denote the partial derivative with respect to the first and the second variables, respectively, and let us denote $\partial_1^{\gamma_1}\partial_2^{\gamma_2}$ by $\partial^\gamma$ when $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$. The following is the corresponding generalization of Definition 2.1 under the multiparameter setting.

***Definition 5.1*** Given a fixed $\epsilon > 0$, for any $\alpha_1, \alpha_2 > 0$, $\mathcal{C}_{\alpha_1,\alpha_2}(\mathbb{R}^2)$ consists of the function $\varphi(x)$ that satisfies, for any $(x_1, x_2) \in \mathbb{R}^2$, $0 \le |\gamma_1| \le [\alpha_1]$ and $0 \le |\gamma_2| \le [\alpha_2]$,

$$\|(\partial_1^{\gamma_1}\varphi)(x_1, \cdot)\|_{\mathcal{C}_{\alpha_2(\mathbb{R})}} \le C, \quad \text{and} \quad \|(\partial_2^{\gamma_2}\varphi)(\cdot, x_2)\|_{\mathcal{C}_{\alpha_1(\mathbb{R})}} \le C, \tag{5.4}$$

where the constant $C > 0$ is independent of $(x_1, x_2)$ and $(\gamma_1, \gamma_2)$, and the *combined Lipschitz condition*:

$$\begin{aligned} &\left|\partial^\gamma\varphi(x_1, x_2) - \partial^\gamma\varphi(x_1, y_2) - \partial^\gamma\varphi(y_1, x_2) + \partial^\gamma\varphi(y_1, y_2)\right| \\ &\qquad \le C|x_1 - y_1|^{\alpha_1-[\alpha_1]}|x_2 - y_2|^{\alpha_2-[\alpha_2]} \times \\ &\qquad\qquad \sup_{\substack{|z_1|\le|x_1-y_1| \\ |z_2|\le|x_2-y_2|}} \frac{1}{(1 + |x_1 - z_1|)^{\alpha_1+1+\epsilon}(1 + |x_2 - z_2|)^{\alpha_2+1+\epsilon}}, \end{aligned} \tag{5.5}$$

where $\gamma_1 = [\alpha_1]$ and $\gamma_2 = [\alpha_2]$.

Define the norm on $\mathcal{C}_{\alpha_1,\alpha_2}(\mathbb{R}^2)$ by the infimum of the constant $C > 0$ that satisfies (5.4) and (5.5).

Similarly, we can define $\mathcal{S}_{\alpha_1,\alpha_2}(\mathbb{R}^2)$ by modifying the definition of $\mathcal{S}_\alpha(\mathbb{R}^n)$ in Definition 2.1. Moreover, we also have

$$\mathcal{S}(\mathbb{R}^2) = \bigcap_{\alpha_1,\alpha_2>0} \mathcal{S}_{\alpha_1,\alpha_2}(\mathbb{R}^2) \quad \text{and} \quad \mathcal{S}'(\mathbb{R}^2) = \bigcup_{\alpha_1,\alpha_2>0} \mathcal{S}_{\alpha_1,\alpha_2}(\mathbb{R}^2)^*,$$

where $\mathcal{S}_{\alpha_1,\alpha_2}(\mathbb{R}^2)^*$ is the dual space of $\mathcal{S}_{\alpha_1,\alpha_2}(\mathbb{R}^2)$. We call the order pair $(\alpha_1, \alpha_2)$, the *multiparameter order (MP order)* of $f \in \mathcal{S}'(\mathbb{R}^2)$ if $f \in \mathcal{S}_{\alpha_1,\alpha_2}(\mathbb{R}^2)^*$. Notice that the MP order can be uniquely defined if we endow $\mathbb{R}^2$ with a well order structure. Whether or not the multiparameter is uniquely defined is not crucial for our results. We therefore do not go into detail about the unique definition.

A Schwartz distribution $f$ of MP order $(\alpha_1, \alpha_2)$ is called *separable* if there exist $f_1, f_2 \in \mathcal{S}'(\mathbb{R})$ of order $\alpha_1$ and $\alpha_2$, respectively, such that $f$ equals the tensor product of $f_1$ and $f_2$, that is, $f = f_1 \otimes f_2$. At the end of this section, we discuss the reason why we abandon the notion of order and use the MP order.

The formulation of the MP Littlewood–Paley analysis follows from the idea of constructing the "ordinary" Littlewood–Paley analysis. We prove the multiparameter versions of Lemma 2.2 and Theorem 2.3 and use them to construct our Littlewood–Paley analysis under the multiparameter setting. We begin with the multiparameter version of Lemma 2.2.

For $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$ and $\varphi \in \mathcal{C}_{\alpha_1,\alpha_2}(\mathbb{R}^2)$, define $\triangle_\alpha$ by

$$(\triangle_\alpha \varphi)(u, v) = \partial^{[\alpha]}\varphi(u_1, u_2) - \partial^{[\alpha]}\varphi(u_1, v_2) - \partial^{[\alpha]}\varphi(v_1, u_2) + \partial^{[\alpha]}\varphi(v_1, v_2),$$

where $[\alpha] = ([\alpha_1], [\alpha_2])$, $u = (u_1, u_2) \in \mathbb{R}^2$, and $v = (v_1, v_2) \in \mathbb{R}^2$.

***Lemma 5.2*** *Let $\kappa_i, N_i > 0$ satisfy $N_i > 1 + \kappa_i$ and $\kappa_i > [\kappa_i]$, for $i = 1, 2$. Suppose that there exists a constant $C > 0$ such that for any $0 \le \gamma_i \le [\kappa_i]$, $i = 1, 2$, $\varphi(x_1, x_2)$ and $\psi(x_1, x_2)$ satisfy*

$$|\partial^\gamma \varphi(x_1, x_2)|, |\partial^\gamma \psi(x_1, x_2)| \le C\Big(\frac{1}{1 + |x_1|}\Big)^{N_1}\Big(\frac{1}{1 + |x_2|}\Big)^{N_2}.$$

*For $\gamma_1 = [\kappa_1]$, $\varphi$ and $\psi$ satisfy*

$$\begin{aligned}(5.6)\qquad &|(\partial_1^{\gamma_1}\varphi)(x_1, x_2) - (\partial_1^{\gamma_1}\varphi)(y_1, x_2)|,\ |(\partial_1^{\gamma_1}\psi)(x_1, x_2) - (\partial_1^{\gamma_1}\psi)(y_1, x_2)|\\ &\qquad \le C|x_1 - y_1|^{\kappa_1 - [\kappa_1]} \sup_{|z_1| \le |x_1 - y_1|}\Big(\frac{1}{1 + |x_1 - z_1|}\Big)^{N_1}\Big(\frac{1}{1 + |x_2|}\Big)^{N_2}.\end{aligned}$$

*For $\gamma_2 = [\kappa_2]$, they satisfy*

$$\begin{aligned}(5.7)\qquad &|(\partial_2^{\gamma_2}\varphi)(x_1, x_2) - (\partial_2^{\gamma_2}\varphi)(x_1, y_2)|,\ |(\partial_2^{\gamma_2}\psi)(x_1, x_2) - (\partial_2^{\gamma_2}\psi)(x_1, y_2)|\\ &\qquad \le C|x_2 - y_2|^{\kappa_2 - [\kappa_2]}\Big(\frac{1}{1 + |x_1|}\Big)^{N_1} \sup_{|z_2| \le |x_2 - y_2|}\Big(\frac{1}{1 + |x_2 - z_2|}\Big)^{N_2}.\end{aligned}$$

*For $\kappa = (\kappa_1, \kappa_2)$, they satisfy the combined Lipschitz condition,*

$$\begin{aligned}(5.8)\qquad &|(\triangle_\kappa \varphi)(x, y)|,\ |(\triangle_\kappa \psi)(x, y)| \le C|x_1 - y_1|^{\kappa_1 - [\kappa_1]}|x_2 - y_2|^{\kappa_2 - [\kappa_2]}\\ &\qquad \times \sup_{\substack{|z_1| \le |x_1 - y_1|\\ |z_2| \le |x_2 - y_2|}}\Big(\frac{1}{1 + |x_1 - z_1|}\Big)^{N_1}\Big(\frac{1}{1 + |x_2 - z_2|}\Big)^{N_2}.\end{aligned}$$

*Furthermore, suppose that $\varphi$ and $\psi$ satisfy the vanishing moment conditions,*

$$(5.9)\qquad \int_{\mathbb{R}} x_1^{\lambda_1}\psi(x_1, x_2)\,dx_1 = \int_{\mathbb{R}} x_1^{\lambda_1}\varphi(x_1, x_2)\,dx_1 = 0, \quad \forall x_2 \in \mathbb{R},$$

$$(5.10)\qquad \int_{\mathbb{R}} x_2^{\lambda_2}\psi(x_1, x_2)\,dx_2 = \int_{\mathbb{R}} x_2^{\lambda_2}\varphi(x_1, x_2)\,dx_2 = 0, \quad \forall x_1 \in \mathbb{R},$$

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*for* $\lambda_i \in \mathbb{N}$ *and* $0 \leq \lambda_i \leq [\kappa_i]$, $i = 1, 2$. *Then there exists a constant* $C > 0$, *independent of* $i$ *and* $j$, *such that*

$$|(\varphi_{i,j} * \psi)(x_1, x_2)| \leq C \min(2^{i(1+\kappa_1)}, 2^{-i\kappa_1}) \min(2^{j(1+\kappa_2)}, 2^{-j\kappa_2}) \times \Big(1 + \frac{|x_1|}{\max(1, 2^{-i})}\Big)^{-N_1} \Big(1 + \frac{|x_2|}{\max(1, 2^{-j})}\Big)^{-N_2}. \tag{5.11}$$

**Proof** The proof is an "iteration" of the result in Lemma 2.2; therefore, we will not present the details of the proof. On the other hand, there are some crucial modifications. Furthermore, our sketch of the proof will illustrate the use of the Lipschitz conditions (5.6)–(5.8), and the use of the vanishing moment conditions (5.9) and (5.10).

If $i \geq 0$ and $j \leq 0$, we have

$$|(\varphi_{i,j} * \psi)(x_1, x_2)| = 2^i \Big| \int_{\mathbb{R}^2} \varphi(2^i(x_1 - y_1), y_2) \psi(y_1, x_2 - 2^{-j} y_2)\, dy_1 dy_2 \Big|.$$

For any fixed but arbitrary $y_1$ and $x_1$, let $\breve{\varphi}(y_2) = \varphi(2^i(x_1 - y_1), y_2)$ be a function of $y_2$. Similarly, for any fixed but arbitrary $y_2$ and $x_2$, define $\breve{\psi}(y_1) = \psi(y_1, x_2 - 2^{-j} y_2)$.

Let $\breve{\varphi}^{(r)}$ and $\breve{\psi}^{(s)}$, $r, s \in \mathbb{N}$, denote the ordinary derivatives of the single variable functions $\breve{\varphi}$ and $\breve{\psi}$, respectively. We assert that

$$|(\varphi_{i,j} * \psi)(x_1, x_2)| = 2^i \Big| \int_{\mathbb{R}^2} \Big[\breve{\varphi}(y_2) - \sum_{0 \leq r \leq [\kappa_2]} \frac{\breve{\varphi}^{(r)}(2^j x_2)}{r!}(y_2 - 2^j x_2)^r\Big] \times \Big[\breve{\psi}(y_1) - \sum_{0 \leq s \leq [\kappa_1]} \frac{\breve{\psi}^{(s)}(x_1)}{s!}(y_1 - x_1)^s\Big]\, dy_1 dy_2 \Big|.$$

The above identity is valid because of the vanishing moment conditions (5.9) and (5.10). For example, we find that

$$\begin{aligned} &\int_{\mathbb{R}^2} \breve{\varphi}^{(r)}(2^j x_2)(y_2 - 2^j x_2)^r \breve{\psi}^{(s)}(x_1)(y_1 - x_1)^s dy_1 dy_2 \\ &\quad = \Big(\partial_2^r \int_{\mathbb{R}} \varphi(2^i(x_1 - y_1), z_2)(y_1 - x_1)^s\, dy_1\Big) \Big|_{z_2 = 2^j x_2} \\ &\qquad \times \Big(\partial_1^s \int_{\mathbb{R}} \psi(z_1, x_2 - 2^{-j} y_2)(y_2 - 2^j x_2)^r\, dy_2\Big) \Big|_{z_1 = x_1} = 0. \end{aligned}$$

Let $\delta_2 = 2^j$, $\delta_1 = 1$, and

$$\begin{aligned} D_{\nu,1} &= \{y_\nu \in \mathbb{R} : |y_\nu - \delta_\nu x_\nu| \leq 3\}, \\ D_{\nu,2} &= \{y_\nu \in \mathbb{R} : |y_\nu - \delta_\nu x_\nu| > 3| \text{ and } y_\nu| \leq |\delta_\nu x_\nu|/2\}, \\ D_{\nu,3} &= \{y_\nu \in \mathbb{R} : |y_\nu - \delta_\nu x_\nu| > 3 \text{ and } |y_\nu| > |\delta_\nu x_\nu|/2\}. \end{aligned}$$

We have $|(\varphi_{i,j} * \psi)(x_1, x_2)| \leq \sum_{1\leq l,m\leq 3} I_{l,m}$, where

$$I_{l,m} = 2^i \int_{D_{1,l}\times D_{2,m}} \Big| \check{\varphi}(y_2) - \sum_{0\leq r\leq[\kappa_2]} \frac{\check{\varphi}^{(r)}(2^j x_2)}{r!}(y_2 - 2^j x_2)^r \Big| \times \Big| \check{\psi}(y_1) - \sum_{0\leq s\leq[\kappa_1]} \frac{\check{\psi}^{(s)}(x_1)}{s!}(y_1 - x_1)^s \Big| \, dy_1 dy_2.$$

When $y_1 \in D_{1,1}$ and $y_2 \in D_{2,1}$, by (5.6), (5.7) and using the idea from [10, Lemma B.1], we obtain

$$I_{1,1} \leq C 2^{-i\kappa_1} 2^{(1+\kappa_2)j} \Big(\frac{1}{1+|x_1|}\Big)^{N_1} \Big(\frac{1}{1+|2^j x_2|}\Big)^{N_2}.$$

In fact, when $(x_1, x_2) \in D_{1,1} \times D_{2,1}$, we estimate $I_{1,1}$ by iterating the estimate for $\int_A$ in [10, Lemma B.1]. If $(x_1, x_2)$ belongs to the other domains $D_{1,l} \times D_{2,m}$, we use the corresponding results for $\int_A$, $\int_B$ and $\int_C$ in [10, Lemma B.1]. Thus, when $i \geq 0$ and $0 \geq j$, we assert that

$$|(\varphi_{i,j} * \psi)(x_1, x_2)| \leq C 2^{-i\kappa_1} 2^{(1+\kappa_2)j} \Big(\frac{1}{1+|x_1|}\Big)^{N_1} \Big(\frac{1}{1+|2^j x_2|}\Big)^{N_2}.$$

For $i \leq 0$ and $j \leq 0$, by (5.9), we find that

$$|(\varphi_{i,j}*\psi)(x_1, x_2)| = \Big| \int_{\mathbb{R}^2} \Big[ \varphi(y_1, y_2) - \sum_{0\leq\gamma_1\leq[\kappa_1]} \frac{(\partial_1^{\gamma_1}\varphi)(2^i x_1, y_2)}{\gamma_1!}(y_1 - 2^i x_1)^{\gamma_1} \Big] \times \psi(x_1 - 2^{-i} y_1, x_2 - 2^{-j} y_2)\, dy_1 dy_2 \Big|.$$

For any fixed but arbitrary $x_1$, let

$$R(y_1, y_2) = \frac{1}{([\kappa_1]-1)!} \int_{2^i x_1}^{y_1} (y_1 - t)^{[\kappa_1]-1} (\partial_1^{[\kappa_1]}\varphi)(t, y_2)\, dt - \frac{(\partial_1^{[\kappa_1]}\varphi)(2^i x_1, y_2)}{[\kappa_1]!}(y_1 - 2^i x_1)^{[\kappa_1]}.$$

The first term on the right-hand side of the above identity is the remainder term of the Taylor expansion of $\varphi(y_1, y_2)$ on the first variable, $y_1$, in integral form. After the proof, we expand why we use the integral form of the remainder instead of following common practice, using the differential form of the remainder term.

We have

(5.12)
$$|(\varphi_{i,j}*\psi)(x_1,x_2)| = \Big| \int_{\mathbb{R}^2} \Big[ R(y_1,y_2) - \sum_{0\le\gamma_2\le[\kappa_2]} \frac{(\partial_2^{\gamma_2} R)(y_1,2^j x_2)}{\gamma_2!}(y_2-2^j x_2)^{\gamma_2} \Big] \times \psi(x_1-2^{-i}y_1, x_2-2^{-j}y_2)\, dy_1 dy_2 \Big|$$

because of Fubini's theorem and the vanishing moment condition (5.10).

By applying the differential form of the remainder term for the Taylor expansion and the mean-value theorem for the integral, we find that there exist $u=(u_1,u_2)\in\mathbb{R}^2$ satisfying $|u_1-y_1|\le|2^i x_1-y_1|$, $|u_2-y_2|\le|2^j x_2-y_2|$, and a constant $C>0$ such that

$$\Big| R(y_1,y_2) - \sum_{0\le\gamma_2\le[\kappa_2]} \frac{(\partial_2^{\gamma_2} R)(y_1,2^j x_2)}{\gamma_2!}(y_2-2^j x_2)^{\gamma_2} \Big| \le C|y_1-2^i x_1|^{[\kappa_1]}|y_2-2^j x_2|^{[\kappa_2]}|(\triangle_\kappa\varphi)(u_1,u_2,x_1,x_2)|, \quad x=(x_1,x_2). \tag{5.13}$$

By using (5.8), (5.12), and (5.13), we obtain

$$|(\varphi_{i,j}*\psi)(x_1,x_2)| \le C 2^{i(\kappa_1+1)}2^{j(\kappa_2+1)}\Big(\frac{1}{1+2^i|x_1|}\Big)^{N_1}\Big(\frac{1}{1+2^j|x_2|}\Big)^{N_2}.$$

The estimates for the other cases, namely, $i\le 0$, $j>0$, and $i\ge 0$, $j\ge 0$, follow similarly. ■

We use the integral form of the remainder term to define $R(y_1,y_2)$ instead of using the differential form

$$\tilde{R}(y_1,y_2) = \frac{(\partial_1^{[\kappa_1]}\varphi)(w,y_2) - (\partial_1^{[\kappa_1]}\varphi)(2^i x_1, y_2)}{[\kappa_1]!}(y_1-2^i x_1)^{[\kappa_1]}$$

for some $|w-y_1|\le|2^i x_1-y_1|$, because, in general, $w$ depends on $y_2$. The existence of $\partial_2^{\gamma_2}\tilde{R}$ relies on the differentiability of $w$ as a function of $y_2$. Since $w$ is not necessarily a smooth function of $y_2$, we cannot use the Taylor expansion of $\tilde{R}(y_1,y_2)$ to establish (5.12).

The above sketch of the proof shows us why we cannot directly quote the result from [10]. Although the results are similar, there are some major modifications on the multiparameter setting.

One of the main differences between the multiparameter setting and the one parameter setting is the use of the combined Lipschitz condition (5.8). On the other hand, if $\varphi$ is separable, that is, $\varphi(x_1,x_2)=\varphi^1(x_1)\varphi^2(x_2)$, we find that

$$(\triangle_\kappa\varphi)(x,y) = \big[(\partial^{[\kappa_1]}\varphi^1)(x_1) - (\partial^{[\kappa_1]}\varphi^1)(y_1)\big]\big[(\partial^{[\kappa_2]}\varphi^2)(x_2) - (\partial^{[\kappa_2]}\varphi^2)(y_2)\big],$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Thus, the combined Lipschitz condition (5.8) for separable analyzing functions reduces to the "ordinary" Lipschitz conditions (5.6) and (5.7).

The reader may have the wrong impression that the assumptions in Lemma 5.2 can be simplified. For example, it seems that we can develop our MP Littlewood–Paley analysis by imposing the same order of smoothness on the first and the second variables in Lemma 5.2, that is, $\kappa_1 = \kappa_2$. The proof of the following theorem shows us why we need to have Lemma 5.2 in the most general form.

With Lemma 5.2, we can obtain the multiparameter version of Theorem 2.3. The proof of the following theorem is similar to the proof of Theorem 2.3. For the sake of brevity, we only provide an outline of the proof.

***Theorem 5.3*** *Let $(\alpha_1, \alpha_2)$, $(\beta_1, \beta_2)$, and $\epsilon > 0$ satisfy $\alpha_i > [\alpha_i]$, $[\alpha_i] = [\beta_i]$, and $\alpha_i > \beta_i + \epsilon$, $i = 1, 2$. Suppose that $\varphi \in \mathcal{S}(\mathbb{R}^2)$ satisfies* (5.3)*, and there exists a constant $C > 0$ such that*

$$\sum_{i,j\in\mathbb{Z}} |\hat{\varphi}(2^i\xi_1, 2^j\xi_2)| < C, \quad \text{if } \xi_1\xi_2 \neq 0. \tag{5.14}$$

*Then for any $\psi \in \mathcal{S}(\mathbb{R}^2)$ satisfying*

$$\int_{\mathbb{R}} x_1^{\lambda_1} \psi(x_1, x_2)\, dx_1 = \int_{\mathbb{R}} x_2^{\lambda_2} \psi(x_1, x_2)\, dx_2 = 0, \quad i = 1, 2, \tag{5.15}$$

*with $\lambda_i \in \mathbb{N}$, $0 \le \lambda_i \le [\alpha_i]$, we have*

$$\psi = \lim_{M,M',N,N'\to\infty} \sum_{i=-M'}^{M} \sum_{j=-N'}^{N} (\varphi_{i,j} * \psi) \tag{5.16}$$

*in $\mathcal{C}_{\beta_1,\beta_2}(\mathbb{R}^2)$.*

*Moreover, we have a constant $C(\alpha, \beta, \epsilon) > 0$, independent of $M$, $M'$, $N$, and $N'$, such that for any $\psi$ satisfying* (5.15)*,*

$$\Big\| \psi - \sum_{i=-M'}^{M} \sum_{j=-N'}^{N} (\varphi_{i,j} * \psi) \Big\|_{\mathcal{C}_{\beta_1,\beta_2}} \le C(\alpha, \beta, \epsilon) \|\psi\|_{\mathcal{C}_{\alpha_1,\alpha_2}} \tag{5.17}$$
$$\times \left( 2^{(-M'+1)(\alpha_1-\beta_1-\epsilon)} + 2^{(-M-1)(\alpha_1-\beta_1)} + 2^{(-N'+1)(\alpha_2-\beta_2-\epsilon)} + 2^{(-N-1)(\alpha_2-\beta_2)} \right).$$

**Proof** Condition (5.14) guarantees the absolute convergence of (5.3) and, hence, the absolute convergence of (5.3) enables us to conclude that for any $(\xi_1, \xi_2) \in \mathbb{R}^2$ with $\xi_1\xi_2 \neq 0$, we have

$$\hat{\varphi}(0, \xi_2) = \lim_{i\to-\infty} \hat{\varphi}(2^i\xi_1, \xi_2) = 0, \quad \text{and} \quad \hat{\varphi}(\xi_1, 0) = \lim_{j\to-\infty} \hat{\varphi}(\xi_1, 2^j\xi_2) = 0.$$

That is, $\varphi$ satisfies

$$\int_{\mathbb{R}} \varphi(x_1, x_2), dx_1 = 0, \quad \text{and} \quad \int_{\mathbb{R}} \varphi(x_1, x_2)\, dx_2 = 0. \tag{5.18}$$

With these vanishing moment conditions for $\varphi$, the proof of Theorem 5.3 is similar to the proof of Theorem 2.3. For instance, given integers $L > M$, $L' > M'$, $K > N$, and $K' > N'$, if we consider the $\gamma = (\gamma_1, \gamma_2)$-order partial derivative of

$$\Psi = \sum_{i=-L'}^{L} \sum_{j=-K'}^{K} (\varphi_{i,j} * \psi) - \sum_{i=-M'}^{M} \sum_{j=-N'}^{N} (\varphi_{i,j} * \psi),$$

we have

$$|\partial^\gamma \Psi| \le \Big( \sum_{i=M}^{L} \sum_{j=0}^{K} + \sum_{i=0}^{M} \sum_{j=N}^{K} \Big) |(\varphi_{i,j} * \partial^\gamma \psi)| \tag{5.19a}$$

$$+ \Big( \sum_{i=M}^{L} \sum_{j=-K'}^{0} + \sum_{i=0}^{M} \sum_{j=-K'}^{-N'} \Big) 2^{j\gamma_2} |(\partial_2^{\gamma_2} \varphi)_{i,j} * (\partial_1^{\gamma_1} \psi)| \tag{5.19b}$$

$$+ \Big( \sum_{i=-L'}^{-M'} \sum_{j=0}^{K} + \sum_{i=-M'}^{0} \sum_{j=N}^{K} \Big) 2^{i\gamma_1} |(\partial_1^{\gamma_1} \varphi)_{i,j} * (\partial_2^{\gamma_2} \psi)| \tag{5.19c}$$

$$+ \Big( \sum_{i=-L'}^{-M'} \sum_{j=-K'}^{0} + \sum_{i=-M'}^{0} \sum_{j=-K'}^{-N'} \Big) 2^{i\gamma_1 + j\gamma_2} |(\partial^\gamma \varphi)_{i,j} * \psi)|. \tag{5.19d}$$

The first row (5.19a) contains those $\varphi_{i,j}$ with $i \ge 0$ and $j \ge 0$, and the partial derivative is taken on $\psi$. We estimate (5.19a) by Lemma 5.2 with $N_i = 1 + \beta_i + \epsilon$ and $\kappa_i = \alpha_i - [\beta_i]$, $i = 1, 2$. Expression (5.19b) includes those $\varphi_{i,j}$ such that $i \ge 0$ and $j \le 0$. We take the partial derivative of the second variable on $\varphi$ and the partial derivative of the first variable on $\psi$ and apply Lemma 5.2 with $N_i = 1+\beta_i+\epsilon$, $i = 1, 2$, $\kappa_2 = \alpha_2 - \gamma_2$, and $\kappa_1 = \alpha_1 - [\beta_1]$. On (5.19c) and (5.19d), the roles of $\varphi$ and $\psi$ are interchanged with respect to (5.19b) and (5.19a), respectively. In view of (5.11), we can obtain the required estimate of $|\partial^\gamma \Psi|$. Similarly, we have the corresponding estimates for the Lipschitz continuities of $\partial^\gamma \Psi$. We leave the details to the reader. ■

As $\varphi$ only satisfies the zeroth-order moment condition (5.18), we see that when $ij < 0$, we need to have Lemma 5.2 with independent smoothness assumptions for the first and the second variables.

The above conditions for $\varphi$ can be further relaxed, we will briefly discuss the possible relaxation at the end of this section.

We can now construct the MP Littlewood–Paley analysis from our previous results.

***Theorem 5.4*** *Suppose that $\varphi \in \mathcal{S}(\mathbb{R}^2)$ satisfy* (5.3) *and* (5.14). *Then for any $f \in \mathcal{S}'(\mathbb{R}^2)$ of MP order $\omega = (\omega_1, \omega_2)$, there exist two sequences of distributions*

$$F_{M,M'N,N',\lambda_1}, S_{M,M'N,N',\lambda_2} \in \mathcal{S}'(\mathbb{R})$$

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*such that*

$$f(x_1,x_2)=\lim_{\substack{M,M'\to\infty\\ N,N'\to\infty}}\sum_{i=-M'}^{M}\sum_{j=-N'}^{N}\Big\{(\varphi_{i,j}*f)(x_1,x_2) - \sum_{|\lambda_1|\le[\omega_1]} x_1^{\lambda_1}\otimes F_{M,M'N,N',\lambda_1}(x_2)-\sum_{|\lambda_2|\le[\omega_2]} S_{M,M'N,N',\lambda_2}(x_1)\otimes x_2^{\lambda_2}\Big\} \tag{5.20}$$

*in the inductive limit topology of* $\mathcal{S}'(\mathbb{R}^2)$, i.e., *the one induced by the inductive system,* $\{\mathcal{C}_{\alpha_1,\alpha_2}(\mathbb{R}^2)^*\}$.

**Proof** We consider the family of function spaces, $\mathcal{C}_{\alpha_1,\alpha_2}(\mathbb{R}^2)$, with

$$\epsilon=\min_{i=1,2}([\omega_i]+1-\omega_i)/4.$$

Let $\beta=(\beta_1,\beta_2)$ satisfy $\beta_i>\omega_i$ and $[\omega_i]+1>\beta_i+2\epsilon$, $i=1,2$.

For any $\lambda\in\mathbb{N}$, let $\theta_{1,\lambda}(x)\in\mathcal{S}(\mathbb{R}^2)$ and $\theta_{2,\lambda}(x)\in\mathcal{S}(\mathbb{R}^2)$ satisfy

$$\int_{\mathbb{R}} x_2^{\gamma}\theta_{1,\lambda}(x_1,x_2)\,dx_2=0 \quad\text{and}\quad \int_{\mathbb{R}} x_1^{\gamma}\theta_{1,\lambda}(x_1,x_2)\,dx_1=\delta_{\gamma\lambda}$$
$$\int_{\mathbb{R}} x_1^{\gamma}\theta_{2,\lambda}(x_1,x_2)\,dx_1=0 \quad\text{and}\quad \int_{\mathbb{R}} x_2^{\gamma}\theta_{2,\lambda}(x_1,x_2)\,dx_2=\delta_{\gamma\lambda}.$$

Let $(\eta_1,\eta_2)=(\beta_1+2\epsilon,\beta_2+2\epsilon)$. For any $\psi\in\mathcal{S}(\mathbb{R}^2)$, we define

$$\tilde\psi(x_1,x_2)=\psi(x_1,x_2)-\sum_{0\le|\lambda_1|\le[\omega_1]} c_{1,\lambda_1}(x_2)\theta_{1,\lambda_1}(x_1,x_2) - \sum_{0\le|\lambda_2|\le[\omega_2]} c_{2,\lambda_2}(x_1)\theta_{2,\lambda_2}(x_1,x_2),$$

where $c_{1,\lambda_1}(x_2)=\int_{\mathbb{R}} x_1^{\lambda_1}\psi(x_1,x_2)\,dx_1$ and $c_{2,\lambda_2}(x_1)=\int_{\mathbb{R}} x_2^{\lambda_2}\psi(x_1,x_2)\,dx_2$.

With $\alpha$ replaced by $\omega$, $\tilde\psi\in\mathcal{S}(\mathbb{R}^2)$ satisfies the vanishing moment condition (5.15) in Theorem 5.3. By applying Theorem 5.3, we obtain

$$\Big\|\tilde\psi-\sum_{i=-M'}^{M}\sum_{j=-N'}^{N}(\varphi_{i,j}*\tilde\psi)\Big\|_{\mathcal{C}_{\beta_1,\beta_2}} \le C(\beta,\epsilon)\|\psi\|_{\mathcal{C}_{\eta_1,\eta_2}}\big(2^{(-M'+1)\epsilon}+2^{(-M-1)2\epsilon}+2^{(-N'+1)\epsilon}+2^{(-N-1)2\epsilon}\big). \tag{5.21}$$

For $f\in\mathcal{S}'(\mathbb{R}^2)$ of MP order $(\omega_1,\omega_2)$ and $\psi\in\mathcal{S}(\mathbb{R}^2)$, define the Schwartz distribution with independent variable $x_2$, $\langle\cdot,\cdot\rangle_1$, and the Schwartz distribution with independent variable $x_1$, $\langle\cdot,\cdot\rangle_2$, by

$$\langle f,\psi\rangle_1(x_2)=\langle f,\psi(\cdot,x_2)\rangle \quad\text{and}\quad \langle f,\psi\rangle_2(x_1)=\langle f,\psi(x_1,\cdot)\rangle,$$

respectively. In other words, $\langle \cdot, \cdot \rangle_1$ is the pairing of $f$ and $\psi$ with respect to the first variable and $\langle \cdot, \cdot \rangle_2$ is the pairing of $f$ and $\psi$ with respect to the second variable. Obviously, $\langle f, \psi \rangle_1$ is of order $\omega_2$ and $\langle f, \psi \rangle_2$ is of order $\omega_1$.

Define

$$F_{M,M',N,N',\lambda_1} = -\langle f, \theta_{1,\lambda_1} \rangle_1 + \sum_{i=-M'}^{M} \sum_{j=-N'}^{N} \langle \varphi_{i,j} * f, \theta_{1,\lambda_1} \rangle_1 \tag{5.22}$$

$$S_{M,M',N,N',\lambda_2} = -\langle f, \theta_{2,\lambda_2} \rangle_2 + \sum_{i=-M'}^{M} \sum_{j=-N'}^{N} \langle \varphi_{i,j} * f, \theta_{2,\lambda_2} \rangle_2. \tag{5.23}$$

Because of $\beta_i > \omega_i$, we find that $f \in \mathcal{C}_{\beta_1,\beta_2}(\mathbb{R}^2)^*$ and hence,

$$\begin{aligned} \Big| \Big\langle \Big( f - \sum_{i=-M'}^{M} \sum_{j=-N'}^{N} (\varphi_i * f)(x_1, x_2) + \sum_{|\lambda_1| \le [\omega_1]} x_1^{\lambda_1} \otimes F_{M,M'N,N',\lambda_1}(x_2) \\ + \sum_{|\lambda_2| \le [\omega_2]} S_{M,M'N,N',\lambda_2}(x_1) \otimes x_2^{\lambda_2} \Big), \psi \Big\rangle \Big| \\ \le \|f\|_{\mathcal{C}^*_{\beta_1,\beta_2}} \Big\| \tilde{\psi} - \sum_{i=-M'}^{M} \sum_{j=-N'}^{N} (\varphi_{i,j} * \tilde{\psi}) \Big\|_{\mathcal{C}_{\beta_1,\beta_2}}. \end{aligned}$$

Similar to the proof of Theorem 3.1, our desired result follows from the above inequality and (5.21). ■

We call

$$R_{M,M'N,N'}(x_1, x_2) = \sum_{|\lambda_1| \le [\omega_1]} x_1^{\lambda_1} \otimes F_{M,M'N,N',\lambda_1}(x_2) + \sum_{|\lambda_2| \le [\omega_2]} S_{M,M'N,N',\lambda_2}(x_1) \otimes x_2^{\lambda_2}$$

the *renormalization* of $f$. The distribution $F_{M,N'N,N',\lambda_1}$ given by (5.22) is of order $\omega_2$, and the distribution $S_{M,N'N,N',\lambda_1}$ given by (5.23) is of order $\omega_1$. Therefore, $R_{M,N'N,N'}$ is of MP order $(\omega_1, \omega_2)$.

From the above result, we see that Lemma 2.2 and the order of the Schwartz distributions defined in Section 2 are insensitive with respect to the multiparameter setting. For instance, suppose $f_1 \in \mathcal{S}'(\mathbb{R})$ and $f_2 \in \mathcal{S}'(\mathbb{R})$ are of orders $\omega_1$ and $\omega_2$, respectively. Then their tensor product $f = f_1 \otimes f_2 \in \mathcal{S}'(\mathbb{R}^2)$ is of order $\max(\omega_1, \omega_2)$. On the other hand, $f$ is of MP order $(\omega_1, \omega_2)$. Thus, if we construct the renormalization, $R_{M,N'N,N'}$, by using the notion of order instead of the MP order, the highest degree for the monomials $x_1^{\lambda_1}$ and $x_2^{\lambda_2}$ in (5.20) is $[\max(\omega_1, \omega_2)]$ and this is definitely not the optimal result. Furthermore, in Theorem 5.4, we obtain the best result for the degrees of the monomials $x_1^{\lambda_1}$ and $x_2^{\lambda_2}$. This fact can be proved by considering the MP Littlewood–Paley analysis of the separable Schwartz distributions, $x_1^{k_1} \otimes x_2^{k_2}$, $k_1, k_2 \in \mathbb{N}$.

For Theorem 5.4, there are some interesting examples that cannot be covered by the assumption $\varphi \in \mathcal{S}(\mathbb{R}^2)$. For instance, let $\varphi$ be a Daubechies wavelet in $\mathbb{R}$. We are allowed to use the function $\Theta(x_1, x_2) = \Phi(x_1)\Phi(x_2)$ as the analyzing function for the MP Littlewood–Paley analysis where $\hat{\Phi} = |\hat{\varphi}|^2$, because it satisfies (5.3). However, it does not belong to $\mathcal{S}(\mathbb{R}^2)$. Similar to the one parameter case, the condition $\varphi \in \mathcal{S}(\mathbb{R}^2)$ can be relaxed if we are interested in the MP Littlewood–Paley analysis for $f \in \mathcal{S}'(\mathbb{R}^2)$ with MP order $(\omega_1, \omega_2)$. It is sufficient to assume that $\varphi \in \mathcal{C}_{\eta_1,\eta_2}(\mathbb{R}^2)$ with $\eta_i > \omega_i + \epsilon$ and $[\eta_i] = [\omega_i]$ satisfies

$$\|\varphi(x_1 - \cdot, x_2) - \varphi(y_1 - \cdot, x_2)\|_{\mathcal{C}_{\eta_1,\eta_2}} \le C|x_1 - y_1|^{\eta_1 - [\eta_1]},$$

$$\|\varphi(x_1, x_2 - \cdot) - \varphi(x_1, y_2 - \cdot)\|_{\mathcal{C}_{\eta_1,\eta_2}} \le C|x_2 - y_2|^{\eta_2 - [\eta_2]},$$

$$\|\varphi((x_1, x_2) - \cdot) - \varphi((x_1, y_2) - \cdot) - \varphi((y_1, x_2) - \cdot) + \varphi((y_1, y_2) - \cdot)\|_{\mathcal{C}_{\eta_1,\eta_2}} \le C|x_1 - x_2|^{\eta_1 - [\eta_1]}|y_1 - y_2|^{\eta_2 - [\eta_2]},$$

where the constant $C > 0$ is independent of $x_i$ and $y_i$, $i = 1, 2$. It is clear that the above conditions also guarantee the convergence (5.16) and the inequality (5.17) in Theorem 5.3, if $\alpha$ and $\beta$ are replaced by $\eta$ and $\omega$, respectively. Compared to the corresponding results for the one parameter Littlewood–Paley analysis, we impose a stronger condition on $\varphi$. The main obstacle for the multiparameter version is that we cannot estimate $((\partial^\gamma\varphi)_{i,j} * \psi)(x) - ((\partial^\gamma\varphi)_{i,j} * \psi)(y)$ and $(\varphi_{i,j} * \partial^\gamma\psi)(x) - (\varphi_{i,j} * \partial^\gamma\psi)(y)$ by using different representations as (4.4) and (4.7), respectively, in the cases $i \ge 0$, $j < 0$ and $i < 0$, $j \ge 0$. (See Remark 4.1).

Finally, note that the non-separable analyzing function arises naturally from the separable analyzing function. Let $\Theta(x_1, x_2)$ define as above and $B(x_1, x_2)$ be a smooth non-separable function satisfying $|\hat{B}(\xi_1, \xi_2)| = 1$ almost everywhere. Then the function $(\Theta * B)(x_1, x_2)$ is a non-separable analyzing function.

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## References

[1] M. Bownik and K. P. Ho, *Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces.* Trans. Amer. Math. Soc. **358**(2006), no. 4, 1469–1510.

[2] S.-Y. Chang and R. Fefferman, *A continuous version of duality of $H^1$ with BMO on the bidisc.* Ann. of Math. **112**(1980), no. 1, 179–201.

[3] ———, *The Calderón-Zygmund decomposition on product domains.* Amer. J. Math. **104**(1982), no. 3, 455–468.

[4] ———, *Some recent developments in Fourier analysis and $H^p$-theory on product domains.* Bull. Amer. Math. Soc. **12**(1985), no. 1, 1–43.

[5] R. Fefferman, *The atomic decomposition of $H^1$ in product spaces.* Adv. in Math. **55**(1985), no. 1, 90–100.

[6] ———, *Singular integrals on product $H^p$ spaces.* Rev. Mat. Iberoamericana **1**(1985), no. 2, 25–31.

[7] ———, *Fourier analysis in several parameters.* Rev. Mat. Iberoamericana **2**(1986), no. 1-2, 89–98.

[8] ———, *Harmonic analysis on product spaces.* Ann. of Math. **126**(1987), no. 1, 109–130.

[9] ———, *Multiparameter Calderón-Zygmund theory.* In: Harmonic Analysis and Partial Differential Equations, 207–221, University of Chicago Press, Chicago, IL, 1999, pp. 207–221.

[10] M. Frazier, and B. Jawerth, *A discrete transform and decomposition of distribution spaces.* J. Funct. Anal. **93**(1990), no. 1, 34–170.

https://doi.org/10.4153/CJM-2008-055-x Published online by Cambridge University Press

[11] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces.* CBMS Regional Conference Series in Mathematics 79, American Mathematical Society, Providence, RI, 1991.
[12] R. F. Gundy, and E. M. Stein, $H^p$ *theory for the poly-disc.* Proc. Nat. Acad. Sci. U.S.A. **76**(1979), no. 3, 1026–029.
[13] K.-P. Ho, *Frames associated with expansive matrix dilations.* Collect. Math. **54**(2003), no. 3, 217–254.
[14] J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series. I.* J. London Math. Soc. **6**(1931), 230–233.
[15] Y. Meyer, *Wavelets and Operators.* Cambridge Studies in Advanced Mathematics 37, Cambridge University Press, Cambridge, 1992.
[16] ———, *Wavelets, Vibrations and Scalings.* CRM Monograph Series 9, American Mathematical Society, Providence, RI, 1998.
[17] J. Peetre, *New Thoughts on Besov Spaces.* Duke University Mathematics Series 1, Mathematics Department, Duke University, Durham, NC. 1976.
[18] W. Rudin, *Functional Analysis.* Second edition. McGraw-Hill, New York, 1991.

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