ON THE STEINBERG CHARACTER OF A FINITE SIMPLE GROUP OF LIE TYPE

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To Bernhard Hermann Neumann on his 60th birthday

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1. Introduction

Let K be an algebraically closed field of characteristic $p > 0$. If G is a connected, simple connected, semisimple linear algebraic group defined over K and σ an endomorphism of G onto G such that the subgroup G_σ of fixed points of σ is finite, Steinberg ([6], [7]) has shown that there is a complex irreducible character χ of G_σ with the following properties. χ vanishes at all elements of G_σ which are not semisimple, and, if $x \in G$ is semisimple, $\chi(x) = \pm n(x)$ where $n(x)$ is the order of a Sylow p-subgroup of $(Z_G(x))_\sigma$ ($Z_G(x)$ is the centraliser of x in G). If G is simple he has, in [6], identified the possible groups G_σ; they are the Chevalley groups and their twisted analogues over finite fields, that is, the 'simply connected' versions of finite simple groups of Lie type. In this paper we show, under certain restrictions on the type of the simple algebraic group G and on the characteristic of K, that χ can be expressed as a linear combination with integral coefficients of characters induced from linear characters of certain naturally defined subgroups of G_σ. This expression for χ gives an explanation for the occurence of $n(x)$ in the formula for $\chi(x)$, and also gives an interpretation for the ± 1 occuring in the formula in terms of invariants of the reductive algebraic group $Z_G(x)$.

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2. Notation and preliminary lemmas

In this section, unless otherwise stated, G denotes a connected reductive linear algebraic group defined over an algebraically closed field K of characteristic $p > 0$, and σ an endomorphism of G onto G such that G_σ is finite. T is a maximal torus of G such that both T and a Borel subgroup contaning T are both fixed by σ; T is unique up to conjugacy by an element of G_σ (see [6]). N is the normaliser of T in G and $N/T = W$. Then σ induces an automorphism of the finite group W.

The connected component of the identity of any closed subgroup H of G will be denoted by H_0. We refer to [1] for the relevant facts about algebraic groups. For any finite group H, $|H|$ denotes its order.

DEFINITION. Two elements $n_1, n_2 \in N$ are said to be σ-conjugate if there exists an element $n \in N$ such that $n_2 = nn_1 n^{-\sigma}$. For any $n \in N$ let

$Z(n)$ be the centraliser of n in N,

$$Z'(n) = [n' \in N | n'nn'^{-\sigma} = n]$$

and
$$Z''(n) = [n_0 \in N | n_0^{-1} nn_0 = n^{\sigma}].$$

$Z'(n)$ is a subgroup of N. $Z''(n)$ may be empty, but if there is an element $n_0 \in Z''(n)$ then $Z''(n) = Z(n)\, n_0$. Let

$$K(n) = [n' \in Z(n) | n' \text{ centralises } (Z(n) \cap T)_0].$$

If $w \in W$, the subgroup $Z'(w)$ and the subset $Z''(w)$ of W are defined similarly as

$$Z'(w) = [w' \in W | w'ww'^{-\sigma} = w]$$

and
$$Z''(w) = [w_0 \in W | w_0^{-1} ww_0 = w^{\sigma}].$$

$Z(w)$ is the centraliser of w in W. $K(w)$ is the subgroup of W of all elements of $Z(w)$ leaving elementwise fixed the connected component of the identity of the subgroup of T of fixed points of w.

We first state a result of Steinberg [6] which is a generalization of a well-known result of Lang [3].

LEMMA 1. (Steinberg) *Let H be a connected linear algebraic group over K and σ an endomorphism of H onto H such that H_σ is finite. Then the map $x \to xx^{-\sigma}$ of H into H is surjective.*

Our aim now is to define certain 'Cartan subgroups' of G_σ and to set up a one-to-one correspondence between the G_σ-conjugacy classes of these subgroups and σ-conjugacy classes of W. This correspondence is known in the case when G is defined and split over a finite field of q elements and σ is the q^{th} power map, but we have not been able to find a description of it in the literature.

LEMMA 2. *If two elements $n_0, n_1 \in N$ are σ-conjugate modulo T, they are σ-conjugate in N.*

PROOF. Let $n_1 = nn_0 n^{-\sigma} t$ where $n \in N$, $t \in T$. By Lemma 1 $n_1 = aa^{-\sigma}$ where $a \in G$. Hence

$$a^{-1} nn_0 n^{-\sigma} a^{\sigma} = a^{-\sigma} t^{-1} a^{\sigma}. \tag{2.1}$$

Since n_1 normalizes T, $a^{-\sigma} t^{-1} a^{\sigma} \in a^{-1} Ta$. Also $a^{-1} Ta$ is connected. Thus by Lemma 1 we can write

$$a^{-\sigma} t^{-1} a^{\sigma} = a^{-1} t_1 aa^{-\sigma} t_1^{-\sigma} a^{\sigma} = a^{-1} t_1 n_1 t_1^{-\sigma} a^{\sigma}$$

for some $t_1 \in T$. Together with (2.1) this yields $n_1 = t_1^{-1} n n_0 n^{-\sigma} t_1^{\sigma}$, showing that n_1, n_0 are σ-conjugate in N. This proves the lemma.

Suppose $n_0 \in N$ and $n_0 = aa^{-\sigma}$ $(a \in G)$. Then the maximal torus $a^{-1}Ta$ and its normaliser $a^{-1}Na$ are fixed by σ. Conversely if the maximal torus $a^{-1}Ta$ is fixed by σ then $aa^{-\sigma} \in N$. We investigate the G_σ-conjugacy of such maximal tori in the next lemma.

LEMMA 3. *Let $n_0, n_1 \in N$, $n_0 = aa^{-\sigma}$, $n_1 = bb^{-\sigma}(a, b \in G)$. Then $a^{-1}Ta$ and $b^{-1}Tb$ are conjugate by an element of G_σ if and only if n_0, n_1 are σ-conjugate in N (and hence, by Lemma 2, if and only if n_0T and n_1T are σ-conjugate in W).*

PROOF. If $n_0 = nn_1n^{-\sigma}$ $(n \in N)$ then $a^{-1}nb = g \in G_\sigma$ and $g^{-1}a^{-1}Tag = b^{-1}Tb$. Conversely let $g^{-1}a^{-1}Tag = b^{-1}Tb$ where $g \in G_\sigma$. Then $agb^{-1} = n \in N$, and $n_0 = nn_1n^{-\sigma}$.

It follows from Lemma 2 and 3 that to each maximal torus $a^{-1}Ta$ fixed by σ there corresponds the element $(aa^{-\sigma})T$ of W which is unique up to σ-conjugacy in W. Furthermore there is a one-to-one correspondence between G_σ-conjugacy classes of maximal tori of G fixed by σ and σ-conjugacy classes of W. For these maximal tori $a^{-1}Ta$ we now consider the subgroups $(a^{-1}Ta)_\sigma$ ('Cartan subgroups' of G_σ) and the subgroups $(a^{-1}Na)_\sigma$.

LEMMA 4. *Let $a^{-1}Ta$ be a maximal torus fixed by σ, and let $aa^{-\sigma} = n_0$, $n_0T = w_0 \in W$. Then*

$$(a^{-1}Na)_\sigma/(a^{-1}Ta)_\sigma$$

is isomorphic to $Z'(w_0)$.

PROOF. Let $a^{-1}na \in (a^{-1}Na)_\sigma$ $(n \in N)$ and consider the map $a^{-1}na \to nT = w$ of $(a^{-1}Na)_\sigma$ into W. Then $ww_0w^{-\sigma} = w_0$ and hence $(a^{-1}Na)_\sigma$ is mapped into $Z'(w_0)$. By the proof of Lemma 2 $(a^{-1}Na)_\sigma$ is actually mapped onto $Z'(w_0)$ and this proves the lemma.

The next lemma is basic for our construction of the Steinberg character of G_σ.

LEMMA 5. *Let $a_i^{-1}Na_i$ $(i = 1, 2, \cdots)$ be a set of representatives for the G_σ-conjugacy classes of subgroups $a^{-1}Na$ of G which are fixed by σ. Let $w_i = a_ia_i^{-\sigma}T$, so that $\{w_i\}$ is a set of representatives for the σ-conjugacy classes of W. Then*

$$\sum_i \frac{\varepsilon(w_i)}{|(a_i^{-1}Na_i)_\sigma|} = \frac{p^m}{|G_\sigma|}$$

where ε is the alternating character of W and p^m is the highest power of p dividing $|G_\sigma|$.

PROOF. σ induces a linear transformation σ^* of the real vector space V generated by the character group of T. We may regard W as acting on V. Let I be the ideal of W-invariant elements without constant term in the algebra of polyno-

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mials on V, and $J = I/I^2$. Steinberg ([6], 11.19) has shown, if G is semisimple, that

$$\det(\sigma_J^* - 1_J)^{-1} = \frac{p^m}{|G_\sigma|}. \tag{2.2}$$

We can extend this to the case when G is reductive, as follows. Let G' be the derived group of G, so that G' is semisimple. Then $(G/G')_\sigma \cong (T/T \cap G')_\sigma$ and, as G' is connected, $G_\sigma/G'_\sigma \cong (G/G')_\sigma$ ([6], 10.11). Hence $(G_\sigma : G'_\sigma) = |(T/T \cap G')_\sigma|$. Now $V = U \oplus V_1$ where $U(V_1)$ isomorphic to the real vector space generated by the character group of $T/T \cap G'$ $(T \cap G')$, and W acts trivially on U. Hence

$$\det(\sigma_J^* - 1_J)^{-1} = \frac{p^m}{|G'_\sigma|} \det(\sigma_U^* - 1_U)^{-1}.$$

However, by duality $\det(\sigma_U^* - 1_U) = |(T/T \cap G')|_\sigma$ and this yields the result.

Again, Steinberg ([6], 14.6) has shown that

$$\frac{1}{|W|} \sum_{w \in W} \det(\sigma^* w - 1)^{-1} = \det(\sigma_J^* - 1_J)^{-1}. \tag{2.3}$$

Suppose a maximal torus $a^{-1}Ta$ of G is fixed by σ and let $n_0 = aa^{-\sigma}$. Let $T_0 = [t \in T | n_0^{-1} t n_0 = t^\sigma]$. Clearly $(a^{-1}Ta)_\sigma = a^{-1}T_0 a$. Thus if $n_0 T = w_0$ then $|T_0| = |\text{Ker}(\sigma - w_0)|$ where w_0 is regarded as acting on T. But $|\text{Ker}(\sigma - w_0)| = \det(\sigma^* - w_0)$ where w_0 is regarded on the right hand side as acting on V. Thus, by (2.2), (2.3) and Lemma 4 we have

$$\frac{p^m}{|G_\sigma|} = \sum_i \frac{\varepsilon(w_i)}{|Z'(w_i)|} \det(\sigma^* - w_i) = \sum_i \frac{\varepsilon(w_i)}{|(a_i^{-1} N a_i)_\sigma|}$$

This proves the lemma.

REMARK. In the case when G is defined and split over a field of q elements and σ is the q^{th} power map, (2.3) follows from a classical formula of Molien (see eg [4]). In that case $|(a^{-1}Ta)_\sigma| = f(q)$ where $f(x)$ is the characteristic polynomial of $w = aa^{-\sigma}T$ acting on V. See also [5], p. 62.

3. Preparatory results

In this section our aim is to consider certain characters of the subgroups $(a_i^{-1} N a_i)_\sigma$ defined in Lemma 5 and the characters of G_σ induced from them. The results given below are preparatory to computing the values of these induced characters.

Until the end of Lemma 7, G satisfies the same assumptions as in § 2. Let $n \in N$, then $n \in (a^{-1}Na)_\sigma$ if and only if $aa^{-\sigma} \in Z''(n)$. Suppose $Z''(n) \neq \phi$ and let n_0 be a fixed element of $Z''(n)$, so that $Z''(n) = Z(n)n_0$. If now $n' \in Z(n)$ and $n_1 \in Z''(n)$ then $n' n_1 n'^{-\sigma} \in Z''(n)$ showing that $Z(n)$ acts on $Z''(n)$ by σ-conjugation.

Consider the equivalence relation R on $Z''(n)$ given by

(3.1) $(n_1 n_2) \in R$ if and only if $n_1 = n_2 t$ $((n_1, n_2) \in Z''(n) \times Z''(n))$,

where $t \in n_0^{-1}(Z(n) \cap T)_0 n_0$. It is easy to see that the action of $Z(n)$ on $Z''(n)$ by σ-conjugation induces an action of the finite group $U = Z(n)/(Z(n) \cap T)_0$ on the finite set B which is the quotient of $Z''(n)$ by the relation R.

LEMMA 6. *The number of orbits of U acting on B is equal to the number of $(Z(n), \sigma)$-conjugacy classes of $Z''(n)$ i.e. the number of orbits of $Z(n)$ acting on $Z''(n)$ by σ-conjugation.*

PROOF. Suppose $n_1, n_2 \in Z''(n)$ are such that for some $n' \in Z(n)$ we have $(n_2, n'n_1 n'^{-\sigma}) \in R$. Then we show that n_1 and n_2 are σ-conjugate by an element of $Z(n)$. We have $n_2 = n'n_1 n'^{-\sigma} t$ where $t \in n_0^{-1}(Z(n) \cap T)_0 n_0$. If $n_2 = cc^{-\sigma}$ $(c \in G)$ then $c^{-\sigma} t^{-1} c^{\sigma} = c^{-1} n' n_1 n'^{-\sigma} c^{\sigma}$. Now $n_2 n_0^{-1} \in Z(n)$ and hence $c^{-\sigma} t^{-1} c^{\sigma} \in c^{-1}(Z(n) \cap T)_0 c$. By arguing as in Lemma 2 we see that $n_2 = t_1^{-1} n' n_1 n'^{-\sigma} t_1$ for some $t_1 \in (Z(n) \cap T)_0$ and this proves the lemma.

We now suppose that the subgroups $a_i^{-1} N a_i$ $(i = 1, 2, \cdots)$ are chosen as in Lemma 5.

LEMMA 7. *Let $n \in N$ and suppose that $Z''(n) \neq \emptyset$. Then there exists a one-to-one correspondence between $(Z(n), \sigma)$-conjugacy classes of $Z''(n)$ and conjugacy classes of the $(a_i^{-1} N a_i)$ containing elements of the form $a_i^{-1} n'^{-1} n n' a_i$ $(n' \in N)$. Furthermore, if $a_i a_i^{-\sigma} = n_i$, the order of the centraliser of $a_i^{-1} n'^{-1} n n' a_i$ in $(a_i^{-1} N a_i)_\sigma$ is | $|Z(n) \cap Z'(n' n_i n'^{-\sigma})|$.*

PROOF. We note that $a_i^{-1} n a_i \in (a_i^{-1} N a_i)_\sigma$ if an only if $a_i a_i^{-\sigma} \in Z''(n)$. Thus, by replacing the subgroups $a_i^{-1} N a_i$ by conjugates by elements of G_σ if necessary, we see that to each (N, σ)-conjugacy class of $Z''(n)$ there corresponds a unique subgroup $(a_i^{-1} N a_i)_\sigma$ which contains $a_i^{-1} n a_i$. Choose a fixed such i, let $a_i = a$ and let $aa^{-\sigma} = n_0 \in Z''(n)$. We show that each $(Z(n), \sigma)$-conjugacy class of $Z''(n)$ contained in the (N, σ)-conjugacy class of n_0 gives rise to a conugacy class of $(a^{-1} N a)_\sigma$ containing an element of the form $a^{-1} n'^{-1} n n' a$ $(n' \in N)$ and conversely. Let $n_1 \in Z''(n)$ be such that $n_1 = n' n_0 n'^{-\sigma}$ where $n' \in N$, Since $n_1 \in Z''(n)$ we have $n_1^{-1} n n_1 = n^{\sigma}$, but this is equivalent to $a^{-1} n'^{-1} n n' a \in (a^{-1} N a)_\sigma$. Suppose n_1, $n_2 \in Z''(n)$, $n_i = n_i' n_0 n_i'^{-\sigma}$ $(n_i' \in N, i = 1, 2)$ and that $n_1 = n' n_2 n'^{-\sigma}$ where $n' \in Z(n)$. Then $x = a^{-1} n_1'^{-1} n' n_2' a \in (a^{-1} N a)_\sigma$ and $a^{-1} n_2'^{-1} n n_2 a = x^{-1} a^{-1} n_1'^{-1} n n_1' a\, x$. Conversely if $a^{-1} n_2'^{-1} n\, n_2 a = a^{-1} y^{-1} n_1'^{-1} n n_1' y a$ where $a^{-1} y a \in (a^{-1} N a)_\sigma$ then $n_1 = n' n_2 n'^{-\sigma}$ where $n' = n_1' y n_2'^{-1} \in Z(n)$. This proves the first part of the lemma.

Now let $a^{-1} n'^{-1} n n' a \in (a^{-1} N a)_\sigma$ and suppose $n' n_0 n'^{-\sigma} = bb^{-\sigma}$ $(b \in G)$. Then $b^{-1} n' a = g \in G_\sigma$ and $g a^{-1} n'^{-1} n n' a g^{-1} \in (b^{-1} N b)_\sigma$. Thus the centraliser of $a^{-1} n'^{-1} n n' a$ in $(a^{-1} N a)_\sigma$ is conjugate in G_σ to the centraliser of $b^{-1} n b$ in

$(b^{-1}Nb)_\sigma$, and it is sufficient to show that the order of the centraliser of $a^{-1}na$ in $(a^{-1}Na)_\sigma$ is $|Z(n) \cap Z'(n_0)|$. But this is clear and the lemma is proved.

For the rest of this section we will assume that G is simply connected and semisimple. For each subgroup $a^{-1}Na$ of G which is fixed by σ our aim is to define a certain linear character of $(a^{-1}Na)_\sigma$ which is trivial on $(a^{-1}Ta)_\sigma$. Since each such subgroup is conjugate by an element of G_σ to one of the $a_i^{-1}Na_i$, it is sufficient to define such characters of the $(a_i^{-1}Na_i)_\sigma$. For this it is sufficient, by Lemma 4, to define a linear character ψ_w of $Z'(w)$ for each w in a set of representatives for the σ-conjugacy classes of w.

Assume for the moment that we have chosen a linear character ψ_{w_i} of $Z'(w_i)$ (where, as before, $w_i = a_i a_i^{-\sigma} T$) and denote the corresponding character of $(a_i^{-1}Na_i)_\sigma$ also by ψ_{w_i}. Let

$$\phi = \sum_i \varepsilon(w_i)\psi^*_{w_i} \tag{3.2}$$

where $\psi^*_{w_i}$ is the character of G_σ induced from ψ_{w_i}.

In order that ϕ is the Steinberg character of G_σ the characters ψ_{w_i} should satisfy certain conditions which will be stated in Lemma 8. Finally the characters will be defined under certain assumptions on G in § 4 and ϕ will then be proved to be the Steinberg character of G_σ.

Lemma 8. *Let p be prime to $|W|$. Suppose we have, for each $w \in W$, a linear character ψ_w of $Z'(w)$ such that these characters have the following properties. If $w_2 = w'w_1w'-\sigma$ for some $w' \in W$ (and hence $Z'(w_2) = w'Z'(w_1)w'^{-1}$) then $\psi_{w_1}(w) = \psi_{w_2}(w'^{-1}ww')$ for any w in $Z'(w_1)$. Furthermore*

(3.3) *If $n \in N-T$, $Z''(n) = Z(n)n_0$, $Z(n)\,T/T = Y \subseteq W$ and $K(n)\,T/T = Y' \subseteq Y$, then for each coset $Y'c$ of Y' in Y,*

$$\sum_{w \in Y'cw_0} \psi_w(z)\varepsilon(w) = 0, \tag{3.4}$$

where $nT = z$, $n_0T = w_0$.

Using these characters let ϕ be defined as in (3.2). Then the contribution to ϕ from the elements of the $(a_i^{-1}Na_i)_\sigma$ which lie outside $(a_i^{-1}Ta_i)_\sigma$ is zero.

Proof. Let $n \in N-T$ and let n_0, w_0, Y, Y' be as in the statement of the lemma. We will consider all the conjugacy classes of each $(a_i^{-1}Na_i)_\sigma$ which contain elements of the form $a_i^{-1}n'^{-1}n\,n'a_i$ for some $n' \in N$. Since p is prime to $|W|$ these elements are semisimple. Hence, since these elements are all conjugate in G and G is simply connected they are conjugate in G_σ([6], 12.5) to, say, $x \in G_\sigma$. We show that the contribution to $\phi(x)$ from all such classes is zero. Since the set of all conjugacy classes of the $(a_i^{-1}Na_i)_\sigma$ which are contained in the conjugacy class of x in G is a union of sets of classes of this type, the lemma will then be proved.

We have the formula

$$\phi(x) = |Z_{G_\sigma}(x)| \sum_i \varepsilon(w_i) \sum_j \frac{1}{|Z_{(a_i^{-1}Na_i)_\sigma}(n_{ij})|} \psi_{w_i}(n_{ij})$$

where the second sum is over a set of representatives of conjugacy classes of $(a_i^{-1}Na_i)_\sigma$ containing elements conjugate to x in G_σ. Thus the contribution to $\phi(x)$ from a class of $(a_i^{-1}Na_i)_\sigma$ containing an element of the form $a_i^{-1}n'^{-1}nn'a_i$ is

$$\begin{aligned} &\frac{\varepsilon(w_i)|Z_{G_\sigma}(x)}{|Z_{(a_i^{-1}Na_i)_\sigma}(a_i^{-1}n'^{-1}nn'a_i)|} \psi_{w_i}(a_i^{-1}n'^{-1}nn'a_i) \\ &= \frac{\varepsilon(w_i)|Z_{G_\sigma}(x)|}{|Z(n) \cap Z'(n'n_i n'^{-\sigma})|} \psi_{w_i}(a_i^{-1}n'^{-1}nn'a_i), \end{aligned}$$

where $n_i = a_i a_i^{-\sigma}$, by Lemma 7.

Furthermore, let $n'T = w'$. Then $\psi_{w_i}(a_i^{-1}n'^{-1}nn'a_i) = \psi_{w_i}(w'^{-1}ww') = \psi_{w'w_iw'^{-\sigma}}(w)$, since $w'^{-1}Z'(w'w_iw'^{-\sigma})w' = Z'(w_i)$. We note that in the one-to-one correspondence given by Lemma 7, the class of $(a_i^{-1}Na_i)_\sigma$ containing $a_i^{-1}n'^{-1}nn'a_i$ corresponds to the $(Z(n), \sigma)$-class of $Z''(n)$ containing $n'n_in'^{-\sigma}$. Thus the required contribution to $\phi(x)$ from the classes of the $(a_i^{-1}Na_i)_\sigma$ containing elements of the form $a_i^{-1}n'^{-1}nn'a_i$ is

$$|Z_{G_\sigma}(x)| \sum_{n''} \frac{\varepsilon(w'')}{|Z(n) \cap Z'(n'')|} \psi_{w''}(z) \tag{3.5}$$

where the summation is over a set of representatives of the $(Z(n), \sigma)$-conjugacy classes of $Z''(n)$, and $n''T = w''$.

Now let U, B be as in Lemma 6, and for $n'' \in Z''(n)$ let $b'' \in B$ be the residue class of n'' and $C(b'') \subseteq U$ the stabilizer of b'' under the action of U on B. We show that the natural map of $Z(n) \cap Z'(n'')$ into $C(b'')$ is surjective. Indeed, let $n_1 \in Z(n)$ be such that $n'' = n_1 n'' n_1^{-\sigma} t$ for some $t \in (Z(n) \cap T)_0$. An argument similar to that of Lemma 2 yields that $n'' = t_1^{-1} n_1 n'' n_1^{-\sigma} t_1^\sigma$ and hence that $t_1^{-1} n_1 \in Z(n) \cap Z'(n'')$ for some $t_1 \in (Z(n) \cap T)_0$. Thus (3.5) can be written as

$$|Z_{G_\sigma}(x)| \sum_{n''} \frac{\varepsilon(w'')}{|(Z(n) \cap T)_0 \cap Z'(n'')||C(b'')|} \psi_{w''}(z).$$

Since, by Lemma 6, the number of $(Z(n), \sigma)$-classes of $Z''(n)$ is also the number of orbits of U acting on B, this expression is equal to

$$\frac{|Z_{G_\sigma}(x)|}{|U|} \sum_{n''} \frac{\varepsilon(w'')}{|(Z(n) \cap T)_0 \cap Z'(n'')|} \psi_{w''}(z)$$

where now the summation is over a set of representatives from the equivalence classes of $Z''(n)$ given by (3.1). Since $(Z(n) \cap T : (Z(n) \cap T)_0)$ is finite we can rewrite this expression as

$$\frac{|Z_{G_\sigma}(x)|}{|Y|} \sum_{w''} \frac{\varepsilon(w'')}{|(Z(n) \cap T)_0 \cap Z'(n'')|} \psi_{w''}(z)$$

where the sum is over a set of representatives for Yw_0 in $Z''(n)$.

For any two elements $n_1, n_2 \in Z''(n)$, $(Z(n) \cap T)_0 \cap Z'(n_1) = (Z(n) \cap T)_0 \cap Z'(n_2)$ if and only if $K(n)n_1 = K(n)n_2$. Hence the above expression will be zero if for each coset $Y'c$ of Y' in Y

$$\sum_{w \in Y'cw_0} \varepsilon(w)\psi_w(z) = 0.$$

But this is the assumption of the lemma, and the lemma is proved.

The next lemma is concerned with elements of the $(a_i^{-1}Na_i)_\sigma$ which lie in $(a_i^{-1}Ta_i)_\sigma$.

LEMMA 9. *Let $a^{-1}Ta$ be a maximal torus of G fixed by σ, let $s \in (a^{-1}Ta)_\sigma$ and $H = Z_G(s)$. Then there exists a one-to-one correspondence between the conjugacy classes of $(a^{-1}Na)_\sigma$ which contain elements of $(a^{-1}Ta)_\sigma$ that are conjugate to s in G_σ and the H_σ-conjugacy classes of maximal tori of H which are fixed by σ and are G_σ-conjugate to $a^{-1}Ta$.*

PROOF. Since G is simply connected, H is a connected reductive group by a theorem of Steinberg ([6], 8.2). The maximal tori of H are precisely those maximal tori of G which contain s. It follows then that the element $g^{-1}sg$ ($g \in G_\sigma$) lies in $(a^{-1}Ta)_\sigma$ if and only if $ga^{-1}Tag^{-1} \subseteq H$. Furthermore, two elements $g_1^{-1}sg_1$ and $g_2^{-1}sg_2$ ($g_1, g_2 \in G_\sigma$) are conjugate in $(a^{-1}Na)_\sigma$ if and only if $g_1a^{-1}Tag_1^{-1}$ and $g_2a^{-1}Tag_2^{-1}$ are conjugate by an element of H_σ, as was to be shown.

4. The main theorem

We now state the main theorem. The notation is that of § 2.

THEOREM. *Let G be simply connected and semisimple. Suppose p does not divide $|W|$ and let the subgroups $a_i^{-1}Na_i$ ($i = 1, 2, \cdots$) be chosen as in Lemma 5. For each $w \in W$ suppose we have a linear character ψ_w of $Z'(w)$ such that these characters have the properties described in Lemma 8, and let ϕ be then defined by (3.2). Then ϕ is the Steinberg character of G_σ. Thus, if s is a semisimple element of G_σ and $H = Z_G(s)$ then $\phi(s) = \varepsilon(s)\, n(s)$ where $n(s)$ is the order of a Sylow p-subgroup of H_σ and $\varepsilon(s) = \pm 1$ has the following interpretation. Let Q be a maximal torus of H such that Q and a Borel subgroup of H containing Q are both fixed by σ. Regard Q as a maximal torus of G, and suppose the element $w(s)$ of W corresponds to Q (in the sense of the remarks following Lemma 3). Then $\varepsilon(s) = \varepsilon(w(s))$.*

PROOF. Let s be a semisimple element of G_σ. By Lemma 8 it is sufficient to consider the contributions to $\phi(s)$ from elements of the form $g^{-1}sg \in (a_i^{-1}Ta_i)_\sigma$ ($g \in G_\sigma$). The order of the centraliser of $g^{-1}sg$ in $(a_i^{-1}Na_i)_\sigma$ is

$$|(a_i^{-1}Na_i)_\sigma \cap Z_G(g^{-1}sg)| = |(ga_i^{-1}Na_ig)_\sigma \cap H| = |(ga_i^{-1}Na_ig \cap H)_\sigma|$$

Hence

$$\phi(s) = |H_\sigma| \sum_i \varepsilon(w_i) \sum_g \frac{1}{|(ga_i^{-1}Na_ig \cap H)_\sigma|}$$

where the second sum is over a set of representatives of the conjugacy classes of $(a_i^{-1}Na_i)_\sigma$ containing elements of the form $g^{-1}sg$ $(g \in G_\sigma)$. Under the correspondence mentioned in Lemma 9, the conjugacy class of $g^{-1}sg$ in $(a_i^{-1}Na_i)_\sigma$ corresponds to the H_σ-conjugacy class of the maximal torus $g^{-1}a_i^{-1}Ta_ig$ of H. Using this correspondence we see that

$$\phi(s) = |H_\sigma| \sum_{c^{-1}Tc} \frac{\varepsilon(cc^{-\sigma}T)}{|(c^{-1}Nc \cap H)_\sigma|} \tag{4.1}$$

where the sum is over a set of representatives of the H_σ-conjugacy classes of maximal tori of H fixed by σ.

Consider the maximal torus Q of H and let $Q = x^{-1}Tx (x \in G)$. Let ε' be the alternating character of $x^{-1}Nx/x^{-1}Tx$; thus if $y \in N$ then $\varepsilon'(x^{-1}yxQ) = \varepsilon(yT)$. By Lemma 5 applied to the connected reductive group H we have

$$\sum_{b^{-1}Qb} \frac{\varepsilon'(bb^{-\sigma}Q)}{|(b^{-1}x^{-1}Nxb \cap H)_\sigma|} = \frac{n(s)}{|H_\sigma|} \tag{4.2}$$

where again the sum is over a set of representatives of the H_σ-conjugacy classes of maximal tori of H fixed by σ. Now

$$\begin{aligned}\varepsilon(xbb^{-\sigma}x^{-\sigma}T) &= \varepsilon'(bb^{-\sigma}x^{-\sigma}x\, Q) = \varepsilon'(bb^{-\sigma}Q)\,\varepsilon'(x^{-\sigma}xQ)\\ &= \varepsilon'(bb^{-\sigma}Q)\,\varepsilon(xx^{-\sigma}T) = \varepsilon'(bb^{-\sigma}Q)\,\varepsilon(w(s)).\end{aligned}$$

By (4.1) and (4.2) we than get $\phi(s) = \varepsilon(w(s))n(s)$ as required.

Finally, ϕ is zero at elements of G_σ which are not semisimple. For any semisimple element s the absolute value of $\phi(s)$ is the absolute value of the value of Steinberg character of G_σ at s. This shows that ϕ is irreducible. Since the degree of ϕ is the order of a Sylow p-subgroup of G_σ, ϕ is the Steinberg character of G_σ. This proves the theorem.

Remarks

1. We note that the condition that p does not divide $|W|$ was used in Lemma 8 to derive the following result.

(4.3) Let $n \in N - T$ be such that $Z''(n) \neq \phi$.

If $a_i^{-1}n'^{-1}nn'a_i \in (a_i^{-1}Na_i)_\sigma$ and $a_j^{-1}n''^{-1}nn''a_j \in (a_j^{-1}Na_j)_\sigma$ for some i, j and n', $n'' \in N$, then $a_i^{-1}n'^{-1}nn'a_i$ and $a_j^{-1}n''^{-1}nn''a_j$ are conjugate in G_σ. Thus the condition that p does not divide $|W|$ can be dispensed with if (4.3) is satisfied in G_σ.

2. Suppose G is defined and split over $k = GF(q)$ and σ is the q^{th} power map.

Then T is a k-split maximal torus of G. In this case, for a semisimple element s, $w(s)$ is the element of W corresponding to a maximal torus of H containing a maximal k-split torus of H, when regarded as a maximal torus of G. If s is regular then H is a maximal torus of G and $w(s)$ is the element of W corresponding to H.

3. The characters we have defined of the subgroups $(a^{-1}Na)_\sigma$ are all trivial on $(a^{-1}Ta)_\sigma$. It is likely that this can be modified as follows. Take any character η of T and let N' be the subgroup of N which fixes η. If characters of the subgroups $(a^{-1}N'a)_\sigma$ can be defined which have properties similar to (3.3) and which take the value $\eta(t)$ at an element $a^{-1}ta$ of $(a^{-1}Ta)_\sigma$ then these could be used to construct characters of G_σ analogous to ϕ. For example, suppose η is a character of T which is not fixed by any $w \neq 1$ of W. For each $w \in W$ choose $a \in G$ such that $aa^{-\sigma}T = w$ and let η_w be the character defined by $\eta_w(a^{-1}ta) = \eta(t)$ of $(a^{-1}Ta)_\sigma$. Let $\chi = \Sigma_{w \in W}\, \varepsilon(w)\eta_w^*$. Then χ is a character of G_σ of degree $|W|\, n(1)$. In this case we need not put any restriction on p as Lemma 8 is not used. It seems likely that the characters of G obtained in this way are some of the principal indecomposable characters (for p) of G_σ.

5. Construction of certain characters of subgroups of W

In this section we give illustrations of when the main theorem can be applied, by actually constructing the characters ψ_w in certain cases.

Assume that G is simple. We have the following possibilities for G_σ [6]:

(1) *G is of type $A_l, B_l, C_l \cdots, E_8$ and G_σ is a Chevalley group.*

(2) (i) *G is of type A_l ($l \geqq 2$), D_l ($l \geqq 4$) or E_6 and G_σ is a twisted analogue of a Chevalley group.*

(ii) *G is of type D_4 and G_σ is a 'triality form' of G.*

(3) (i) *G is of type B_2, $p = 2$ and G_σ is Suzuki group.*

(ii) *G is of type G_2 and $p = 3$ or G is of type F_4 and $p = 2$ and G_σ is a Ree group.*

In the case when $W = W_\sigma$ we have $Z'(w) = Z(w) = Z''(w)$ for each $w \in W$. Then we have to define characters ψ_w of $Z(w)$ for each $w \in W$ having the property (3.3) where we may take $w_0 = 1$. In Case 1, $W = W_\sigma$. In Case 2 we make the following remark. Suppose there is an element w' in W such that $w'^{-1}ww' = w^\sigma$ for all $w \in W$. This means that $Z'(w') = W$. By Lemma 4 there is a maximal torus $a^{-1}Ta$ of G fixed by σ such that $(a^{-1}Na)_\sigma/(a^{-1}Ta)_\sigma$ is isomorphic to W. Let $T' = a^{-1}Ta$, $N' = a^{-1}Na$, $N'/T' = W'$, so that $W' = W'_\sigma$. Suppose we replace T by T', and N by N' and define the required characters ψ_w on subgroups $Z'(w)$ of W' ($w \in W'$). Then the expression on the left side of (3.4) will be replaced by the same expression where the elements are now taken form W', possibly multiplied by -1 since T' corresponds to the element $aa^{-\sigma}T$ of W. Hence in this case we can assume that $W = W_\sigma$ and define the characters ψ_w as in Case 1. We remark that W contains such an element w' if G is of type A_l, D_l (l odd) or E_6.

In Case 3 the characteristic of K divides $|W|$ and so Lemma 8 cannot be used. However, in the case of types B_2 and G_2, if we suitably define the characters ψ_w and then define ϕ as in (3.2), we can check directly using the results of Suzuki [8] and Ward [9] that ϕ is the Steinberg character of G_σ.

(5.1) The subgroup Y in Lemma 8 depends on the choice of n and not merely on $z = nT \in W$. However, suppose z lies in a direct product W' of reflection subgroups of W, each subgroup being generated by reflections corresponding to a closed set of roots (with respect to T) of G. For a root α let Z_α be the centraliser in G of the connected component of the kernel of α. As α runs over the roots considered above, let H be the reductive subgroup of G generated by the Z_α. Then W' is the Weyl group of H and $n \in H$. From the structure of H ([1], 17) it follows that if $z = z_1 z_2 \cdots$ according to the decomposition of W', there exist elements $n_1, n_2, \cdots$ in $Z(n)$ such that $n_1 T = z_1$, $n_2 T = z_2, \cdots$. Hence Y contains at least the cyclic groups generated by $z_1, z_2, \cdots$. This fact will be used later.

In order to define the characters ψ_w we consider in turn the possible types for W. The groups W for each of the possible types for G are described in ([1], 19). In Cases (i), (ii) and (iii) σ is the identity automorphism of W.

(i) *Type* A_l. W is the symmetric group S_{l+1}. Let $z \in W$ be written as a product of disjoint cycles. Then $K(z)$ is the direct product of the cyclic groups generated by these cycles. By applying the argument of (4.1) we see that $K(z) \subseteq Y$. Hence it is sufficient to show (3.4) with $Y' = K(z)$. The structure of $Z(z)$ is well known and can be described as follows. Let $z = w_1 w_2 \cdots$ where $w_m = (b_1^1 b_2^1 \cdots b_m^1)(b_1^2 b_2^2 \cdots b_m^2) \cdots (b_1^r b_2^r \cdots b_m^r)$ is the product of those cycles occuring in z which are of length m. Let T_m be the group $V_1 V_2 \cdots V_r M$ where V_i is the cyclic subgroup generated by $b_i = (b_1^i b_2^i \cdots b_m^i)$, and M is generated by elements of W of the form $c_1 c_2 \cdots c_m$ where c_1 is a permutation of $b_1^1, \cdots b_1^r$ and c_i is got from c_1 by replacing b_1^j by b_i^j. The $Z(z) = T_1 \times T_2 \times \cdots \times T_m \times \cdots$. Define the character ψ_z of $Z(z)$ by putting $\psi_z(y) = 1$ for all y in M, and defining it on T_m by $\psi_z(b_i) = \theta$, where, if $m = 2^a k$ (k odd) then θ is a primitieve k^{th} root of unity.

It can then be verified that these characters satisfy (3.4).

(ii) *Types* B_l ($l \geqq 2$), C_l ($l \geqq 3$). W is a semidirect product of a normal subgroup P which is elementary abelian of order 2^l generated by reflections $a_1, a_2, \cdots a_l$, with a subgroup W_0 which is generated by reflections $(1i)$ ($i = 2, \cdots, l$). W_0 is isomorphic to S_l and we have

$$(1i)a_j(1i) = \begin{cases} a_j & \text{if } j \neq i, 1 \\ a_1 & \text{if } j = i \\ a_i & \text{if } j = 1 \end{cases}$$

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The conjugacy classes of W are described, e.g. in [10].

As in (i) we can show that if $z \in W$ then $K(z) \subseteq Y$. Suppose $z \in W_0$ is of the form

$$(5.2) \qquad (b_1^1 b_2^1 \cdots b_m^1) \cdots (b_1^r b_2^r \cdots b_m^r)$$

as in (i). Then we define the character ψ_z of $Z(z)$ by defining ψ_z on $Z(z) \cap W_0$ as in (i) and by stipulating that $\psi_z(y) = 1$ for y in $Z(z) \cap P$. Now suppose

$$(5.3) \qquad z = a_{\alpha_1}(b_1^1 b_2^1 \cdots b_m^1)a_{\alpha_2}(b_1^2 b_1 \cdots b_m^2) \cdots a_{\alpha_r}(b_1^r b_2^r \cdots b_m^r),$$

where $\alpha_i = b_1^i$. Let V_i be the cyclic group generated by $b_i = a_{\alpha_i}(b_1^i b_2^i \cdots b_m^i)$ and M the subgroup of $Z(z)$ defined as in (i). Define ψ_z on $A_1 A_2 \cdots A_r M$ by $\psi_z(y) = 1$ for y in M and $\psi_z(b_i) = \theta$ where θ is a primitive m^{th} root of unity.

We can then define the characters ψ_z for a general element z, and it can be verified that they have the required property.

(iii) *Type* D_l $(l \geqq 4)$. The Weyl group W' in this case is the subgroup of index 2 of the group W of (ii) consisting of all elements of the form $a_1^{\delta_1} a_2^{\delta_2} \cdots a_l^{\delta_l} y$ $(y \in W_0)$ where $\Sigma\delta_i$ is even. If two elements of W' are such that the corresponding elements y in W_0 fix at least one symbol from 1 to l they are conjugate in W' if and only if they are conjugate in W. However, if $x = a_1^{\delta_1} \cdots a_l^{\delta_l} y$ and y moves all the symbols from 1 to l then the conjugacy class of x in W splits into two classes in W'. For any element $z \in W$ we now define a character ψ_z of its centraliser in W. If $z \in W'$, the required character of its centraliser in W' is defined to be the restriction of this character. If z is of the form (5.2) we define ψ_z as in (ii). If z is of the form (5.3) define $\psi_z(y) = 1$ for $y \in M$ and $\psi_z(b_i) = \theta$ where θ is a primitive $(2m)^{\text{th}}$ root of unity.

We remark that in this case it is not clear whether we always have, for $z \in W$, $K(z) \subseteq Y$. However, using the remarks (5.1) we can again verify that the characters ψ_z have the required properties.

(iv) *Type* D_l $(l \geqq 4)$, with σ the automorphism of W' which interchanges (12) and $a_1 a_2$ (12). By the remarks at the beginning of the section we can assume that l is even. We regard W' as a subgroup of the Weyl group W of the type D_{l+1} in the obvious way. Since $l+1$ is odd there is an element $w_0 \in W$ (in fact $w_0 = a_2 a_3 \cdots a_{l+1}$) such that $w_0^{-1} w w_0 = w^\sigma$ for all $w \in W$. Then for any $w \in W$, $Z'(w) = Z(ww_0^{-1})$. Thus if $z \in W'$, $Z'(z) \cap W' \subseteq Z(zw_0^{-1})$. In order to define the character ψ_z of $Z'(z) \cap W'$ we define it on $Z(zw_0^{-1})$ as in (iii) and restrict it to $Z'(z) \cap W'$. These characters again have the required properties.

(v) *Type* D_4, with σ the automorphism of W' which maps $(12) \to a_1 a_2(12)$, $a_1 a_2(12) \to (34)$, $(34) \to (12)$. The σ-conjugacy classes of W' can be calculated. We give in the colums of table 1 from left to right, representatives z of the classes, $Z'(z)$, and the characters to be chosen. We put $x = a_1 a_2 a_3 a_4$.

TABLE 1

1	$\{a_1 a_2(234), (23)\}$	Trivial Character
x	$\{a_1 a_2(234), (23)\}$	Trivial Character
(12)	$\{x, a_3 a_4 (34)\}$	Trivial Character
$a_2 a_3(12)$	$\{x, (12)\}$	Trivial Character
$a_3 a_4(123)$	$\{(123), a_1 a_4(12)(34)\}$	$(123) \to \omega$, $a_1 a_4(12)(34) \to 1$
(123)	$\{a_1 a_4(12)(34)\}$	$a_1 a_4(12)(34) \to -1$
$a_1 a_2(123)$	$\{(123), a_1 a_4(12)(34)\}$	$(123) \to \omega$, $a_1 a_4(12)(34) \to 1$

Here, and in the rest of the section, ω and i denote a primitive cube root and a fourth root of unity respectively.

(vi) *Type* G_2. W is generated by reflections b and c such that $a = bc$ is of order 6.

1. σ is the identity. Let ψ_a be the character $a \to \omega$ of $Z(a) = \{a\}$. For the centralisers of elements of W not conjugate to a we take the trivial character.

2. σ is the automorphism which interchanges b and c. We have $Z'(1) = \{a^3\}$, $Z'(b) = Z'(a^2 b) = Z'(a^3 b) = \{a\}$. We choose the trivial character of $Z'(1)$ $Z'(b)$ and the character $a \to \omega$ of $Z'(a^2 b)$ and $Z'(a^3 b)$.

(vii) *Type* B_2. W is generated by two reflections b and c such that $a = bc$ is of order 4. Let σ be the automorphism interchanging b and c. We have $Z'(1) = \{a^2\}$, $Z'(b) = Z'(a^2 b) = \{a\}$. Choose the character $a^2 \to -1$ of $Z'(1)$ and the character $a \to i$ of $Z'(b)$ and $Z'(a^2 b)$.

(viii) *Type* F_4. W is a group of order 1152, the conjugacy classes and characters of which are given in [2]. σ is the identity automorphism of W. It can be shown, by a lengthy computation, that there exist characters ψ_w of $Z(w)$ for $w \in W$ such that $\psi_{w'}(w) = \psi_w(w')$ for any pair $w, w' \in W$. These characters have the required properties.

In the case of the Weyl group of type E_6 we can define characters ψ_w having properties (3.3) provided $Y' = K(z)$ in (3.4). However, we have not been able to show that $Y' = K(z)$ holds for each z.

Summarizing, the main theorem is valid in the following cases.

(1) *G is of type A_l, B_l, C_l, D_l, G_2, F_4 and G_σ is a Chevalley group.*

(2) *G is of type A_l, D_l and G_σ is a twisted form of a Chevalley group.*

(3) *G is of type D_4 and G_σ is a triality form of G.*

(4) *G_σ is a Suzuki group or a Ree group of type G_2.*

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Added in proof. A description of the one-to-one correspondence mentioned after Lemma 1 is contained in

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