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# SPHERE THEOREM BY MEANS OF THE RATIO OF MEAN CURVATURE FUNCTIONS

SUNG-EUN KOH
*Department of Mathematics, Konkuk University, Seoul, 143-701, Korea*
*e-mail:sekoh@kkucc.konkuk.ac.kr*

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**Abstract.** It is well known that a compact embedded hypersurface of the Euclidean space without boundary is a round sphere if one of mean curvature functions is constant. In this note, we show that a compact embedded hypersurface of the Euclidean space (and other constant curvature spaces) without boundary is a round sphere if the ratio of some two mean curvature functions is constant.

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**1. Introduction.** Let $M^n$ be an emdedded submanifold of $N^{n+1}$ and let $H_k$ denote the $k$-th mean curvature function of $M^n$, that is, $H_k$ is the $k$-th elementary symmetric polynomial of principal curvatures of $M^n$ divided by $\binom{n}{k}$, and $H_0$ is defined to be 1. For instance, $H_1$ is the usual mean curvature and $H_n$ is the Gauss-Kronecker curvature.

Alexandrov's well-known sphere theorem [**1**] states that, when $N^{n+1}$ is the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$ or the open half sphere $\mathbb{S}^{n+1}_+$, $M^n$ is a round sphere if $H_1$ is constant. This theorem was generalized in [**3**] in the following way.

Theorem A. *Let $N^{n+1}$ be one of $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$ or $\mathbb{S}^{n+1}_+$ and let $\phi : M^n \to N^{n+1}$ be an isometric embedding of a compact oriented n-dimensional manifold without boundary $M^n$. If $H_k$ is constant for some $k = 1, 2, \ldots, n$, then $\phi(M^n)$ is a geodesic hypersphere.*

In this note, we generalize Theorem A in the following way.

Theorem B. *Let $N^{n+1}$ be one of $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$ or $\mathbb{S}^{n+1}_+$ and $\phi : M^n \to N^{n+1}$ be an isometric embedding of a compact oriented n-dimensional manifold without boundary $M^n$. If the ratio $H_k/H_l$ is constant for some $k, l = 0, 1, 2 \ldots, n,\ k > l$ and $H_l$ does not vanish on $M^n$, then $\phi(M^n)$ is a geodesic hypersphere.*

As $H_0$ is defined to be 1, the above theorem reduces to Theorem A if $l = 0$. Theorem B is a generalization of Theorem A in this sense. Note also that we cannot expect the result for the whole sphere $\mathbb{S}^{n+1}$. For example, $H_1$ and $H_2$ of the embedding

$$\mathbb{S}^1(a) \times \mathbb{S}^1(b) \subset \mathbb{S}^3, \quad a^2 + b^2 = 1, \quad a \neq b,$$

are nonzero constants.

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**2. Proof.** We use the hyperboloid model for $\mathbb{H}^{n+1}$ and the usual embedding of $\mathbb{S}^{n+1}$ into $\mathbb{R}^{n+2}$. Let $\eta$ denote a unit normal field on $M^n$. We use the following Minkowski formula (for proof, see [**3**]) where $\langle\ ,\ \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^{n+1}$ (on $\mathbb{R}^{n+2}$) when $N^{n+1}$ is $\mathbb{R}^{n+1}$ (when $N^{n+1}$ is $\mathbb{S}_+^{n+1}$) and the Lorentzian inner product on $\mathbb{R}^{n+2}$ when $N^{n+1}$ is $\mathbb{H}^{n+1}$.

Lemma A. *The following identities hold for every $k = 1, \ldots, n$.*

(i) *When $N^{n+1}$ is $\mathbb{R}^{n+1}$,*

$$\int_M (H_{k-1} + H_k\langle\phi, \eta\rangle)\, dM = 0.$$

(ii) *When $N^{n+1}$ is $\mathbb{H}^{n+1}$,*

$$\int_M (H_{k-1}\langle\phi, p\rangle + H_k\langle\eta, p\rangle)\, dM = 0 \quad \text{for any } p \in \mathbb{R}^{n+2}.$$

(iii) *When $N^{n+1}$ is $\mathbb{S}_+^{n+1}$,*

$$\int_M (H_{k-1}\langle\phi, p\rangle - H_k\langle\eta, p\rangle)\, dM = 0 \quad \text{for any } p \in \mathbb{R}^{n+2}.$$

We also use the following inequalities for higher order mean curvatures.

Lemma B. *Suppose $H_k > 0$ for some $k = 1, 2, \ldots, n$. Then the following hold.*

(i) $H_k^{\frac{k-1}{k}} \leq H_{k-1}$*; hence every $H_l$, $l \leq k$, is positive.*
(ii) $H_k/H_{k-1} \leq H_{k-1}/H_{k-2}$.
(iii) *For every $l < k$,* $H_k/H_l \leq H_{k-1}/H_{l-1}$.

*Proof of Lemma B.* For (i), (ii), see, for example, [**2**, Section 12].
From (ii), we have

$$H_k/H_{k-1} \leq H_{k-1}/H_{k-2} \leq \cdots \leq H_{l+1}/H_l \leq H_l/H_{l-1},$$

which is equivalent to (iii). □

Now, assume

$$H_k/H_l = \alpha$$

for a constant number $\alpha$.

**(2.1). Proof when** $N^{n+1} = \mathbb{R}^{n+1}$. Since $M^n$ is compact, one can find a point in $M^n$ where all the principal curvatures are positive. Then $H_k, H_l$ are positive at that point. Since $H_k/H_l$ is constant on $M^n$ and since $H_l$ does not vanish on $M^n$ by assumption, $H_k$ and $H_l$ are positive on $M^n$. Then $\alpha > 0$ and from the inequality (ii) of Lemma B, we have

$$0 < \alpha = H_k/H_l \leq H_{k-1}/H_{l-1}. \qquad (*)$$

Since $H_k = \alpha H_l$, we have by Lemma A,

$$\begin{aligned} 0 &= \int_M (H_{k-1} + H_k\langle \phi, \eta\rangle)\, dM \\ &= \int_M (H_{k-1} + \alpha H_l\langle \phi, \eta\rangle)\, dM, \end{aligned}$$

that is,

$$\int_M H_{k-1}\, dM = \int_M (-\alpha H_l\langle \phi, \eta\rangle)\, dM. \qquad (1)$$

On the other hand, since $\alpha$ is constant, we also have by Lemma A,

$$\int_M \alpha(H_{l-1} + H_l\langle \phi, \eta\rangle)\, dM = 0,$$

that is,

$$\int_M \alpha H_{l-1}\, dM = \int_M (-\alpha H_l\langle \phi, \eta\rangle)\, dM. \qquad (2)$$

From (1) and (2), we have

$$\int_M (H_{k-1} - \alpha H_{l-1})\, dM = 0.$$

Since we have from $(*)$

$$H_{k-1} - \alpha H_{l-1} \geq 0,$$

it follows that

$$H_{k-1}/H_{l-1} = \alpha = H_k/H_l$$

everywhere on $M^n$. Thus, proceeding inductively, we have finally

$$H_{k-l} = H_{k-l}/H_0 = \alpha$$

everywhere on $M^n$. Thus, by Theorem A, $\phi(M^n)$ is a geodesic hypersphere. □

**(2.2) Proof when $N^{n+1} = \mathbb{H}^{n+1}$** : At a point of $M^n$ where the distance function of $\mathbb{H}^{n+1}$ attains its maximum, all the principal curvatures are positive. Then $H_k$, $H_l$ are positive on $M^n$ and $(*)$ also holds in this case. Since $H_k = \alpha H_l$, we have

$$0 = \int_M (H_{k-1}\langle \phi, p\rangle + H_k\langle \eta, p\rangle)\,dM$$
$$= \int_M (H_{k-1}\langle \phi, p\rangle + \alpha H_l\langle \eta, p\rangle)\,dM,$$

that is,

$$\int_M H_{k-1}\langle \phi, p\rangle\,dM = \int_M (-\alpha H_l\langle \eta, p\rangle)\,dM.$$

Since $\alpha$ is constant, we also have

$$\int_M \alpha(H_{l-1}\langle \phi, p\rangle + H_l\langle \eta, p\rangle)\,dM = 0,$$

so that

$$\int_M (H_{k-1} - \alpha H_{l-1})\langle \phi, p\rangle\,dM = 0.$$

Now, if we take $p = (1, 0, \ldots, 0) \in \mathbb{R}^{n+2}$, then the sign of $\langle \phi, p\rangle$ does not change on $M^n$. Since $H_{k-1} - \alpha H_{l-1} \geq 0$ from $(*)$, we have

$$H_{k-1}/H_{l-1} = \alpha = H_k/H_l$$

everywhere on $M^n$. Thus, proceeding inductively, we have finally

$$H_{k-l} = H_{k-l}/H_0 = \alpha$$

everywhere on $M^n$. Thus, by Theorem A, $\phi(M^n)$ is a geodesic hypersphere. □

**(2.3) Proof when** $N^{n+1} = \mathbb{S}^{n+1}_+$. Let $c \in \mathbb{S}^{n+1}$ be the centre of $\mathbb{S}^{n+1}_+$. Then at a point of $M^n$ where the height function $\langle \phi, c\rangle$ attains its maximum, all the principal curvatures are positive because $M^n$ lies in the open half sphere with the centre $c$. Now proceeding as in (2.2), we have

$$\int_M (H_{k-1} - \alpha H_{l-1})\langle \phi, p\rangle\,dM = 0.$$

Since $M^n$ lies in the open half sphere, one can find a vector $p \in \mathbb{R}^{n+2}$ so that $\langle \phi, p\rangle$ is positive on $M^n$. Since $H_{k-1} - \alpha H_{l-1} \geq 0$ by $(*)$, arguing in the same way as before, we can see that $H_{k-l} = \alpha$, that is, $\phi(M^n)$ is a geodesic hypersphere. □

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