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IDEALS OF FREE INVERSE SEMIGROUPS

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Abstract

It is shown that no proper ideal of a free inverse semigroup is free and that every isomorphism between ideals is induced by a unique automorphism of the whole semigroup. In addition, necessary and sufficient conditions are given for two principal ideals to be isomorphic.

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This paper is concerned with various properties of (two-sided) ideals of a free inverse semigroup S, the investigation being based on Scheiblich's structure theorem for S (Scheiblich (1972), (1973)).

We show first that no proper ideal of S is a free inverse subsemigroup of S (Theorem 1). A complementary result, stating that every ideal of S contains an isomorphic copy of S, is a consequence of a theorem established by the author in a separate note (Munn (to appear)).

The second topic discussed is that of isomorphisms between ideals of S. Every isomorphism from one ideal of S to another can be extended to a unique automorphism of S (Theorem 2): consequently, the automorphism group of an ideal of S is a subgroup of the automorphism group of S itself. A description of the automorphism group of S has been provided by O'Carroll (1974).

Necessary and sufficient conditions are given for two principal ideals of S to be isomorphic (Theorem 3) and for the inverse subsemigroups of S generated by two $\mathscr{J}$-classes to be isomorphic (Theorem 4). Theorems 3 and 4 answer two questions raised by Reilly (1976).

The notation and terminology throughout will be that of Howie (1976).

Let S be an inverse semigroup. The semilattice of S and the automorphism group of S will be denoted by $E(S)$ and aut S respectively. Also, if A is a nonempty subset of S then A^{-1} denotes $\{a^{-1}: a \in A\}$ and so the inverse subsemigroup of S generated by A is just $\langle A \cup A^{-1}\rangle$, the subsemigroup generated by $A \cup A^{-1}$.

The following easily-checked facts will be used below without subsequent comment. If V is an inverse subsemigroup of S then any $\mathscr{R}$-class [$\mathscr{J}$-class] of S contained in V is also an $\mathscr{R}$-class [$\mathscr{J}$-class] of V. Moreover, if T and U are inverse subsemigroups of S and $t \in T$ is such that $R_t \subseteq T$ then, for any isomorphism θ: $T \to U$ for which $R_{t\theta} \subseteq U$, we have that $R_t\theta = R_{t\theta}$.

For the remainder of the paper X will denote a nonempty set. Let $G = \mathscr{F}\mathscr{G}_X$, the free group on X, and let 1 denote the identity of G. The *length* $l(a)$ of $a \in G$ is defined by

$$l(a) = \begin{cases} n & \text{if the reduced form of } a \text{ is } x_1 x_2 \dots x_n \quad (x_i \in X \cup X^{-1}), \\ 0 & \text{if } a = 1. \end{cases}$$

For all $a \in G$ the set of all initial segments (including 1) of the reduced form of a will be denoted by $\bar{a}$; further, for all nonempty subsets A of G we write

$$\bar{A} = \{\bar{a}\colon a \in A\}$$

and we say that A is *left closed* if and only if $\bar{A} = A$.

We now describe the construction for the free inverse semigroup on X given by Scheiblich ((1972), (1973)). Let $\mathscr{Y}$ denote the set of all finite left closed subsets of G with at least two elements. Write

$$S = \{(A, g) \in \mathscr{Y} \times G\colon g \in A\}.$$

It is readily verified that if (A, g) and (B, h) are in S then $A \cup gB \in \mathscr{Y}$; hence we can define a multiplication on S by the rule that

$$(A, g)(B, h) = (A \cup gB, gh).$$

With respect to this multiplication S is an inverse semigroup in which

$$(\forall (A, g) \in S), \quad (A, g)^{-1} = (g^{-1}A, g^{-1})$$

and

$$E(S) = \{(A, 1)\colon A \in \mathscr{Y}\}.$$

Let us write

$$W = \{(\bar{x}, x)\colon x \in X\}.$$

(Note that, for all $x \in X$, $\bar{x} = \{1, x\}$.) Then $S = \langle W \cup W^{-1} \rangle$ and each mapping from W to an inverse semigroup T extends to a unique homomorphism from S to T. Accordingly, S is the free inverse semigroup on X and W is a set of free generators of S (see Reilly (1972), (1973)). We denote S, as constructed above, by $\mathscr{F}\mathscr{I}_X$. The cardinal $|X|$ of X is termed the *rank* of S.

The mapping π: $S \to G$ defined by

$$(A, g)\pi = g$$

is evidently a surjective homomorphism. Now suppose that T is an inverse

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subsemigroup of S. It is straightforward to prove that

(P1) $(\forall a, b \in T), \quad a\pi = b\pi \Leftrightarrow ea = eb \quad$ for some $e \in E(T)$.

Thus π induces the least group congruence on T(Munn (1961)). Moreover, if T is an ideal of S then $T\pi = G$.

It will be convenient to denote the $\mathscr{Y}$-component of a typical element a of S by $\mathscr{S}(a)$. Thus

$$(\forall a \in S), \quad a = (\mathscr{S}(a), a\pi).$$

Green's relations on S are characterized in Reilly (1972) and further properties of S are listed in Reilly (1976). We note, in particular, that S is combinatorial and completely semisimple and that its partially ordered set of $\mathscr{J}$-classes satisfies the maximal condition. Since elements a and b of S are $\mathscr{R}$-equivalent if and only if $\mathscr{S}(a) = \mathscr{S}(b)$, we have that

(P2) $(\forall a \in S), \quad R_a \pi = \mathscr{S}(a)$.

Each $\mathscr{J}$-class of S is finite: specifically,

(P3) $(\forall a \in S), \quad |J_a| = |\mathscr{S}(a)|^2$.

Reilly (1972) has shown that every set of free generators of S is contained in $W \cup W^{-1}$. The following property therefore follows from (P3).

(P4) If a belongs to a set of free generators of S then $|J_a| = 4$.

For ease of reference we also record that

(P5) $(\forall a, b \in S), \quad (a, b) \in \mathscr{J} \Leftrightarrow \mathscr{S}(a) = g^{-1}\mathscr{S}(b) \quad$ for some $g \in \mathscr{S}(b)$.

Unlike the corresponding situation for free groups, not every inverse subsemigroup of a free inverse semigroup S is free; for example, $E(S)$ is not a free inverse semigroup. By a *proper* ideal of S we mean an ideal other than S itself. We now establish

THEOREM 1. *No proper ideal of a free inverse semigroup S is a free inverse subsemigroup of S.*

PROOF. Take $S = \mathscr{F}\mathscr{I}_X$. Suppose that M is an ideal of S which is also a free inverse subsemigroup of S. We shall show that $M = S$.

Let K be a set of free generators of M and let $a \in K$. Then, by (P4) (with M replacing S), the $\mathscr{J}$-class of M containing a has exactly 4 elements. Since M is an ideal of S this means that $|J_a| = 4$. Consequently, by (P3), $|\mathscr{S}(a)| = 2$ and so, since $a^2 \neq a$, we have that $a \in W \cup W^{-1}$. Thus $K \subseteq W \cup W^{-1}$ and therefore $K \cup K^{-1} \subseteq W \cup W^{-1}$. Hence $(K \cup K^{-1})\pi \subseteq (W \cup W^{-1})\pi = X \cup X^{-1}$.

Now suppose that there exists $b \in (W \cup W^{-1}) \backslash (K \cup K^{-1})$. Then $b\pi \in (X \cup X^{-1}) \backslash (K \cup K^{-1})\pi$, since $\pi|_{W \cup W^{-1}}$ is injective. Let $e \in E(M)$. Then $eb \in M$ and so there exist elements $k_1, k_2, \ldots, k_n$ in $K \cup K^{-1}$ such that

$$k_1 k_2 \ldots k_n = eb.$$

Since $e\pi = 1$ it follows that

$$(k_1 \pi)(k_2 \pi) \ldots (k_n \pi)(b\pi)^{-1} = 1.$$

But

$$k_i \pi \in (K \cup K^{-1})\pi \subseteq X \cup X^{-1} \backslash \{b\pi, (b\pi)^{-1}\} \quad (i = 1, 2, \ldots, n)$$

and so we have a contradiction. Thus $K \cup K^{-1} = W \cup W^{-1}$. Hence $M = \langle K \cup K^{-1} \rangle = \langle W \cup W^{-1} \rangle = S$ and the proof is complete.

REMARK. Although a proper ideal of a free inverse semigroup S is not itself a free inverse semigroup, it contains an isomorphic copy of S (Munn (to appear) Remark 3).

We now examine isomorphisms between ideals of a free inverse semigroup. To save repetition we shall assume that $S = \mathscr{F}\mathscr{I}_X$ and $G = \mathscr{F}\mathscr{G}_X$ in the three lemmas below.

LEMMA 1. *Let T and U be inverse subsemigroups of S and let θ: $T \to U$ be an isomorphism. Then there exists an isomorphism ϕ: $T\pi \to U\pi$ such that, for all $a \in T$, $a\pi\phi = a\theta\pi$. Suppose, further, that $t \in T$ is such that $R_t \subseteq T$ and $R_{t\theta} \subseteq U$. Then*

$$(\forall a \in R_t), \quad a\theta = (\mathscr{S}(a)\varphi, a\pi\varphi).$$

(*Note that if $a \in R_t$ then $\mathscr{S}(a) \subseteq T\pi$, since $\mathscr{S}(a) = R_t\pi$, by* (P2).)

PROOF. Let $a, b \in T$ be such that $a\pi = b\pi$. Then, by (P1), there exists $e \in E(T)$ such that $ea = eb$ and so $e\theta a\theta = e\theta b\theta$. Since $e\theta \in E(U)$, this shows that $a\theta\pi = b\theta\pi$. Thus we can define a mapping φ: $T\pi \to U\pi$ by the rule that

$$(1) \qquad (\forall a \in T), \quad a\pi\varphi = a\theta\pi.$$

Since θ is surjective, φ is surjective. Now suppose that $a, b \in T$ are such that $a\pi\phi = b\pi\varphi$. Then, by (P1), there exists $f \in E(U)$ such that $f(a\theta) = f(b\theta)$ and so $(ea)\theta = (eb)\theta$, where $e = f\theta^{-1} \in E(T)$. Thus, since θ is injective, $ea = eb$. Consequently, by (P1), $a\pi = b\pi$. Hence φ is injective. Since θ and π are homomorphisms, so also is φ. Thus φ is an isomorphism.

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Next, let $a \in R_t$. Since $R_t \subseteq T$ and $R_{t\theta} \subseteq U$ it follows that $R_a\theta = R_t\theta = R_{t\theta} = R_{a\theta}$ and so

$$\begin{aligned} a\theta &= (\mathscr{S}(a\theta), a\theta\pi) \\ &= (R_{a\theta}\pi, a\theta\pi) && \text{by (P2)} \\ &= (R_a\theta\pi, a\theta\pi) \\ &= (R_a\pi\varphi, a\pi\varphi) && \text{by (1)} \\ &= (\mathscr{S}(a)\varphi, a\pi\varphi) && \text{by (P2).} \end{aligned}$$

This completes the proof.

DEFINITION. An automorphism φ of $G = \mathscr{F}\mathscr{G}_X$ is *special* if and only if $(X \cup X^{-1})\varphi = X \cup X^{-1}$.

Let the set of all special automorphisms of G be denoted by $\operatorname{aut}^* G$. It is clear that $\operatorname{aut}^* G$ is a subgroup of $\operatorname{aut} G$. If X is finite with exactly n elements then $|\operatorname{aut}^* G| = 2^n n!$.

Theorem 2 of O'Carroll (1974) shows, in effect, that

$$\operatorname{aut} S \cong \operatorname{aut}^* G.$$

LEMMA 2. *Let M and N be ideals of S and let θ: $M \to N$ be an isomorphism. Then there exists $\varphi \in \operatorname{aut}^* G$ such that*

$$(\forall a \in M), \quad a\theta = (\mathscr{S}(a)\varphi, a\pi\varphi).$$

PROOF. Since M and N are ideals of S, each is a union of $\mathscr{R}$-classes of S. Also $M\pi = G = N\pi$. Hence, by Lemma 1, there exists $\varphi \in \operatorname{aut} G$ such that

$$(2) \qquad (\forall a \in M), \quad a\theta = (\mathscr{S}(a)\varphi, a\pi\varphi).$$

It remains to show that φ is special.

Suppose that there exists $x \in X \cup X^{-1}$ such that $l(x\varphi) > 1$. Let $e \in E(M)$ and let $k = |\mathscr{S}(e)|$. Since $\mathscr{S}(e)$ is finite and left closed there exists a nonnegative integer r such that $x^n \in \mathscr{S}(e)$ if $0 \leqslant n \leqslant r$ and $x^n \notin \mathscr{S}(e)$ if $n > r$. (By convention, $x^0 = 1$.) Let us write $f = eg$, where $g = (\overline{x^{r+2k}}, 1)$. Then $f \in M$, since M is an ideal of S. Now let $A = \{x^{r+1}, x^{r+2}, \ldots, x^{r+2k}\}$. We have that

$$\mathscr{S}(f) = \mathscr{S}(e) \cup A, \quad \mathscr{S}(e) \cap A = \emptyset$$

and so $|\mathscr{S}(f)| = 3k$. Thus, since φ is injective,

$$(3) \qquad |\mathscr{S}(f)\varphi| = 3k.$$

But $A\varphi \subseteq \mathscr{S}(f)\varphi$; also $\mathscr{S}(f)\varphi$ is left closed since $\mathscr{S}(f)\varphi = \mathscr{S}(f\theta)$, by (2). Hence $\overline{A\varphi} \subseteq \mathscr{S}(f)\varphi$. Thus, from (3),

$$|\overline{A\varphi}| \leqslant 3k. \tag{4}$$

Let $u, v \in X \cup X^{-1}$ be the first and last letters, respectively, of $x\varphi$. Then, for all $n \in \mathbf{N}$, u and v are the first and last letters, respectively, of $(x\varphi)^n$. Consider the elements listed below:

$$\begin{aligned} &(x\varphi)^{r+1}, (x\varphi)^{r+2}, \ldots, (x\varphi)^{r+2k}, \\ &(x\varphi)^{r+1}u, (x\varphi)^{r+2}u, \ldots, (x\varphi)^{r+2k-1}u. \end{aligned} \tag{5}$$

Let $p, q \in \mathbf{N}$. If $p \neq q$ then $(x\varphi)^p \neq (x\varphi)^q$ and $(x\varphi)^p u \neq (x\varphi)^q u$. Now suppose that $(x\varphi)^p = (x\varphi)^q u$. Then $p \neq q$, since $u \neq 1$, and so $l((x\varphi)^{p-q}) > 1$. But $(x\varphi)^{p-q} = u$ and $l(u) = 1$, which is a contradiction. Thus the $4k-1$ elements in the list (5) are distinct. Furthermore, for all integers p such that $r+1 \leqslant p \leqslant r+2k-1$, $(x\varphi)^p u$ is an initial segment of $(x\varphi)^{p+1}$ if $u \neq v^{-1}$ and is an initial segment of $(x\varphi)^p$ if $u = v^{-1}$. It follows that all the elements in the list (5) lie in $\overline{A\varphi}$. Hence $|\overline{A\varphi}| \geqslant 4k-1$. But this contradicts (4), since $k > 1$. Consequently, $l(x\varphi) = 1$ for all $x \in X \cup X^{-1}$; that is, $(X \cup X^{-1})\varphi \subseteq X \cup X^{-1}$.

Suppose that $(X \cup X^{-1})\varphi \neq X \cup X^{-1}$. Then there exists $y \in X \cup X^{-1}$ such that $y\varphi^{-1} \notin X \cup X^{-1}$. Hence

$$l(y\varphi^{-1}) > 1. \tag{6}$$

Now θ^{-1} is an isomorphism from N to M. By analogy with (2), there exists $\psi \in \operatorname{aut} G$ such that

$$(\forall b \in N), \quad b\theta^{-1} = (\mathscr{S}(b)\psi, b\pi\psi).$$

Thus, for all $a \in M$, $a\pi = (a\theta)\theta^{-1}\pi = (a\theta)\pi\psi$ and so $a\pi\psi^{-1} = a\theta\pi = a\pi\varphi$. Since $M\pi = G$ this implies that $\psi^{-1} = \varphi$. Hence $\psi = \varphi^{-1}$. The same argument as before, with $N, M, \theta^{-1}, \varphi^{-1}$ replacing M, N, θ, φ respectively, now shows that (6) leads to a contradiction. Hence $(X \cup X^{-1})\varphi = X \cup X^{-1}$; that is, φ is special.

Lemma 3. *Let $\varphi \in \operatorname{aut}^* G$. Then there exists an automorphism α of S such that*

$$(\forall a \in S), \quad a\alpha = (\mathscr{S}(a)\varphi, a\pi\varphi).$$

Proof. We first note that, for all $g \in G$, $\overline{g\phi} \sim \bar{g}\phi$ and so, for all $a \in S$, $\mathscr{S}(a)\phi$ is left closed. Hence we can define a mapping α: $S \to S$ by

$$(\forall a \in S), a\alpha = (\mathscr{S}(a)\phi, a\pi\phi).$$

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Similarly, we can define $\alpha'\colon S \to S$ by

$$(\forall a \in S), \quad a\alpha' = (\mathscr{S}(a)\phi^{-1}, a\pi\phi^{-1}).$$

Then $\alpha'\alpha = \iota = \alpha\alpha'$, where ι is the identity mapping on S. Hence α is bijective. Moreover, α is a homomorphism, since ϕ is a homomorphism. Thus $\alpha \in \operatorname{aut} S$.

REMARK. From Lemma 2 (with $M = N = S$) and Lemma 3 we can recover O'Carroll's theorem linking $\operatorname{aut} S$ and $\operatorname{aut}^* G$.

We now come to the second main result.

THEOREM 2. *Let S be a free inverse semigroup, let M and N be ideals of S and let $\theta\colon M \to N$ be an isomorphism. Then there exists a unique automorphism α of S such that $\alpha|_M = \theta$.*

PROOF. Let $S = \mathscr{F}\mathscr{I}_X$, as before. By Lemma 2, there exists $\varphi \in \operatorname{aut}^* G$ such that

$$(\forall a \in M), \quad a\theta = (\mathscr{S}(a)\varphi, a\pi\varphi).$$

Hence, by Lemma 3, there exists $\alpha \in \operatorname{aut} S$ such that

$$(\forall a \in S), \quad a\alpha = (\mathscr{S}(a)\varphi, a\pi\varphi).$$

Thus $\alpha|_M = \theta$.

Now suppose that $\beta \in \operatorname{aut} S$ is such that $\beta|_M = \theta$. Then, by Lemma 2 (with $M = N = S$), there exists $\psi \in \operatorname{aut}^* G$ such that

$$(\forall a \in S), \quad a\beta = (\mathscr{S}(a)\psi, a\pi\psi).$$

Hence, since $a\alpha = a\beta$ for all $a \in M$, we have that

$$(\forall a \in M), \quad a\pi\varphi = a\pi\psi.$$

But $M\pi = G$. Consequently $\phi = \psi$ and so $\alpha = \beta$.

By specializing to the case $M = N$ we readily obtain the following corollary concerning $\operatorname{aut} M$.

COROLLARY. *Let M be an ideal of a free inverse semigroup of S. Then*

$$\operatorname{aut} M \cong \{\alpha \in \operatorname{aut} S\colon M\alpha = M\}.$$

Thus, since $\operatorname{aut} S \cong \operatorname{aut}^* G$, we see that $\operatorname{aut} M$ is isomorphic to a subgroup of

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aut* G. In particular, if S has finite rank then aut M is finite.

As an application of the corollary above, consider the following sequence of ideals of S. For each $n \in \mathbf{N}$ let us write

$$S_n = \{a \in S: |\mathscr{S}(a)| \geqslant n+1\}.$$

It is almost immediate that each S_n is an ideal of S and that

$$S = S_1 \supset S_2 \supset S_3 \supset \dots.$$

Now, for all $n \in \mathbf{N}$ and all $\alpha \in \operatorname{aut} S$, we have that $S_n\alpha = S_n$, as can easily be verified with the help of (P3). Hence, from the corollary,

$$(\forall n \in \mathbf{N}), \quad \operatorname{aut} S_n \cong \operatorname{aut} S.$$

On the other hand, since the number of elements in a maximal $\mathscr{J}$-class of S_k is $(k+1)^2$ it follows that

$$(\forall m, n \in \mathbf{N}), \quad S_m \cong S_n \Leftrightarrow m = n.$$

In the special case where S has rank 1 the ideals S_n $(n \in \mathbf{N})$ are the only ideals of S and hence all ideals have the same automorphism group, namely the group of order 2. It will be shown later that if S has finite rank greater than 1 then S possesses a principal ideal whose automorphism group is trivial.

Next we give a method for testing whether two principal ideals of a free inverse semigroup are isomorphic.

Theorem 3. *Let $S = \mathscr{F}\mathscr{I}_X$, let $G = \mathscr{F}\mathscr{G}_X$ and let $a, b \in S$. Then $SaS \cong SbS$ if and only if there exist $\varphi \in \operatorname{aut}^* G$ and $g \in \mathscr{S}(b)$ such that $\mathscr{S}(a)\varphi = g^{-1}\mathscr{S}(b)$.*

Proof. Suppose first that there exists an isomorphism $\theta: SaS \to SbS$. Then, by Lemma 2, there exists $\varphi \in \operatorname{aut}^* G$ such that

$$(\forall c \in SaS), \quad c\theta = (\mathscr{S}(c)\varphi, c\pi\varphi).$$

In particular, $a\theta = (\mathscr{S}(a)\varphi, a\pi\varphi)$ and so $\mathscr{S}(a\theta) = \mathscr{S}(a)\varphi$. But J_a and J_b are the greatest $\mathscr{J}$-classes of SaS and SbS respectively and so $(a\theta, b) \in \mathscr{J}$. Thus, by (P5), there exists $g \in \mathscr{S}(b)$ such that $\mathscr{S}(a\theta) = g^{-1}\mathscr{S}(b)$. Consequently, $\mathscr{S}(a)\varphi = g^{-1}\mathscr{S}(b)$.

Conversely, suppose that there exist $\varphi \in \operatorname{aut}^* G$ and $g \in \mathscr{S}(b)$ such that $\mathscr{S}(a)\varphi = g^{-1}\mathscr{S}(b)$. By Lemma 3, there exists $\alpha \in \operatorname{aut} S$ such that

$$(\forall a \in S), \quad a\alpha = (\mathscr{S}(a)\varphi, a\pi\varphi).$$

Now $b = (\mathscr{S}(b), b\pi)$ and so, since there exists $g \in \mathscr{S}(b)$ such that $g^{-1}\mathscr{S}(b) = \mathscr{S}(a)\varphi = \mathscr{S}(a\alpha)$, it follows from (P5) that $(b, a\alpha) \in \mathscr{J}$. Thus $(SaS)\alpha = S(a\alpha)S = SbS$; hence $SaS \cong SbS$.

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The result can be expressed in a simple form making use of the author's concept of a 'word-tree' (Munn (1974)). Each $\mathcal{J}$-class of $S = \mathcal{F}\mathcal{I}_X$ corresponds to an unrooted word-tree and two principal ideals of S are isomorphic if and only if the word-trees corresponding to their generating $\mathcal{J}$-classes are obtainable from each other by reversing the orientation of those edges labelled by elements of some subset of X and then relabelling all the edges by applying a permutation to X.

The argument used in the first part of the proof of Theorem 3 enables us to show that a free inverse semigroup of finite rank greater than 1 has a principal ideal whose automorphism group is trivial. Let $S = \mathcal{F}\mathcal{I}_X$ and $G = \mathcal{F}\mathcal{G}_X$, where $2 \leqslant |X| = n \in \mathbf{N}$, and let the elements of X be $x_1, x_2, \ldots, x_n$. Take

$$a = (\bar{x}_1 \cup \overline{x_2^2} \cup \overline{x_3^3} \cup \ldots \cup \overline{x_n^n}, 1)$$

and let $\theta \in \operatorname{aut} SaS$. Then there exist $\varphi \in \operatorname{aut}^* G$ and $g \in \mathcal{S}(a)$ such that

(i) $(\forall c \in SaS), \quad c\theta = (\mathcal{S}(c)\varphi, c\pi\varphi)$

and

(ii) $\mathcal{S}(a)\varphi = g^{-1}\mathcal{S}(a)$.

The elements of $\mathcal{S}(a)\varphi$ of length 1 are $x_1\varphi, x_2\varphi, \ldots, x_n\varphi$. Suppose that $g = x_i^r$, where $1 \leqslant i \leqslant n$ and $1 \leqslant r \leqslant i$. If $r \neq i$ then $g^{-1}\mathcal{S}(a)$ has precisely two elements of length 1, namely x_i^{-1} and x_i, while if $r = i$, it has precisely one of length 1, namely x_i^{-1}. In either case we obtain a contradiction from (ii). Consequently, $g = 1$. Hence, since φ is length-preserving, it follows from (ii) that φ is the identity automorphism of G. Thus, from (i), θ is the identity automorphism of SaS.

Theorem 3 provides a solution to a problem suggested by Reilly (1976). A second problem, also posed by Reilly in the same paper and related to the first, can be solved by similar techniques. We state the result as

THEOREM 4. *Let $S = \mathcal{F}\mathcal{I}_X$ and let $a, b \in S$. Then $\langle J_a \rangle \cong \langle J_b \rangle$ if and only if there exists an isomorphism*

$$\varphi\colon \langle J_a \rangle \pi \to \langle J_b \rangle \pi$$

and an element g in $\mathcal{S}(b)$ such that $\mathcal{S}(a)\varphi = g^{-1}\mathcal{S}(b)$. (Note that $\mathcal{S}(a) \subseteq \langle J_a \rangle \pi$, since $\mathcal{S}(a) = R_a\pi$, by (P2).)

PROOF. Write $T = \langle J_a \rangle$ and $U = \langle J_b \rangle$. Since $J_a^{-1} = J_a$ we see that T is an inverse subsemigroup of S. Similarly, U is an inverse subsemigroup of S.

Suppose that there exists an isomorphism $\theta\colon T \to U$. Since J_a and J_b are, respectively, the greatest $\mathcal{J}$-classes of T and U, it follows that $J_a\theta = J_b$. Hence, in

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particular, $R_{a\theta} \subseteq U$. Thus, by Lemma 1, there exists an isomorphism φ: $T\pi \to U\pi$ such that

$$a\theta = (\mathscr{S}(a)\varphi, a\pi\varphi).$$

Hence $\mathscr{S}(a\theta) = \mathscr{S}(a)\varphi$. But, by (P5), since $(a\theta, b) \in \mathscr{J}$ there exists $g \in \mathscr{S}(b)$ such that $\mathscr{S}(a\theta) = g^{-1}\mathscr{S}(b)$. Thus $\mathscr{S}(a)\varphi = g^{-1}\mathscr{S}(b)$.

Conversely, suppose that there exists an isomorphism φ: $T\pi \to U\pi$ and an element g in $\mathscr{S}(b)$ such that $\mathscr{S}(a)\varphi = g^{-1}\mathscr{S}(b)$. First, we note that, for all $h \in \mathscr{S}(a)$,

$$\begin{aligned}(h^{-1}\mathscr{S}(a))\varphi &= (h\varphi)^{-1}\mathscr{S}(a)\varphi \\ &= (h\varphi)^{-1}g^{-1}\mathscr{S}(b) \\ &= (g(h\varphi))^{-1}\mathscr{S}(b);\end{aligned}$$

furthermore, $g(h\varphi) \in \mathscr{S}(b)$, since $h\varphi \in \mathscr{S}(a)\varphi = g^{-1}\mathscr{S}(b)$. Combining this with (P5) we see that, for all $h \in \mathscr{S}(a)$ and all $k \in h^{-1}\mathscr{S}(a)$,

(7) $$((h^{-1}\mathscr{S}(a))\varphi, k\varphi) \in J_b.$$

Now consider an element $c \in T$. By the definition of T there exist $c_1, c_2, \ldots, c_r \in J_a$ such that $c = c_1c_2 \ldots c_r$. But $\mathscr{S}(c_i) = h_i^{-1}\mathscr{S}(a)$ for some h_i in $\mathscr{S}(a)$, by (P5), and so, by (7), $(\mathscr{S}(c_i)\varphi, c_i\pi\varphi) \in J_b$ $(i = 1, 2, \ldots, r)$. Thus

$$(\mathscr{S}(c)\varphi, c\pi\varphi) = (\mathscr{S}(c_1)\varphi, c_1\pi\varphi) \ldots (\mathscr{S}(c_r)\varphi, c_r\pi\varphi) \in \langle J_b \rangle = U.$$

We can therefore define a mapping θ: $T \to U$ by the rule that

(8) $$(\forall c \in T), \quad c\theta = (\mathscr{S}(c)\varphi, c\pi\varphi).$$

Clearly θ is a homomorphism; moreover, since φ has an inverse it follows that θ is injective. We show that θ is surjective. It will suffice to prove that $J_b \subseteq J_a\theta$.

Let $d \in J_b$. By (P5), there exists $k \in \mathscr{S}(b)$ such that $\mathscr{S}(d) = k^{-1}\mathscr{S}(b)$. But $\mathscr{S}(b) = g(\mathscr{S}(a)\varphi)$, by hypothesis. Hence $k = g(h\varphi)$ for some $h \in \mathscr{S}(a)$ and so

$$k^{-1}\mathscr{S}(b) = (h\varphi)^{-1}g^{-1}[g(\mathscr{S}(a)\varphi)];$$

that is,

(9) $$\mathscr{S}(d) = (h^{-1}\mathscr{S}(a))\varphi.$$

Also $d\pi \in \mathscr{S}(d)$ and so, by (9), $d\pi = m\varphi$ for some $m \in h^{-1}\mathscr{S}(a)$. But $(h^{-1}\mathscr{S}(a), m) \in J_a$, by (P5). Hence, by (9) and (8),

$$d = ((h^{-1}\mathscr{S}(a))\varphi, m\varphi) = (h^{-1}\mathscr{S}(a), m)\theta \in J_a\theta.$$

Thus $J_b \subseteq J_a\theta$, as required.

The mapping θ is therefore an isomorphism and the proof is complete.

To conclude, we give an example to illustrate Theorems 3 and 4. Let $X = \{x, y, z\}$, let $S = \mathscr{F}\mathscr{I}_X$ and let $a, b \in S$ be defined as follows:

$$a = (\{1, x, xy, xyx\}, 1), \quad b = (\{1, x, z^{-1}, z^{-1}x\}, 1).$$

Let $G = \mathscr{F}\mathscr{G}_X$. Then it can be verified, by exhaustion of cases, that there does not exist (φ, g) in aut* $G \times \mathscr{S}(b)$ such that $\mathscr{S}(a)\varphi = g^{-1}\mathscr{S}(b)$. Thus, by Theorem 3, $SaS \not\cong SbS$.

Now $\langle J_a \rangle \pi$ and $\langle J_b \rangle \pi$ are, respectively, the free groups on $\{x, y\}$ and $\{x, z\}$. Let $\varphi\colon \langle J_a \rangle \pi \to \langle J_b \rangle \pi$ be the isomorphism defined by

$$x\varphi = x, \quad y\varphi = x^{-1}z.$$

Then

$$\mathscr{S}(a)\varphi = \{1, x, z, zx\} = z\mathscr{S}(b)$$

and $z^{-1} \in \mathscr{S}(b)$. Hence, by Theorem 4, $\langle J_a \rangle \cong \langle J_b \rangle$.

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