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ON LUCAS-SETS FOR VECTOR-VALUED ABSTRACT POLYNOMIALS IN K-INNER PRODUCT SPACES

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Introduction. The fact that Rolle's theorem on critical points of a real differentiable function does not hold in general for analytic functions of a complex variable raises a natural question [**7**, p. 21] as to whether or not it can be generalized to polynomials, the simplest subclass of analytic functions. While attempting to answer this and other related problems in [**4**, **5**, **6**], Lucas proved that all the critical points of a nonconstant polynomial f lie in the convex hull of the set of zeros of f (see Theorem (6.1) in [**7**]). Walsh [**12**] has shown that Lucas' theorem is equivalent to the following result [**7**, Theorem (6.2)], namely: Any convex circular region which contains all the zeros of a polynomial f also contains all the zeros of its derivative f'.

It is trivial to see that this result holds also for convex sets in general. Let us agree to call any subset A of the complex plane a *Lucas-set* for polynomials if, whenever A contains all the zeros of any polynomial f, A also contains all the zeros of f'. Convex circular regions and convex sets, in general, may then be viewed as Lucas-sets in our terminology. In the present paper we intend to tap the possibilities (if any) for the existence of an analogous terminology in an arbitrary K-inner product space E over K (K being an algebraically closed field of characteristic zero), in respect to abstract polynomials [**2**, **10**, **14**] and a suitably defined notion of their derivatives. In order to be able to do just this, we bring in the notion of generalized circular regions of E in Section 1 and introduce in Section 2 the concept of pseudo-derivatives of abstract polynomials from E to K. In Section 3 we successfully employ these concepts in formulating our main theorems, which generalize Lucas' theorem to abstract polynomials (including the vector-valued ones) on E and which essentially state that generalized circular regions are indeed the kind of Lucas-sets (for abstract polynomials) that we were looking for. Finally, in Section 4, we give examples to demonstrate the validity of hypotheses in and the degree of generality of our main theorems.

1. Preliminaries and notations. Throughout this paper **C** (resp. **R**) will denote the field of complex (resp. real) numbers of K an algebraically closed field of characteristic zero, with K_0 as a maximal ordered

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subfield of K such that

$$K = K_0(i) = \{a + ib : a, b \in K_0\},$$

where $-i^2 = 1$. (For details, see [**1**, pp. 38–40], [**3**, p. 56], [**11**, pp. 248–255]). Also, K_{0+} (resp. $\mathbf{R}_+$) will denote the set of all nonnegative elements of K_0 (resp. $\mathbf{R}$). It is known [**1**, p. 39] that every element in K_{0+} has a unique square root in K_{0+}. For every element $z = a + ib \in K$ (with a, b in K_0) we may, in analogy with the complex plane, define the conjugate of z, the real part of z, the imaginary part of z, and the K_0-absolute value of z by $\bar{z} = a - ib$, $\operatorname{Re}(z) = (z + \bar{z})/2$, $\operatorname{Im}(z) = (z - \bar{z})/2i$, and $|z| = +(a^2 + b^2)^{1/2}$, respectively. With such mechanisms available to K (wherein K_0 plays the role of 'reals', except for completeness), the definition of a complex (resp. real) inner product (see [**13**, p. 120]) in a complex (resp. real) vector space can be immediately extended to inner products in a vector space E over K (resp. K_0), which will be referred to as K-(resp. K_0-) *inner products*. A vector space E over K (resp. K_0), together with a K-(resp. K_0-) inner product $\langle . , . \rangle$, will be termed as a K-(resp. K_0-) *inner product space*. Similarly, the definitions of 'metric', 'convexity', and 'norm' in respect to a complex (or real) vector space extend automatically to the corresponding notions in the spaces E under discussion, if we only replace in those definitions the field $\mathbf{R}$ by the field K_0. The resulting terminology will be 'K_0-metric', 'K_0-convexity', and 'K_0-norm'. Now the terms 'K_0-metric space' and 'K_0-normed vector space' have obvious meanings.

For convenience of future reference, we record certain facts in the following

Remark 1.1.(I) If H is a Hamel basis for a vector space E over K (resp. K_0), every element $x \in E$ can be written as $x = \sum \alpha h$ (uniquely, except for the order of its terms), where the summand ranges over all elements $h \in H$ and the coefficients in K (resp. K_0) being zero for all but a finite number of the h (see [**13**, pp. 16-17]). For each $x = \sum \alpha h$ and $y = \sum \beta h$ in E, we may then define

$$\begin{array}{ll}(1.1) \\ (1.1)'\end{array} \quad \langle x, y \rangle = \begin{cases} \sum \alpha\bar{\beta} \text{ if } E \text{ is a vector space over } K, \\ \sum \alpha\beta \text{ if } E \text{ is a vector space over } K_0. \end{cases}$$

It is with this K-(resp. K_0-) inner product (and it is indeed one) that we shall refer to E as a *natural* K-(resp. K_0-) *inner product space*. Thus: any vector space E over K (resp. K_0) can be given a K-(resp. K_0)-inner mapping product.

(II) If $(E, \langle . , . \rangle)$ is a K-(resp. K_0-) inner product space, then the mapping $\| . \| : E \to K_{0+}$, defined by

$$(1.2) \qquad \|x\| = \langle x, x \rangle^{1/2},$$

makes E a K_0-normed vector space in its own right. (The proof is similar to the one in [**13**, p. 121].)

Throughout the rest of this section $(E, \langle . , . \rangle)$ will denote a K-inner product space with a built-in K_0-norm on it defined by (1.2). Let K_∞ denote the projective field (see [**18**, p. 352] or [**15**, p. 116]) got by adjoining to K an element ∞ having the properties of "scalar infinity". Also, as in [**18**, p. 372], adjoin to E an element ω and furnish $E \cup \{\omega\}$ (more precisely, denoted by E_ω) with the following structure: (1) the subset E of E_ω preserves its initial vector space structure; and (2) $a + \omega = \omega + a = \omega$ for every $a \in E$, $\lambda\omega = \omega$ for every $\lambda \in K - \{0\}$, and $\infty . a = \omega$ for every $a \in E - \{0\}$. Let us denote by ψ the permutation of E_ω (it is indeed one) defined by

$$(1.3) \qquad \psi(x) = \begin{cases} x/\|x\|^2 & \text{if } x \in E - \{0\} \\ \omega & \text{if } x = 0 \\ 0 & \text{if } x = \omega. \end{cases}$$

The mapping ψ is called the *inversion* of E_ω with origin as pole and has the property that $\psi(\psi(x)) = x$ for $x \in E_\omega$ and that

$$(1.4) \qquad \psi(\lambda x) = \frac{1}{\bar{\lambda}}\psi(x) \ \forall \lambda \in K_\infty, x \in E_\omega,$$

provided we agree to write $\bar{\omega} = \omega$ and provided the scalar product λx is not of the form $0 . \omega$, $\infty . 0$, or $\infty . \omega$. Next, we associate, to each fixed element $b \in E$, a permutation ψ_b of E_ω given by

$$(1.5) \qquad \psi_b(x) = \psi(x - b) = \begin{cases} (x - b)/\|x - b\|^2 & \text{if } x \in E - \{b\} \\ \omega & \text{if } x = b \\ 0 & \text{if } x = \omega. \end{cases}$$

The mapping ψ_b is called an *inversion* of E_ω with *pole* b. Since $\psi(\psi(x)) = x$, we see that

$$(1.6) \qquad y = \psi_b(x) \quad \text{if and only if } x = \psi(y) + b.$$

Note also that

$$(1.7) \qquad \psi_b(\omega) = 0 \quad \text{and} \quad \psi_b(b) = \omega.$$

Definition 1.2. A subset A of E_ω is called a *generalized circular region* of E_ω (abbreviated as g.c.r.) if either A is one of the sets $\emptyset$, E, E_ω, or A satisfies the following two conditions:

(i) $\psi_b(A)$ is K_0-convex for every $b \in E - A$, where ψ_b is as defined by (1.5);

(ii) $\omega \in A$ if A is not K_0-convex.

We shall denote by $D(E_\omega)$ the class of all g.c.r.'s of E_ω. The empty set $\emptyset$, E, E_ω, and single-point sets (and their complements in E_ω) are examples

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of trivial g.c.r.'s of E_ω. The fact that there exist abundantly many non-trivial g.c.r.'s of E_ω is established by Proposition 1.5 below.

Definition 1.3. Let $B_g(E_\omega)$ denote the class consisting of the following subsets of E_ω:

(1) $\emptyset$, E, E_ω, and $\{\omega\}$;

(2) All sets which are of the form

$$\{x \in E : \alpha\|x\|^2 + \operatorname{Re}\langle x, a\rangle + \gamma \leqq 0\}, \tag{1.8}$$

or of the form

$$\{x \in E : \alpha\|x\|^2 + \operatorname{Re}\langle x, a\rangle + \gamma < 0\} \tag{1.9}$$

for some $\alpha \in K_{0+}$, $\gamma \in K_0$, and $a \in E$.

(3) The complements in E_ω of the sets in (2). Each member of $B_g(E_\omega)$ is called a *generalized ball* of E_ω. Note that the expression $\operatorname{Re}\langle x, a\rangle$ disappears in (1.8) and (1.9) if $a = 0$.

PROPOSITION 1.4. *Every set of the form* (1.8) *or* (1.9) *is K_0-convex.*

Proof. It is sufficient to prove the statement for sets of the form (1.8), the proof for the other form being similar. Let A be any set of the form (1.8). We devide the proof into three cases.

Case I. Suppose $\alpha = 0$, so that

$$A = \{x \in E : \operatorname{Re}\langle x, a\rangle \leqq -\gamma\}$$

for some $a \in E$ and $\gamma \in K_0$. If $x_i \in A$ and $t_i \in K_{0+}$ for $i = 1, 2, \ldots, n$ with $t_1 + t_2 + \ldots + t_n = 1$, and if we put $x = t_1x_1 + \ldots + t_nx_n$, then

$$\begin{aligned}\operatorname{Re}\langle x, a\rangle = \operatorname{Re}\left[\sum_{i=1}^{n} t_i\langle x_i, a\rangle\right] &= \sum_{i=1}^{n} t_i \operatorname{Re}\langle x_i, a\rangle \\ &\leqq \sum_{i=1}^{n} t_i(-\gamma), \text{ since } x_i \in A, t_i \in K_{0+} = -\gamma.\end{aligned}$$

Therefore $x \in A$, and A is K_0-convex.

Case II. Suppose $\alpha > 0$. If we put $c = -(1/2\alpha)a$, then (cf. (1.2))

$$\begin{aligned}A &= \{x \in E : \langle x, x\rangle - 2\operatorname{Re}\langle x, c\rangle + \gamma/\alpha \leqq 0\} \\ &= \{x \in E : \langle x - c, x - c\rangle \leqq \|c\|^2 - \gamma/\alpha\} \\ &= \{x \in E : \|x - c\|^2 \leqq \|c\|^2 - \gamma/\alpha \\ &= \begin{cases}\emptyset & \text{if } \|c\|^2 - \gamma/\alpha < 0 \\ \{c\} & \text{if } \|c\|^2 - \gamma/\alpha = 0 \\ \{x \in E : \|x - c\| \leqq r\} & \text{if } \|c\|^2 - \gamma/\alpha = r^2 \text{ (say)} > 0.\end{cases}\end{aligned}$$

In either case, we find that A is K_0-convex.

PROPOSITION 1.5. $B_g(E_\omega) \subseteq D(E_\omega)$.

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Proof. Let $A \in B_g(E_\omega)$. From Proposition 1.4 it is obvious that A satisfies condition (ii) in Definition 1.2. Therefore, in order to show that $A \in D(E_\omega)$, we have only to prove that $\psi_b(A)$ is K_0-convex for every $b \in E - A$ and we break up the proof as follows:

(i) If A is any one of the sets $\emptyset$, E, E_ω, or $\{\omega\}$, it is trivial to see that $\psi_b(A)$ is K_0-convex for every $b \in E - A$.

(ii) If A is any set of the form (1.8) and $b \in E - A$, the relations (1.3) and (1.6) imply that

$$(1.10) \quad \psi_b(A) = \{y \in E_\omega : y = \psi_b(x) \quad \text{for some } x \in A\} \\ = \{y \in E_\omega : \psi(y) + b \in A\}.$$

Since $\omega \notin A$ in the case under consideration and since $x \neq b$ for any $x \in A$, we see from (1.7) that $0, \omega \notin \psi_b(A)$ and (hence) that $\|y\| \neq 0$ for every $y \in \psi_b(A)$. Consequently,

$$\psi_b(A) = \{y \in E : b + y/\|y\|^2 \in A\}.$$

That is, $\psi_b(A)$ consists of all elements $y \in E$ for which

$$\alpha \langle b + y/\|y\|^2, b + y/\|y\|^2\rangle + \operatorname{Re} \langle b + y/\|y\|^2, a\rangle + \gamma \leqq 0,$$

or (equivalently),

$$\alpha\left[\|b\|^2 + \frac{1}{\|y\|^2} + \frac{2}{\|y\|^2} \operatorname{Re} \langle y, b\rangle\right] + \operatorname{Re} \langle b, a\rangle + \frac{1}{\|y\|^2} \operatorname{Re} \langle y, a\rangle \\ + \gamma \leqq 0.$$

Therefore,

$$\psi_b(A) = \{y \in E : \delta\|y\|^2 + \operatorname{Re} \langle y, c\rangle + \alpha \leqq 0\},$$

where

$$\delta = \alpha\|b\|^2 + \operatorname{Re} \langle b, a\rangle + \gamma \in K_0 \quad \text{and} \quad c = 2\alpha b + a \in E.$$

Since $b \notin A$, we see that $\delta \in K_{0+}$. Hence, $\psi_b(A)$ is again a set of the form (1.8) with α, γ, a replaced by δ, α, c, respectively, and so $\psi_b(A)$ is K_0-convex due to Proposition 1.4.

(iii) If A is any set of the form (1.9), we reproduce the same arguments as in (ii) (with $\leqq$ replaced by $<$) and conclude that $\psi_b(A)$ is again of the form (1.9) for every $b \in E - A$ and, hence, K_0-convex.

(iv) If A is the complement in E_ω of a set of the form (1.8) and if $b \in E - A$, then

$$(1.11) \quad A = \{x \in E : \alpha\|x\|^2 + \operatorname{Re} \langle x, a\rangle + \gamma > 0\} \cup \{\omega\}$$

for some $\alpha \in K_{0+}$, $\gamma \in K_0$, and $a \in E$. Since $0 = \psi_b(\omega) \in \psi_b(A)$ and $\omega = \psi_b(b) \notin \psi_b(A)$, from (1.10) we obtain

$$\psi_b(A) = \{y \in E : y \neq 0; b + y/\|y\|^2 \in A\} \cup \{0\}.$$

Now proceeding as in (ii) above, we obtain

$$(1.12)\quad \psi_b(A) = \{y \in E : y \neq 0; \delta\|y\|^2 + \operatorname{Re}\langle y, c\rangle + \alpha > 0\} \cup \{0\},$$

where

$$\delta = \alpha\|b\|^2 + \operatorname{Re}\langle b, a\rangle + \gamma \leqq 0$$

(since $b \in A$) and

$$c = 2\alpha b + a \in E.$$

In case $\alpha > 0$, we see that

$$\begin{aligned}\psi_b(A) &= \{y \in E : \delta\|y\|^2 + \operatorname{Re}\langle y, c\rangle + \alpha > 0\}\\ &= \{y \in E : \mu\|y\|^2 + \operatorname{Re}\langle y, -c\rangle - \alpha < 0\},\end{aligned}$$

where $\mu = -\delta \in K_{0+}$. Therefore, $\psi_b(A)$ is a set of the form (1.9) and, hence, K_0-convex. In case $\alpha = 0$, we see that

$$\psi_b(A) = \{y \in E : y \neq 0; \nu\|y\|^2 + \operatorname{Re}\langle y, -a\rangle < 0\} \cup \{0\},$$

where $\nu = -\delta = -\operatorname{Re}\langle b, a\rangle - \gamma \in K_{0+}$. Putting $d = a/2\nu$, we get

$$\psi_b(A) = \begin{cases}\{y \in E : y \neq 0; \|y - d\| < \|d\|\} \cup \{0\} & \text{if } \nu > 0\\ \{y \in E : y \neq 0; \operatorname{Re}\langle y, a\rangle > 0\} \cup \{0\} & \text{if } \nu = 0.\end{cases}$$

To show that $\psi_b(A)$ is again K_0-convex, we take any elements $y_i \in \psi_b(A)$ and $t_i \in K_{0+}$ for $i = 1, 2, \ldots, n$, with $t_1 + t_2 + \ldots + t_n = 1$ and show that $t_1y_1 + \ldots + t_ny_n \in \psi_b(A)$, irrespective of whether $\nu > 0$ or $\nu = 0$. This is trivial to verify when all the y_i's happen to be 0 or when all the coefficients t_i corresponding to the nonzero y_i's happen to be zero. However, if $t_k \neq 0$ and $y_k \neq 0$ for some k, we observe that

$$\|y_k - d\| < \|d\| \quad \text{and} \quad \|y_i - d\| \leqq \|d\| \quad \forall\, i \neq k$$

in case $\nu > 0$, and that

$$\operatorname{Re}\langle y_k, a\rangle > 0 \quad \text{and} \quad \operatorname{Re}\langle y_i, a\rangle \geqq 0 \quad \forall\, i \neq k$$

in case $\nu = 0$. In the two cases we respectively have

$$\left\|\left(\sum_{i=1}^n t_iy_i\right) - d\right\| = \left\|\sum_{i=1}^n t_i(y_i - d)\right\| \leqq \sum_{i=1}^n t_i\|y_i - d\| < \|d\|$$

and

$$\operatorname{Re}\left\langle \sum_{i=1}^n t_iy_i, a\right\rangle = \sum_{i=1}^n t_i \operatorname{Re}\langle y_i, a\rangle > 0.$$

That is, in either case, $\sum_{i=1}^n t_iy_i \in A$, and $\psi_b(A)$ is K_0-convex. We have, therefore, shown that $\psi_b(A)$ is K_0-convex for every $\alpha \in K_{0+}$, $\gamma \in K_0$, and $a \in E$.

(v) If A is the complement in E_ω of a set of the form (1.9) and if $b \in E - A$, then A is the set got by replacing the inequality $>$ by $\geqq$ in the expression (1.11). Now proceeding exactly as in (iv) above, with corresponding changes of course, we see that $\psi_b(A)$ is precisely the set got by replacing in (1.12) the inequality $>$ by $\geqq$. But, this time, we have (for every $\alpha \in K_{0+}$)

$$\psi_b(A) = \{y \in E : \delta\|y\|^2 + \operatorname{Re}\langle y, c\rangle + \alpha \geqq 0\}$$
$$= \{y \in E : \mu\|y\|^2 + \operatorname{Re}\langle y, -c\rangle - \alpha \leqq 0\},$$

where $\mu = -\delta > 0$. That is, $\psi_b(A)$ is of the form (1.8) and, hence, K_0-convex.

Since cases (i)–(v) exhaust all members of $B_g(E_\omega)$, our proof is complete.

In what follows we show that, in the special case when E is taken as a natural K-inner product space, the class $D(E_\omega)$ turns out to be essentially the class of subsets introduced earlier by Zervos (see Definition 1 in [**18**, p. 372]) in relation to natural K_0-inner product spaces. To this effect, we proceed as follows: Given a Hamel basis H for a vector space E over K and the corresponding natural K-inner product $\langle . , . \rangle$ on E (cf. Remark 1.1 (I)), we define a mapping $\langle . , . \rangle_0 : E \times E \rightarrow K_0$ by

$$(1.13) \quad \langle x, y\rangle_0 = \operatorname{Re}\langle x, y\rangle = \operatorname{Re}[\sum \alpha\bar{\beta}] \quad \forall\, x = \sum \alpha h, y = \sum \beta h,$$

where $\alpha, \beta \in K$ and $h \in H$ (cf. (1.1)). For every $\alpha \in K_0$ and $x \in E$ if we define αx to be what it is originally in E, then E becomes a vector space over K_0 with $H_1 = H \cup \{iH\}$ as a Hamel basis for it. Thus every element $x = \sum \alpha h$ ($\alpha \in K$, $h \in H$) in E can be expressed as $x = \sum \lambda_1 (\lambda_1 \in K_0, h_1 \in H_1)$, and vice-versa. The passage from one to the other can be effected in the following simple manner: $x = \sum \alpha h = \sum \lambda_1 h_1$ if and only if, for each element $h \in H$, the real (resp. imaginary) part of the coefficient of h in the representation $\sum \alpha h$ is equal to the coefficient of h (resp. ih) in the representation $\sum \lambda_1 h_1$ (note that $H_1 = \cup \{h, ih\}$, the union ranging over all elements $h \in H$). In view of this, we may transform (1.13) in terms of the basis elements h_1 with coefficients in K_0. In fact, if $x = \sum \alpha h = \sum \lambda_1 h_1$, $y = \sum \beta h = \sum \mu_1 h_1$ and if we write $\alpha = a_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ ($\alpha_1, \beta_1, \alpha_2, \beta_2$ being in K_0), then (1.13) implies that

$$\langle x, y\rangle_0 = \sum(\alpha_1\beta_1 + \alpha_2\beta_2) = \sum \lambda_1\mu_1 \quad \forall\, x = \sum \lambda_1 h_1, y = \sum \mu_1 h_1.$$

That is, (1.13) is indeed a natural K_0-inner product (cf. (1.1)′) on E (regarded as a vector space over K_0). Consequently, $(E, \langle . , . \rangle_0)$ becomes a natural K_0-inner product space and, hence (cf. Remark 1.1(II)), a K_0-normed vector space (over K_0) under the K_0-norm $\| . \|_0 : E \rightarrow K_{0+}$ defined by

$$(1.14) \quad \|x\|_0 = \langle x, x\rangle_0^{1/2} \quad \forall\, x \in E.$$

Since $(E, \langle . , . \rangle)$ is also a K-inner product space, E becomes a K_0-normed vector space (over K) with the K_0-norm $\| . \| : E \to K_{0+}$ defined by $\|x\| = \langle x, x\rangle^{1/2}$ (see (1.2)). But from (1.13) and (1.14), we obviously have

$$(1.15) \quad \|x\| = \|x\|_0 \quad \forall\, x \in E,$$

and the two norms are equivalent. That is, the vector space E over K and the vector space E over K_0 are isomorphic (under the identity map) as well as isometric in the usual K_0-metrics induced via the respective norms. If E is regarded as a natural K_0-inner product space and if we replace $\| . \|$ by $\| . \|_0$, the relations (1.3)–(1.5) give exactly the same mappings as were used by Zervos [**18**, p. 372] and, hence, Definition 1.2 turns out to be precisely the definition of g.c.r.'s of E_ω as given by Zervos [**18**, p. 372] (he calls these 'd.e.'). In view of (1.15), since the mappings (1.3), (1.5), and Definition 1.2 remains unaffected when $\| . \|$ is replaced by $\| . \|_0$, we infer that, in the special case when E is taken as a natural K-inner product space, the family $D(E_\omega)$ in Definition 1.2 coincides with the family of all g.c.r.'s of E_ω as introduced by Zervos [**18**, p. 372], when E is regarded as a natural K_0-inner product space in the above manner.

If $E = K$ is taken as a 1-dimensional natural K-inner product space with $H = \{1\}$ (so that $\|z\| = |z|$) and if for each $b \in K$ we define (see [**15**, p. 116]) $\varphi_b(z) = 1/(z - b)$ for $z \in K_\omega$, then the mapping ψ_b in (1.5) and the mapping φ_b are connected through the equations

$$\psi_b(z) = (z - b)/|z - b|^2 = \overline{\varphi_b(z)} \quad \forall\, z \in K_\omega,$$

which amounts to the relation $\overline{\psi_b(z)} = \varphi_b(z)$ for $z \in K_\omega$. Consequently, if $A \subseteq K_\omega$ and $b \in K - A$, then $\psi_b(A)$ is K_0-convex if and only if $\overline{\psi_b(A)} = \varphi_b(A)$ is K_0-convex. That is, we may replace the map ψ_b in Definition 1.2 by the map φ_b without affecting the nature of the set A. We, therefore, have the following

Remark 1.6.(I) If $E = K$ is taken as a natural K-inner product space, then a subset A of K_ω is a g.c.r. in the sense of Definition 1.2 if and only if A is a g.c.r. of the field K_ω in the sense of Zervos (see [**15**, p. 116] or Definition 2 of a d.e. in [**18**, pp. 352–353] with B taken as the family of all K_0-convex subsets of K).

(II) In view of Remark (I) above and the characterization due to Zervos [**18**, pp. 372–387] of the class $D(K_\omega)$ when $K = \mathbf{C}$, we get the following result (see [**18**, p. 352] or [**15**, p. 116]): The nontrivial members of $D(\mathbf{C}_\omega)$ are the open interior (or exterior) of circles or the open half-planes, adjoined with a connected subset (possibly empty) of their boundary. Members of $D(\mathbf{C}_\omega)$, with all or no boundary points included, will be termed as (*classical*) *circular regions* of $\mathbf{C}_\omega$.

2. Pseudo-derivatives of abstract polynomials. Throughout this section E and V will denote vector spaces over the same field K. The

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concept of a polynomial or of a homogeneous polynomial from a Banach (or normed vector) space X over **C** to another such space Y is too well-known (see Definition 26.2.2 in [**2**, p. 760] and Definition 2.2 in [**10**, p. 303]). The term 'abstract homogeneous polynomial' has been used by Hörmander [**3**] (see also [**8**, **9**, **15**, **16**, **17**]) in the same context, but in a much more general situation, when X and Y are taken as vector spaces over K. It is, however, based on these facts that we use the terminology 'abstract polynomials' in the following

Definition 2.1. A mapping $P : E \to V$ is called a *vector-valued abstract polynomial* of degree n if for every $x, y \in E$,

$$(2.1)\qquad P(x + \rho y) = \sum_{k=0}^{n} A_k(x, y)\rho^k \ \forall \rho \in K,$$

where the $A_k(x, y) \in V$ and are independent of ρ and are such that $A_n(x, y) \not\equiv 0$. We shall denote by $\mathscr{P}_n{}^*$ the class of all vector-valued abstract polynomials of degree n from E to V. The term 'abstract polynomial' will be used in the context $V = K$ and, in that case, we shall write $\mathscr{P}_n{}^*$ simply as $\mathscr{P}_n$ to distinguish it from the general case.

Following the same method as given in [**2**, pp. 761–762] or as in [**10**, Theorem (2.2)] we easily see that $A_0(x, y)$ is independent of y, that $A_n(x, y)$ is independent of x, and that $A_k(x, y)$ is a vector-valued abstract polynomial of degree $n - k$ in x (for each fixed y) and also a vector-valued abstract homogeneous polynomial of degree k in y (for each fixed x). (A precise definition of a (vector-valued) abstract homogeneous polynomial can be found in [**3**, pp. 55, 59].) All this implies the existence of at least one nonzero element $h \in E$ for which $A_n(x, h) = A_n(0, h) \neq 0$ for every $x \in E$. An element h with this property will be called *faithful* to P. This terminology is motivated by the observation that, given such an element h, the expression on the right hand side of the equation

$$(2.2)\qquad P(x + \rho h) = \sum_{k=0}^{n} A_k(x, h)\rho^k \ \forall \rho \in K$$

is a polynomial of degree (exactly) n from K to V for each fixed $x \in E$ (provided only that $P \in \mathscr{P}_n{}^*$). The set of all elements faithful to P will be denoted by $F(P)$. Consequently, if $P \in \mathscr{P}_n{}^*$ and is given by (2.1), then

$$(2.3)\qquad F(P) = \{h \in E : h \neq 0; A_n(0, h) \neq 0\} \neq 0.$$

If $P \in \mathscr{P}_n{}^*$, we shall write

$$Z(P) = \{x \in E : P(x) = 0\}$$

and call it the *null-set* of P.

Definition 2.2. Given $P \in \mathscr{P}_n{}^*$ (via (2.1)) and an element $h \in F(P)$, we define for each $k = 1, 2, \ldots, n$, the kth *pseudo-derivative* $P_h{}^{(k)}$ of P

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(relative to h) to be the mapping from E to V given by

(2.4) $\quad P_h^{(k)}(x) = k!A_k(x, h) \quad \forall\, x \in E.$

Indeed, $P_h^{(k)} \in \mathscr{P}_{n-k}{}^*$. The first few members in (2.4) will be written as P_h', P_h'', etc.

PROPOSITION 2.3. *If $P \in \mathscr{P}_n{}^*$ and $h \in F(P)$, then $h \in F(P_h^{(k)})$ for $k = 1, 2, \ldots, n - 1$, and we have*

$$P_h^{(k+1)}(x) = (P_h^{(k)})_h'(x) \quad \forall\, x \in E, k = 1, 2, \ldots, n - 1.$$

Proof. For any choice of k, $1 \leqq k \leqq n - 1$, we know from (2.4) that

$$P_h^{(k)}(x) = k!A_k(x, h) = Q(x), \quad \text{say}.$$

We claim that $h \in F(Q)$ and that $Q_h'(x) = P_h^{(k+1)}(x)$ for every $x \in E$. To this effect, we proceed as follows: For any $x \in E$ and $\rho \in K$, we have

$$Q(x + \rho h) = (k!)A_k(x + \rho h, h).$$

But (2.1) implies that

$$(2.5) \qquad P((x + \rho h) + \mu h) = \sum_{k=0}^{n} A_k(x + \rho h, h)\mu^k \;\forall \mu \in K,$$

$$(2.6) \qquad P(x + (\rho + \mu)h) = \sum_{k=0}^{n} A_k(x, h)(\rho + \mu)^k \;\forall \mu \in K.$$

Since $P((x + \rho h) + \mu h) = P(x + (\rho + \mu)h)$ for all $\mu \in K$, we see that the coefficient of μ^k in each of the expressions on the right hand side of (2.5) and (2.6) are equal. That is,

$$A_k(x + \rho h, h) = \sum_{j=0}^{n-k} C(k + j, k)A_{k+j}(x, h)\rho^j$$

$$= \sum_{j=k}^{n} C(j, k)A_j(x, h)\rho^{j-k}.$$

Therefore,

$$(2.7) \qquad P_h^{(k)}(x + \rho h) = Q(x + \rho h)$$

$$= \sum_{j=k}^{n} j(j - 1) \ldots (j - k + 1)A_j(x, h)\rho^{j-k},$$

wherein the coefficient of ρ^{n-k}, i.e., $k!C(n, k)A_n(x, h)$, does not vanish. Thus $Q \in \mathscr{P}_{n-k}{}^*$, $h \in F(Q)$ (see (2.3) for the definition of $F(Q)$), and the coefficient of ρ in the expansion (2.7) of $Q(x + \rho h)$ is given by $(k + 1)!A_{k+1}(x, h)$. The definition of $Q_h'(x)$ (see (2.4) with $k = 1$) then implies that

$$Q_h'(x) = (k + 1)!A_{k+1}(x, h) = P_h^{(k+1)}(x),$$

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as was to be proved. Since the above arguments are valid for any value of $k = 1, 2, \ldots, n - 1$, the proof is complete.

Remark 2.4. (I) If $P \in \mathscr{P}_n^*$, $h \in F(P)$, and if

$$P(x + \rho h) = \sum_{j=0}^{n} A_j(x, h)\rho^j \ \forall \rho \in K,$$

then (2.7) says that $P_h^{(k)} \in \mathscr{P}_{n-k}^*$, $h \in F(P_h^{(k)})$, and that the expression for $P_h^{(k)}(x + \rho h)$ is precisely what we obtain when we formally differentiate with respect to ρ the expression for $P(x + \rho h)$ (regarded as a polynomial in ρ from K to V (cf. (2.2)). This would serve as a quick means for realizing the expression for $P_h^{(k)}(x + \rho h)$.

(II) Proposition 2.3 suggests that, for a fixed $h \in F(P)$, the vector-valued abstract polynomials $P_h^{(k)}$ $(k = 1, 2, \ldots, n)$ deserve to be called successive pseudo-derivatives of P.

(III) If an nth degree (ordinary) polynomial $f : K \to K$ is given by

$$f(x) = \sum_{j=0}^{n} a_j x^j \ \forall x \in K \ (a_j \in K, a_n \neq 0), \tag{2.8}$$

then (for every $x, y \in K$)

$$\begin{aligned} f(x + \rho y) &= \sum_{j=0}^{n} \left[a_j \sum_{k=0}^{j} C(j, k) x^{j-k} y^k \rho^k \right] \\ &= \sum_{k=0}^{n} \left[y^k \sum_{j=k}^{n} C(j, k) a_j x^{j-k} \right] \rho^k = \sum_{k=0}^{n} A_k(x, y)\rho^k \ \forall \rho \in K, \end{aligned}$$

where

$$A_k(x, y) = (y^k/k!) \cdot \sum_{j=k}^{n} j(j-1) \ldots (j-k+1) a_j x^{j-k}.$$

We notice that the coefficients $A_k(x, y)$ are independent of ρ, that

$$A_n(x, y) \equiv A_n(0, y) = a_n y^n \neq 0 \quad \text{for all } y \neq 0,$$

and that

$$A_k(x, y) = y^k f^{(k)}(x)/k!,$$

where

$$f^{(k)}(x) = \sum_{j=k}^{n} j(j-1) \ldots (j-k+1) a_j x^{j-k} \tag{2.9}$$

defines the kth *formal derivative* (see [**15**, p. 121] or [**17**, p. 553]) of f with respect to x. To sum up: If f is a polynomial of degree n from K to K, then

(i) f is necessarily an abstract polynomial of degree n from K to K (i.e., $f \in \mathscr{P}_n$ with $E = K$),

(ii) every nonzero element of K is faithful to f when f is regarded as a member of $\mathscr{P}_n$. That is, $F(f) = K - \{0\}$,

(iii) for each value of $k = 1, 2, \ldots, n$, the kth pseudo-derivative $f_h^{(k)}$ of the abstract polynomial $f \in \mathscr{P}_n$ (with $E = K$) and the kth formal derivative $f^{(k)}$ of the (ordinary) polynomial f (see (2.9)) satisfy the relationship

$$(2.10) \quad f_h^{(k)}(x) = h^k f^{(k)}(x) \quad \forall\, x \in E, h \in K - \{0\}.$$

In particular (for $h = 1$), $f_1^{(k)} = f^{(k)}$ and the two notions coincide. It may be noted that the discussion in Remark III above applies in particular also when $K = \mathbf{C}$, but in this case the formal derivative $f^{(k)}$ becomes really the kth derivative of f defined via calculus.

3. The main theorems. Having introduced the concepts of g.c.r.'s and pseudo-derivatives of vector-valued abstract polynomials in K-inner product spaces (Sections 2 and 3) and having realized that these are analogous extensions of the corresponding notions of (classical) circular regions and derivatives of (ordinary) polynomials in the complex plane, our attempt now is to see if these concepts really extend their cooperation in letting us formulate a Lucas-type theorem for the class of polynomials in $\mathscr{P}_n^*$. The present section answers this problem in the affirmative and, as a first step in this direction, we prove the following theorem concerning the polynomials in $\mathscr{P}_n$ (the special case when $V = K$). Throughout the discussion in this section E will denote a K-inner product space and V a vector space over K.

THEOREM 3.1. *If $P \in \mathscr{P}_n$ and $S \in D(E_\omega)$ such that $\omega \notin S$ and $Z(P) \subseteq S$, then $Z(P_h') \subseteq S$ for every $h \in F(P)$.*

Proof. Take an arbitrary but fixed element $h \in F(P)$ (see (2.3)). In order to show that $Z(P_h') \subseteq S$, we take any element $x \in Z(P_h')$. In case $P_h'(x) = 0 = P(x)$, then $x \in Z(P) \subseteq S$, and we are done. In case $P_h'(x) = 0 \neq P(x)$, we still prove that $x \in S$ as follows: Suppose on the contrary that $x \notin S$. Since $h \in F(P)$ and K is algebraically closed, we can write (cf. (2.1) and (2.2))

$$P(x + \rho h) = \sum_{k=0}^{n} A_k(x, h)\rho^k \ \forall \rho \in K$$

$$= A_n(x, h) \cdot \prod_{j=1}^{n} [\rho - \rho_j(x, h)] \ \forall \rho \in K,$$

where $A_k(x, h)$ and $\rho_j(x, h)$ belong to K and are independent of ρ such that $A_n(x, h) = A_n(0, h) \neq 0$. The two expressions for $P(x + \rho h)$ being identically equal for all $\rho \in K$, the coefficients of various powers of ρ in the two representations must be equal. If $\Delta(k, n)$ denotes the sum of all possible products that can be formed out of the scalars $\rho_1(x, h)$,

$\rho_2(x, h), \ldots, \rho_n(x, h)$ taken k at a time, then

$$P(x) = A_0(x, h) = (-1)^n A_n(x, h) \cdot \Delta(n, n),$$
$$P_h'(x) = A_1(x, h) = (-1)^{n-1} A_n(x, h) \cdot \Delta(n-1, n).$$

Since in the case under consideration $P(x) \neq 0 = P_h'(x)$, we now conclude that $\rho_j(x, h) \neq 0$ for all j and, hence, that

$$(3.1) \qquad P_h'(x)/P(x) = -\Delta(n-1, n)/\Delta(n, n) = -\sum_{j=1}^{n} 1/\rho_j(x, h) = 0.$$

If we write $\rho_j \equiv \rho_j(x, h)$ and realize that $P(x + \rho_j h) = 0$ for all j, then $x + \rho_j h \in Z(P) \subseteq S$ and (due to (1.3)–(1.5)) we have

$$(3.2) \qquad \psi_x(x + \rho_j h) = (1/\bar{\rho}_j)(h/\|h\|^2) \in \psi_x(S).$$

Since $x \notin S$ and $S \in D(E_\omega)$, we see that $\psi_x(S)$ is K_0-convex and from (3.2) we obtain

$$(3.3) \qquad (1/n)\left[\sum_{j=1}^{n} 1/\bar{\rho}_j\right] \cdot (h/\|h\|^2) \in \psi_x(S).$$

But the hypothesis $\omega \notin S$ and the relation (1.7) imply that

$$0 = \psi_x(\omega) \notin \psi_x(S).$$

The relation (3.3) then suggests that

$$\sum_{j=1}^{n} 1/\bar{\rho}_j \neq 0.$$

That is,

$$\sum_{j=1}^{n} 1/\rho_j(x, h) \sum_{j=1}^{n} 1/\rho_j \neq 0,$$

which contradicts (3.1). This completes our proof.

In the following corollary we shall take $E = K$ as a 1-dimensional natural K-inner product space over K, so that K satisfies the structural properties of the vector space E of Theorem 3.1 and $D(K_\omega)$ makes sense (see Remark 1.1 (I) and Remark 1.6 (I)).

COROLLARY 3.2. ([**18**, Theorem 4, p. 360]). *Let a polynomial $f : K \to K$ be given by* (2.8) *and let f' denote the (first) formal derivative of f (cf.* (2.9) *with $k = 1$). If $A \in D(K_\omega)$ such that $\omega \notin A$ and $Z(f) \subseteq A$, then $Z(f') \subseteq A$.*

Proof. In view of Remark 2.4 (III) we know that $f \in \mathscr{P}_n$ (with $E = K$), that $F(f) = K - \{0\}$, and that $f_h'(x) = hf'(x)$ for every $x \in E$ and $h \in K - \{0\}$. In particular, $f_1'(x) = f'(x)$. Since f and A satisfy the hypotheses of Theorem 3.1, we conclude that $Z(f') \equiv Z(f_1') \subseteq A$.

Remark. 3.3.(I) In the special case when $K = \mathbf{C}$ and A is taken as the interior of a circle (which is indeed a special variety of g.c.r.'s of $\mathbf{C}_\omega$ in view of Remark 1.6(III)), the above corollary reduces essentially to Lucas' theorem (see introduction) and, hence, the set A becomes a Lucas-set for polynomials from $\mathbf{C}$ to $\mathbf{C}$.

(II) In view of Remark (I) above and the terminology used earlier in the introductory material of this paper, we may now define a subset $A \subseteq E$ to be a *Lucas-set for abstract polynomials* in $\mathscr{P}_n$ if, whenever A contains the null-set $Z(P)$ of any abstract polynomial $P \in \mathscr{P}_n$, A also contains the null-set $Z(P_h')$ of the pseudo-derivative P_h' of P (relative to each fixed $h \in F(P)$). Theorem 3.1 then says that every g.c.r. A of E_ω ($\omega \notin A$) is a Lucas-set for abstract polynomials.

Repeated applications of Theorem 3.1 and Proposition 2.3 immediately furnish the following theorem which includes Theorem 3.1 and which deals with the null-sets of successive pseudo-derivatives.

THEOREM 3.4. *If $P \in \mathscr{P}_n$ and $S \in D(E_\omega)$ such that $\omega \notin S$ and $Z(P) \subseteq S$, then $Z(P_h^{(k)}) \subseteq S$ for every $h \in F(P)$, $k = 1, 2, \ldots, n - 1$.*

In the remainder of this section our attempt is to obtain, via application of Theorem 3.4, a formulation of Lucas' theorem that extends Theorem 3.4 and, hence, Theorem 3.1 to the class $\mathscr{P}_n^*$ and that answers the general problem posed in the beginning of this section. In order to be able to formulate a concise theorem, we need the following concepts.

A mapping $L : V \to K$ is called a *linear-form* on V if

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \quad \text{for } u, v \in V \quad \text{and} \quad \alpha, \beta \in K.$$

Translations of maximal subspaces of V are termed *hyperplanes* of V and are characterized (cf. [**13**, p. 40]) by sets of the form $\{v \in V : L(v) = t\}$, where L is a nontrivial linear form on V and t is a fixed element of K. A subset M of V is called (see [**3**, p. 59] or [**14**, p. 95]) *supportable* if for every $\xi \in V - M$, there exists a hyperplane through ξ and the origin which does not intersect M. That is, for every $\xi \in V - M$, there exists a linear form L ($\not\equiv 0$) on V such that $L(\xi) = 0$ but $L(v) \neq 0$ for every $v \in M$. Quite obviously, the origin cannot belong to any supportable subset of V. The following proposition gives a general method for constructing a supportable subset of an arbitrary vector space V.

PROPOSITION 3.5. *The complement of every maximal subspace of V is a supportable subset of V.*

Proof. If S is a maximal subspace of V, then (cf. the method used in the proof of Theorem 4 in [**13**, p. 40]) there exists a nontrivial linear form L on V such that

$$S = \{v \in V : L(v) = 0\}.$$

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Put $M = V - S$. To each $\xi \in V - M$, we see that $L(\xi) = 0$ and that $L(v) \neq 0$ for every $v \in M$ (note that, incidentally, the same L works for each ξ). That is, $V - S$ is a supportable subset of V.

If $P \in \mathscr{P}_n^*$ and is given by (2.1) and if M is a supportable subset of V, we shall write

$$E(P) \equiv E(P, M) = \{x \in E : P(x) \notin M\},$$
$$F^*(P) \equiv F^*(P, M) = \{h \in E : A_n(0, h) \in M\}.$$

Note that $F^*(P) \subseteq F(P)$, because h cannot be zero and $0 \notin M$.

Remark 3.6. In the special case when $V = K$ (so that $\mathscr{P}_n^*$ coincides with $\mathscr{P}_n$) we see that $K - \{0\} = M$ (say) is a supportable subset of K (due to Proposition 3.5), that the corresponding set $E(P) \equiv E(P, M)$ reduces essentially to the null-set $Z(P)$ of P, and that the set $F^*(P) \equiv F^*(P, M)$ coincides with the nonempty set $F(P)$ given by (2.3).

PROPOSITION 3.7. *Let* dim $V \geqq 2$. *For every nonzero element $a \in V$, there exists a supportable subset M of V such that $a \in M$.*

Proof. Since dim $V \geqq 2$, we can always find an element $b \in V$ such that a and b are linearly independent. If B denotes the subspace of V spanned by $\{b\}$, then $a \notin B$. Using the same method of proof as in Theorem 2 of Wilansky [**13**, p. 19], we can obtain a linear form L on V such that $L(a) = 1 \neq 0$ and $L(x) = 0$ for $x \in B$. Since the set

$$S = \{x \in V : L(x) = 0\}$$

is a maximal subspace of V, by Proposition 3.5 its complement $V - S = M$ (say) is a supportable subset of V. Obviously, $a \in M$. This completes the proof.

Remark 3.8. In case $V = K$, Remark 3.6 says that $F^*(P) = F(P) \neq \emptyset$ for every $P \in \mathscr{P}_n^* \equiv \mathscr{P}_n$ when M is taken as $K - \{0\}$. However, in case dim $V \geqq 2$, every element h of the nonempty set $F(P)$ determines a nonzero element $A_n(0, h) \in V$ and, via Proposition 3.7, a supportable subset M such that $A_n(0, h) \in M$. That is for every $h \in F(P)$ there exists an M such that $h \in F^*(P)$. In other words: For every $P \in \mathscr{P}_n^*$, there exists an M such that $F^*(P) \neq \emptyset$.

THEOREM 3.9. *If $P \in \mathscr{P}_n^*$, $S \in D(E_\omega)$ $(\omega \notin S)$ and if M is a supportable subset of V such that $F^*(P) \neq \emptyset$ and $E(P) \subseteq S$, then $E(P_h^{(k)}) \subseteq S$ for every $h \in F^*(P)$, $k = 1, 2, \ldots, n - 1$.*

Proof. For any $h \in F^*(P)$, let us note that $h \in F(P)$ and the $P_h^{(k)}$ are meaningful entities to talk about. Suppose, on the contrary, that $E(P_h^{(k)}) \subseteq S$ for some k in the set $\{1, 2, \ldots, n - 1\}$ and some $h \in F^*(P)$. There must then exist an element $z \notin S$ for which $P_h^{(k)}(z) \notin M$. The definition of M now guarantees the existence of a nontrivial linear form L on V such that $L(P_h^{(k)}(z)) = 0$ but $L(v) \neq 0$ for $v \in M$. If P is given by (2.1) and if we define a mapping $Q : E \to K$ by $Q(x) = L(P(x))$ for $x \in E$, then for every $x, y \in E$,

$$Q(x + \rho y) = L\left[\sum_{k=0}^{n} A_k(x, y)\rho^k\right] = \sum_{k=0}^{n} B_k(x, y)\rho^k \ \forall \rho \in K,$$

where $B_k(x, y) = L(A_k(x, y))$ are elements in K independent of ρ and where

$$B_n(x, y) = L(A_n(x, y)) = L(A_n(0, y) = B_n(0, y).$$

This implies that

$$B_n(x, h) = B_n(0, h) = L(A_n(0, h)) \neq 0$$

(since $h \in F^*(P)$) and that $Z(Q) \subseteq E(P) \subseteq S$. That is: $Q \in \mathscr{P}_n$, $h \in F(Q)$ and $Z(Q) \subseteq S$. From Theorem 3.4 we conclude that $Z(Q_h^{(k)}) \subseteq S$. Finally, since

$$Q_h^{(k)}(x) = k!B_k(x, h) = L(k!\mathrm{A}_k(x, h)) = L(P_h^{(k)}(x)) \ \forall\ x \in E,$$

the property of the point z implies that

$$Q_h^{(k)}(z) = L(P_h^{(k)}(z)) = 0.$$

Therefore, $z \in Z(Q_h^{(k)}) \subseteq S$, which contradicts the fact that $z \notin S$. The proof is now complete.

In the special case when $V = K$ and $M = K - \{0\}$, the above theorem reduces essentially to Theorem 3.4. In fact, Theorem 3.9 is the most general formulation of Lucas' theorem in the present paper. In conformity with our previous terminology, we see that g.c.r.'s of E_ω are indeed the Lucas-sets for vector-valued abstract polynomials from E to V.

4. Examples. In this section we give some interesting examples which fathom the degree of generality of our main theorems of Section 3 and which give assurances about the validity of hypotheses in the theorems that we have established. Let us first recall that the definition of the class $\mathscr{P}_n$ (more generally $\mathscr{P}_n^*$) in Section 2 only requires the scalar field K to be of characteristic zero whereas, in addition, K has to be algebraically closed (in order for $(E, \langle . , . \rangle)$ to be a K-inner product space) for defining in Section 1 the concept of g.c.r.'s of E_ω and (hence) for formulating our main theorems of Section 3. Let us also recall that if $(E, \langle . , . \rangle)$ is a K-inner product space then (using arguments similar to the ones following the proof of Theorem 1.5; see relations (1.13)–(1.15)) $(E, \langle . , . \rangle_0)$ also becomes a K_0-inner product space over K_0. Our first example below shows that Theorem 3.1 (and, hence, Theorems 3.4 and 3.9) cannot be further generalized to abstract polynomials from E to K_0, where E is a

K_0-inner product space and K_0 a maximal ordered field. That is, the theorems mentioned above cannot be generalized to vector spaces over nonalgebraically closed fields of characteristic zero.

Example 4.1. Let K_0 be a maximal ordered field, so that K_0 is a nonalgebraically closed field of characteristic zero (see [**11**, pp. 233, 250]) and $K_0(i) = K$ (say) is algebraically closed and of characteristic zero (cf. [**1**, pp. 38–40] or [**11**, pp. 248–255]). Let us take $E = K$ as a 1-dimensional natural K-inner product space over K with Hamel basis $H = \{1\}$ and $\langle x, y\rangle = x\bar{y}$ for $x, y \in K$. Then (cf. relations (1.13)–(1.15) and the discussions along the way) $E = K_0(i) = K$ becomes a 2-dimensional natural K_0-inner product space over K_0 under the K_0-inner product defined by

$$\langle x, y\rangle_0 = \operatorname{Re}\langle x, y\rangle = s_1s_2 + t_1t_2$$

for elements $x = s_1 + it_1$, $y = s_2 + it_2$ in $K_0(i)$ (with Hamel basis $H_1 = \{1, i\}$). Furthermore, let us also recall that the definition of $D(E_\omega)$ remains unaffected when, instead of treating E as a natural K-inner product space $(K, \langle . , . \rangle)$, we treat E as a natural K_0-inner product space $(K_0(i), \langle . , . \rangle_0)$ and replace (in Definition 2.1) $\langle . , . \rangle$ by $\langle . , . \rangle_0$ and $\| . \|$ by $\| . \|_0$ (see relations (1.4)–(1.5) and the discussion that follows). With this fact in view if we define, for a fixed element $a = 1 - i \in K_0(i) = E$,

$$S = \{x \in E : \langle x, a\rangle_0 \geqq 0\} = \{s + it : s, t \in K_0; s - t \geqq 0\},$$

then Proposition 1.4 and 1.5 imply that $A \in D(E_\omega)$. Next, we define a mapping P from $E = K_0(i)$ to K_0 by

$$P(x) = (s - t)(s^2 + 1) \quad \forall\, x = s + it \in E.$$

Then for elements $x = s + it$ and $y = s_1 + it_1$ in E, we have

$$\begin{aligned} P(x + \rho y) &= P[(s + \rho s_1) + i(t + \rho t_1)]\ \forall \rho \in K_0 \\ &= [(s - t) + \rho(s_1 - t_1)] \cdot [(s + \rho s_1)^2 + 1] \\ &= \sum_{k=0}^{3} A_k(x, y)\rho^k \text{ (say) } \forall \rho \in K_0, \end{aligned}$$

where $A_k(x, y) \in K_0$ and are independent of ρ such that

$$A_3(x, y) = A_3(0, y) = s_1^2(s_1 - t_1) \not\equiv 0.$$

Therefore, P is an abstract polynomial from $E = K_0(i)$ to K_0 and $S \in D(E_\omega)$ with $\omega \notin S$ such that (since $s^2 + 1$ cannot vanish for any $s \in K_0$ (see [**1**, p. 36]))

$$Z(P) = \{s + it : s, t \in K_0; s - t = 0\} \subseteq S.$$

But we can easily verify that $h = 1 + 2i \in F(P)$ (since $A_3(0, h) = -1 \neq 0$) and, for this h, $P_h'(x) = A_1(x, h) = 0$ for $x = (1 + \sqrt{2}) + i \notin S$. That is, Theorem 3.1 does not hold when K is replaced by K_0.

In relation (2.3) we have already shown that $F(P) \neq \emptyset$ for every $P \in \mathscr{P}_n$. Therefore, in order that Theorem 3.1 should not be vacuously true, it is essential that we show the existence of abstract polynomials $P \in \mathscr{P}_n$ and g.c.r.'s $S \in D(E_\omega)$ such that $\omega \notin S$. This is done in the following

Example 4.2. Given a finite or infinite dimensional K-inner product space $(E, \langle . , . \rangle)$ and any fixed nonzero element $a \in E$, let us take

$$S = \{x \in E : \operatorname{Re} \langle x, a \rangle \leqq 0\}.$$

If we define $L(x) = \langle x, a \rangle$ for $x \in E$, then L is a nontrivial linear form on E and the set

$$\{x \in E : L(x) = 0\} = B \text{ (say)}$$

is a maximal subspace of E. Since $a \neq 0$ and $L(a) = \langle a, a \rangle = \|a\|^2 > 0$, we see that $a \notin B$ and (hence) that every element $x \in E$ can be uniquely represented as $x = ta + b$ for some $t \in K$ and $b \in B$. Obviously, $B \subseteq S$ and $x \in S$ (resp. $x \in B$) if and only if $\operatorname{Re}(t) \leqq 0$ (resp. $t = 0$). If λ_j $(j = 1, 2, \ldots, n)$ are arbitrarily selected elements in K such that $\operatorname{Re}(\lambda_j) \geqq 0$ and if we define a mapping $P : E \rightarrow K$ by

$$P(x) = \prod_{j=1}^{n} (t + \lambda_j) \ \forall x = ta + b \in E,$$

we see that for every element $x = ta + b$, $y = t_1 a + b_1$ in E,

$$P(x + \rho y) = \prod_{j=1}^{n} (t + \rho t_1 + \lambda_j) \ \forall \rho \in K.$$

The product on the right hand side of the last equality can obviously be expressed in the form (2.1) with

$$A_n(x, y) = A_n(0, y) = t_1^n \not\equiv 0.$$

Therefore, $P \in \mathscr{P}_n$ and $F(P) = E - B$. Also, $P(x) = 0$ if and only if $x = -\lambda_j a + b$ for some j and some $b \in B$. Since $\operatorname{Re}(-\lambda_j) = -\operatorname{Re}(\lambda_j) \leqq 0$, every element $x \in Z(P)$ must lie in S. Consequently, $P \in \mathscr{P}_n$ with $F(P) = E - B$ such that $Z(P) \subseteq S$, where $\omega \notin S$ and $S \in D(E_\omega)$ in view of Propositions 1.4 and 1.5.

Remark 4.3. Since the choices for a and λ_j in the above example are arbitrary, we have now shown that for every g.c.r. $S \in D(E_\omega)$ determined (in the manner of the above example) by each point $a \in E - \{0\}$, there exist (in fact) infinitely many abstract polynomials $P \in \mathscr{P}_n$ satisfying the hypotheses in Theorem 3.1.

Since Theorem 3.1 is a special case of Theorem 3.9 for $V = K$, Example 4.2 does in fact verify the validity of the hypotheses of the latter when

$V = K$. In the general case when $\dim V \geqq 2$, Remark 3.8 establishes only that the part hypothesis $F^*(P) \neq \emptyset$ in Theorem 3.9 is valid. In the following example, however, we show the validity of all the hypotheses in their totality.

Example 4.4. Given a finite or infinite dimensional K-inner product space $(E, \langle . , . \rangle)$ with Hamel basis H, let us take

$$S = \{x \in E : \operatorname{Re} \langle x, h_1 \rangle \geqq 0\},$$

where h_1 is a fixed element arbitrarily selected from H. Then $S \in D(E_\omega)$ due to Propositions 1.4 and 1.5, and $\omega \notin S$. Next, we take V as a (finite or infinite dimensional) vector space over K of $\dim V \geqq 2$ with a Hamel basis H_1 and select out arbitrarily an element $f_1 \in H_1$. Then $H_1 - \{f_1\} = H_1'$ (say) $\neq \emptyset$. Denote by B the subspace of V spanned by the set H_1'. Since B is a maximal subspace of V, $V - B = M$ (say) is a supportable subset of V (see Proposition 3.5) such that $f_1 \in M$. If we arbitrarily choose elements $b \in B$ and $\lambda_j \in K$ $(j = 1, 2, \ldots, n)$ with $\operatorname{Re}(\lambda_j) \geqq 0$, and if we define a mapping $P : E \rightarrow V$ by

$$P(x) = \left[\prod_{j=1}^{n} (\langle x, h_1 \rangle - \lambda_j)\right] \cdot f_1 + b \ \forall x \in E,$$

then for every $x, y \in E$,

$$\begin{aligned} P(x + \rho y) &= \left[\prod_{j=1}^{n} (\langle x + \rho y, h_1 \rangle - \lambda_j)\right] \cdot f_1 + b \ \forall \rho \in K \\ &= \left[\prod_{j=1}^{n} (\rho \langle y, h_1 \rangle + \langle x, h_1 \rangle - \lambda_j)\right] \cdot f_1 + b. \end{aligned}$$

The last expression can obviously be expressed in the form (2.1) with the coefficient $A_n(x, y)$ satisfying the relations

$$A_n(x, y) = A_n(0, y) = \langle y, h_1 \rangle^n f_1 \not\equiv 0.$$

If we set $h = h_1/\|h_1\|^2$, then

$$A_n(0, h) = \langle h, h_1 \rangle^n f_1 = f_1 \neq 0.$$

Therefore, $P \in \mathscr{P}_n^*$ and, since $A_n(0, h) = f_1 \in M$, we see that $h \in F^*(P)$ and $F^*(P) \neq \emptyset$. Furthermore, $P(x) \notin M$ if and only if $P(x) \in B$, which holds if and only if $\langle x, h_1 \rangle - \lambda_j = 0$ (for, otherwise, f_1 would belong to B, contradicting that $f_1 \notin B$). That is, if $x \in E(P)$ then

$$\operatorname{Re} \langle x, h_1 \rangle = \operatorname{Re}(\lambda_j) \geqq 0 \quad \text{for some } j$$

and so $x \in S$. We have, therefore, shown that all the hypotheses in Theorem 3.9 are satisfied by the g.c.r. S, the supportable subset M, and the polynomial $P \in \mathscr{P}_n^*$ as considered in this example.

Due to the arbitrary nature of the selection of elements h_1, f_1, b, and λ_j, we observe that for every g.c.r. $S \in D(E_\omega)$, determined in the manner of the above example, there exist infinitely many polynomials $P \in \mathscr{P}_n{}^*$ satisfying the hypotheses of Theorem 3.9, including the case when E and or V are infinite dimensional. Consequently, Examples 4.2 and 4.4 together reveal the fact that the hypotheses in Theorem 3.9 (and, hence, in Theorems 3.1 and 3.4) are not a mere assembly of figments of imagination manipulated to produce a vacuously true mathematical statement.

Concluding remarks. Though we have defined the concept of g.c.r.'s of E_ω when E is a K-inner product space, both the definition as well as Theorems 3.1, 3.4, and 3.9 as such remain valid when E is initially taken as a K_0-normed vector space over K. This is obvious in the light of the fact that only the K_0-norm, induced by the initial K-inner product on E, has been used in Definition 2.1 and in the proofs of those theorems (see (1.5)) and, naturally enough, the same analysis goes through if we start initially with a K_0-norm on E. The reason why we have not discussed $D(E_\omega)$ for K_0-normed vector spaces is that the family $D(E_\omega)$ in this case would not be as rich as it is in the case when E is a K-inner product space. For, in the latter case, the sets

$$\{x \in E : \operatorname{Re} \langle x, a\rangle \geqq 0\}$$

are members of $D(E_\omega)$ while, in the former case, these sets are meaningless to talk about.

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