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$(n+1)$-TENSOR NORMS OF LAPRESTÉ'S TYPE

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Abstract. We study an $(n+1)$-tensor norm $\alpha_{\mathbf{r}}$ extending to $(n+1)$-fold tensor products, the classical one of Lapresté in the case $n = 1$. We characterise the maps of the minimal and the maximal multi-linear operator ideals related to $\alpha_{\mathbf{r}}$ in the sense of Defant and Floret (A. Defant and K. Floret, *Tensor norms and operator ideals*, North Holland Mathematical Studies, no. 176 (North Holland, Amsterdam, Netherlands, 1993). As an application we give a complete description of the reflexivity of the $\alpha_{\mathbf{r}}$-tensor product $(\otimes_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}})$.

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1. Introduction. In [**13**] Pietsch proposed building a systematic theory of ideals of multi-linear mappings between Banach spaces, similar to the already well-developed one regarding linear maps, as a first step to study ideals of more general nonlinear operators. Since then several classes of multi-linear operators more or less related to classical absolutely p-summing operators have been studied, although without to deal with aspects derived from a general organised theory.

Having in mind the close connection existing in linear case between problems of this kind and tensor products (see [**3**] for a systematic survey of the actual state of the art), in the present setting it is expected an analogous connection with multiple tensor products. However, a systematic study of this approach has not been initiated until the works of Floret [**4**, **5**], mainly motivated by the potential applications of the new theory to infinite holomorphy. In this way, classical notions of maximal operator ideals and its associated α-tensor norm, dual tensor norm α' and the related α-nuclear and α-integral operators can be extended to the framework of multi-linear operator ideals and multiple tensor products.

However, there are few concrete examples of multi-tensor norms where the general concepts of the theory have been applied and checked. The purpose of this paper is to study an $(n+1)$-tensor norm $\alpha_{\mathbf{r}}$ on tensor products $\bigotimes_{j=1}^{n+1} E_j$, $1 \le n$, of $n+1$ Banach spaces E_j, extending the classical one of Lapresté for $n = 1$, as well as its associated $\alpha_{\mathbf{r}}$-nuclear and $\alpha_{\mathbf{r}}$-integral multi-linear operators. Knowledge of such operators allows us to characterise the reflexivity of the corresponding tensor product $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ of spaces ℓ^{u_j}.

The paper is organised as follows. First we introduce the notation and some general facts to be used. In Section 2 we define the $(n+1)$-fold tensor product $\bigotimes_{\alpha_{\mathbf{r}}}\left(E_1, E_2, \ldots, E_n, F\right)$, $n \in \mathbb{N}$ of type $\alpha_{\mathbf{r}}$ of Banach spaces E_j, $1 \le j \le n$ and F. We find its topological dual introducing the so-called $\mathbf{r}$-dominated maps and obtain multi-linear extensions of the classical theorems of Grothendieck–Pietsch and Kwapien

(Theorem 3). The latter one is the key to approximate **r**-dominated maps by multi-linear maps of finite rank in many usual cases (Theorem 7) and to compare different tensor norms $\alpha_{\mathbf{r}}$, a tool that will be very useful in our applications in the final section of the paper.

The elements of a completed $\alpha_{\mathbf{r}}$-tensor product canonically lead to multi-linear **r**-nuclear operators from $\prod_{j=1}^{n} E_j$ into F, which are considered in Section 3 and characterised by means of suitable factorisations in Theorem 9. According to the pattern of the general theory of multi-tensor norms, the next step must be the study of the so-called **r**-integral multi-linear maps, i.e. the maps in the ideal associated to the $\alpha_{\mathbf{r}}$-tensor norm in the sense of Defant–Floret [**3**]. To do this we need a technical result about the structure of some ultraproducts, which follows easily from the work of Raynaud [**14**]. It will be presented in Section 4 just before its use.

In Section 4 we characterise the **r**-integral operators, obtaining as a main result the 'continuous' version of previous factorisations of **r**-nuclear operators. Finally, in Section 5 we apply the characterisations of Sections 3 and 4 to study the reflexivity of $\alpha_{\mathbf{r}}$-tensor products and, more particularly, to characterise the reflexivity of $\alpha_{\mathbf{r}}$-tensor products of ℓ^u spaces, a result that, as far as we know, is indeed new for classical Lapresté's tensor norms.

We shall deal always with vector spaces defined over the field $\mathbb{R}$ of real numbers. Notation of the paper is standard in general. Some not so usual notations are settled now.

Given a normed space E, we shall denote by B_E its closed unit ball and $J_E : E \longrightarrow E''$ will be the canonical isometric inclusion of E into the bidual space E''. $B_{E'}$ will be considered as a compact topological space $(B_{E'}, \sigma(E', E))$ when provided with the topology induced by the weak*-topology $\sigma(E', E)$. For every $x \in E$, we shall denote by f_x the continuous function defined on $(B_{E'}, \sigma(E', E))$ as $f_x(x') = \langle x, x' \rangle$ for every $x' \in B_{E'}$. The symbol $E \approx F$ will mean that E and F are isomorphic normed spaces. The closed linear span in a Banach space E of a sequence $\{x_m\}_{m=1}^{\infty} \subset E$ (respectively of a single vector x) will be represented by $[x_n]_{m=1}^{\infty}$ (resp. $[x]$).

As usual, $\mathbf{e}_k$ denotes the kth standard unit vector in every ℓ^p, $1 \leq p \leq \infty$. ℓ_h^p, $h \in \mathbb{N}$ will be the ℓ^p-space defined over the set $\{1, 2, .., h\}$ with the standard measure.

Given a normed space E, a sequence $\{x_m\}_{m=1}^{k} \subset E$, $k \in \mathbb{N} \cup \{\infty\}$, and $1 \leq p \leq \infty$, we define in the case $p < \infty$

$$\pi_p\big((x_m)_{j=1}^{k}\big) := \left(\sum_{m=1}^{k} \|x_m\|^p\right)^{\frac{1}{p}}, \quad \varepsilon_p\big((x_m)_{m=1}^{k}\big) := \sup_{x' \in B_{E'}} \left(\sum_{m=1}^{k} \left|\left\langle x_m, x' \right\rangle\right|^p\right)^{\frac{1}{p}}$$

and when $p = \infty$

$$\pi_\infty\big((x_m)_{m=1}^{k}\big) := \varepsilon_\infty\big((x_m)_{m=1}^{k}\big) = \sup_{1 \leq m \leq k} \|x_m\|.$$

A sequence $\{x_m\}_{m=1}^{\infty} \subset E$ is called weakly p-absolutely summable, notation $(x_m)_{m=1}^{\infty} \in \ell^p(E)$ (resp. p-absolutely summable) if $\varepsilon_p\big((x_m)_{m=1}^{\infty}\big) < \infty$ (resp. $\pi_p((x_m)_{m=1}^{\infty}) < \infty$). Given Banach spaces E and F, an operator or linear map $T \in \mathcal{L}(E, F)$ is said to be p-absolutely summable if there exists $C \geq 0$ such that

$$(x_m)_{m=1}^{\infty} \in \ell^p(E) \implies \pi_p\Big(\big(T(x_m)\big)_{m=1}^{\infty}\Big) \leq C\, \varepsilon_p\big((x_m)_{m=1}^{\infty}\big). \tag{1}$$

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The linear space $\mathfrak{P}_p(E,F)$ of all p-absolutely summing operators from E into F becomes a Banach space under the norm $\mathbf{P}_p(T) := \inf\{C \geq 0 \mid (1) \text{ holds}\}$ for every $T \in \mathfrak{P}_p(E,F)$.

We always consider a finite cartesian product $\prod_{m=1}^{h} E_m$ of normed spaces E_m, $1 \leq m \leq h \in \mathbb{N}$ as a normed space provided with the ℓ^∞-norm $\|(x_m)_{m=1}^h\| = \sup_{m=1}^h \|x_m\|$. If F is a Banach space, we shall denote by $\mathcal{L}^h(\prod_{m=1}^h E_m, F)$ the Banach space of all h-linear continuous maps from $\prod_{m=1}^h E_m$ into F. Given $T \in \mathcal{L}^h(\prod_{m=1}^h E_m, F)$, we can define in a natural way the transposed *linear* map $T' : F' \longrightarrow \mathcal{L}^h\left(\prod_{m=1}^h E_m, \mathbb{R}\right)$ putting

$$\forall\, y' \in F' \quad \forall\, (x_m)_{m=1}^h \in \prod_{m=1}^h E_m \quad \left\langle T'(y'), (x_m)_{m=1}^h \right\rangle = \left\langle T\big((x_m)_{m=1}^h\big), y' \right\rangle.$$

Given maps $A_j \in \mathcal{L}(E_j, F_j)$ between normed spaces E_j and F_j, $1 \leq j \leq n$, we write

$$(A_j)_{j=1}^n := (A_1, A_2, \dots, A_n) : \prod_{j=1}^n E_j \longrightarrow \prod_{j=1}^n F_j$$

to denote the *continuous linear* map defined by

$$\forall\, (x_j)_{j=1}^n \in \prod_{j=1}^n E_j \quad (A_j)_{j=1}^n\big((x_1, x_2, \dots, x_n)\big) = \big(A_1(x_1), A_2(x_2), \dots, A_n(x_n)\big).$$

Sometimes we will write (A_j) instead of $(A_j)_{j=1}^n$. Concerning $(n+1)$-tensor norms, $n \geq 1$ (or multi-tensor norms), we refer the reader to the pioneer works [**4**] and [**5**]. If it is needed to emphasise, $\alpha\left(z; \bigotimes_{j=1}^{n+1} M_j\right)$ or similar notations will denote the value of the multi-tensor norm α of $z \in \otimes_{j=1}^{n+1} M_j$.

As customary, for $p \in [1, \infty]$, p' will be the conjugate extended real number such that $1/p + 1/p' = 1$. Given $n \geq 1$, in all the paper we denote by $\mathbf{r}$ an $(n+2)$-pla of extended real numbers $\mathbf{r} = (r_0, r_1, r_2, \dots, r_n, r_{n+1})$ such that $1 < r_0 \leq \infty$, $1 < r_j < \infty$, $1 \leq j \leq n+1$ and

$$1 = \frac{1}{r_0} + \frac{1}{r_1'} + \frac{1}{r_2'} + \cdots + \frac{1}{r_{n+1}'}. \tag{2}$$

Such $\mathbf{r}$ will be called an admissible $(n+2)$-pla. Moreover, we define w such that

$$\frac{1}{w} := \frac{1}{r_1'} + \frac{1}{r_2'} + \cdots + \frac{1}{r_n'}, \tag{3}$$

which gives the equality

$$n = \frac{1}{w} + \sum_{j=1}^n \frac{1}{r_j}. \tag{4}$$

For later use, we note that (2) implies

$$1 = \frac{r_0'}{r_1'} + \frac{r_0'}{r_2'} + \cdots + \frac{r_0'}{r_n'} + \frac{r_0'}{r_{n+1}'} \quad \text{and} \quad \frac{1}{r_{n+1}} = \frac{1}{r_0} + \frac{1}{r_1'} + \frac{1}{r_2'} + \cdots + \frac{1}{r_n'}, \tag{5}$$

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as well as

$$\frac{1}{w}=\frac{1}{r_0'}-\frac{1}{r_{n+1}'}=\frac{1}{r_{n+1}}-\frac{1}{r_0}\quad\Longrightarrow 1=\frac{1}{w}+\frac{1}{r_0}+\frac{1}{r_{n+1}'}\tag{6}$$

and moreover,

$$\forall\, 1\leq j\leq n\qquad r_{n+1}<w<r_j'\tag{7}$$

and

$$\forall\, 1\leq j\leq n+1\quad r_j<r_0.\tag{8}$$

To finish this Introduction we consider the following construction which will be of fundamental importance in all the paper. Given any measure space $(\Omega, \mathcal{A}, \mu)$ and an admissible $(n+2)-$pla $\mathbf{r}$, as a direct consequence of generalised Hölder's inequality and (2), we have a canonical $(n+1)$-linear map $\mathfrak{M}_\mu : L^{r_0}(\Omega, \mathcal{A}, \mu) \times \prod_{j=1}^n L^{r_j'}(\Omega, \mathcal{A}, \mu) \longrightarrow L^{r_{n+1}}(\Omega, \mathcal{A}, \mu)$ defined by the rule

$$\forall\,(f_j)_{j=0}^n\in L^{r_0}(\Omega,\mu)\times\prod_{j=1}^n L^{r_j'}(\Omega,\mu)\qquad \mathfrak{M}_\mu\big((f_j)\big)=\prod_{j=0}^n f_j$$

verifying $\|\mathfrak{M}_\mu\big((f_j)\big)\| \leq \|g\|_{L^{r_0}(\Omega)} \prod_{j=1}^n \|f_j\|_{L^{r_j'}(\Omega)}$. If $(\Omega, \mathcal{A}, \mu)$ is $\mathbb{N}$ with the counting measure, we will write simply $\mathfrak{M}$ instead of $\mathfrak{M}_\mu$. Moreover, given $g \in L^{r_0}(\Omega, \mu)$, we shall write D_g to denote the n-linear map from $\prod_{j=1}^n L^{r_j'}(\Omega, \mu)$ into $L^{r_{n+1}}(\Omega, \mu)$ such that

$$\forall\,(f_j)_{j=1}^n\in\prod_{j=1}^n L^{r_j'}(\Omega,\mu)\qquad D_g\big((f_j)_{j=1}^n\big)=\mathfrak{M}_\mu\big((g,f_1,\ldots,f_n)\big).\tag{9}$$

It will be important for later applications to remark that $\mathfrak{M}_\mu$ induces a linearisation map $\widetilde{\mathfrak{M}}_\mu : \big(L^{r_0}(\Omega, \mu)\widehat{\bigotimes}\big(\widehat{\bigotimes}_{j=1}^n L^{r_j'}(\Omega, \mu)\big), \pi\big) \longrightarrow L^{r_{n+1}}(\Omega, \mu)$ and a canonical map

$$\widehat{\mathfrak{M}}_\mu:\big(L^{r_0}(\Omega,\mu)\widehat{\bigotimes}\big(\widehat{\bigotimes}_{j=1}^n L^{r_j'}(\Omega,\mu)\big),\pi\big)/Ker(\widetilde{\mathfrak{M}}_\mu)\longrightarrow L^{r_{n+1}}(\Omega,\mu)$$

such that $\|\widehat{\mathfrak{M}}_\mu\| \leq 1$. Moreover, by (5) we obtain $f = f^{\frac{r_{n+1}}{r_0}} \prod_{j=1}^n f^{\frac{r_{n+1}}{r_j'}}$ for every $f \geq 0$ in $L^{r_{n+1}}(\Omega, \mu)$. As $f = f^+ - f^-$ for every $f \in L^{r_{n+1}}(\Omega, \mu)$, it turns out that $\widetilde{\mathfrak{M}}_\mu$ is a surjective map and $\widehat{\mathfrak{M}}_\mu$ becomes an isomorphism such that $\|\widehat{\mathfrak{M}}_\mu^{-1}\| \leq 2$.

2. $\alpha_{\mathbf{r}}$-tensor products and r-dominated multi-linear maps. Let E_j, $1 \leq j \leq n+1$ be normed spaces. Using classical methods we can show that

$$\alpha_{\mathbf{r}}\left(z;\bigotimes_{j=1}^{n+1}E_j\right):=\inf \pi_{r_0}\big((\lambda_m)_{m=1}^h\big)\prod_{j=1}^{n+1}\varepsilon_{r_j'}\big((x_m^j)_{m=1}^h\big),\tag{10}$$

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taking the infimum over all representations of z of type

$$z = \sum_{m=1}^{h} \lambda_m \left(\otimes_{j=1}^{n+1} x_{jm}\right), \quad x_{jm} \in E_j \;\; 1 \le j \le n+1,\; 1 \le m \le h,\;\; h \in \mathbb{N},$$

is a norm on $\bigotimes_{j=1}^{n+1} E_j$, which defines an $(n+1)$-tensor norm in the class of normed spaces. It is interesting to note that if $n = 1$, we obtain the classical tensor norm $\alpha_{r_2 r_1}$ of Lapresté (see **[3]** for details).

The just defined normed tensor product space will be denoted by $\left(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\right)$ or $\bigotimes_{\alpha_{\mathbf{r}}}\left(E_1, E_2, \ldots, E_{n+1}\right)$ and its completion by $\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\left(E_1, E_2, \ldots, E_{n+1}\right)$. It is clear that for every permutation σ on the set $\{1, 2, \ldots, n+1\}$ the map

$$I_\sigma : \sum_{i=1}^{m} \lambda_m \otimes_{j=1}^{n+1} x_{jm} \in \left(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\right) \longrightarrow \sum_{i=1}^{m} \lambda_m \otimes_{j=1}^{n+1} x_{\sigma(j)m} \in \left(\bigotimes_{j=1}^{n+1} E_{\sigma(j)}, \alpha_{\mathbf{s}}\right),$$

where $\mathbf{s}$ is the admissible $(n+2)$-pla $s_0 := r_0$ and $s_j = r_{\sigma(j)},\ 1 \le j \le n+1$, is an isometry from $\left(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\right)$ onto $\left(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{s}}\right)$. We shall use this type of isomorphism in Section 5 in the particular case of transpositions σ simply indicating that the transposed indexes $\sigma(j_0) = j_1, \sigma(j_1) = j_0$ in the way $j_0 \to j_1, j_1 \to j_0$.

To compute the topological dual of an $\alpha_{\mathbf{r}}$-tensor product we set a new definition.

DEFINITION 1. Let F and $E_j, 1 \le j \le n$ be normed spaces. A map $T \in \mathcal{L}^n\left(\prod_{j=1}^{n} E_j, F\right)$ is said to be **r**-dominated if there is $C \ge 0$ such that for every $h \in \mathbb{N}$ and every set of finite sequences $\{x_{jk}\}_{k=1}^{h} \subset E_j,\ \ 1 \le j \le n$ and $\{y'_k\}_{k=1}^{h} \subset F'$ the inequality

$$\pi_{r'_0}\left(\left(\left|\left\langle T(x_{1k}, x_{2k}, \ldots, x_{nk}), y'_k\right\rangle\right|\right)_{k=1}^{m}\right) \le C \left(\prod_{j=1}^{n} \varepsilon_{r'_j}\left(\left(x_{jk}\right)_{k=1}^{m}\right)\right) \varepsilon_{r'_{n+1}}\left(\left(y'_k\right)_{k=1}^{h}\right) \quad (11)$$

holds.

It is easy to see that the linear space $\mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^{n} E_j, F\right)$ of **r**-dominated n-linear maps from $\prod_{j=1}^{n} E_j$ into F is normed setting $\mathbf{P}_{\mathbf{r}}(T) := \inf\left\{C \ge 0 \;\middle|\; (11) \text{ holds}\right\}$ for every $T \in \mathfrak{P}_{\mathbf{r}}(\prod_{j=1}^{n} E_j, F)$, becoming a Banach space when F does. The interest on **r**-dominated multi-linear maps follows from the next result.

THEOREM 2. *$(\bigotimes_{\alpha_{\mathbf{r}}}(E_1, E_2, \ldots, E_n, F))' = \mathfrak{P}_{\mathbf{r}}(\prod_{j=1}^{n} E_j, F')$ for all normed spaces F and $E_j,\ \ 1 \le j \le n$.*

Proof. (1) Given $T \in \mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^{n} E_j, F'\right)$ and $z = \sum_{k=1}^{h} \lambda_k \left(\otimes_{j=1}^{n} x_{jk}\right) \otimes y_k$ in $\left(\bigotimes_{j=1}^{n} E_j\right) \bigotimes F$ we define $\varphi_T(z) = \sum_{k=1}^{h} \lambda_k \left\langle T\left((x_{1k}, x_{2k}, \ldots, x_{nk})\right), y_k\right\rangle$. It follows directly from Hölder's inequality, Definition 1 and (10)

$$\left|\varphi_T(z)\right| \le \mathbf{P}_{\mathbf{r}}(T)\, \alpha_{\mathbf{r}}(z) \implies \left\|\varphi_T\right\| \le \mathbf{P}_{\mathbf{r}}(T). \quad (12)$$

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(2) Conversely, let $\psi \in \left(\bigotimes_{\alpha_{\mathbf{r}}}(E_1, E_2, \dots, E_n, F)\right)'$. We define $T_\psi \in \mathcal{L}^n(\prod_{j=1}^n E_j, F')$ as

$$\forall\ (x_j)_{j=1}^n \in \prod_{j=1}^n E_j,\ \forall\ y \in F\quad \left\langle T_\psi\left((x_j)_{j=1}^n\right), y\right\rangle = \psi\left(x_1 \otimes x_2 \otimes \dots x_n \otimes y\right).$$

Given $\{x_{jk}\}_{k=1}^h \subset E_j,\ 1 \leq j \leq n$ and $\{y_k\}_{k=1}^h \subset F,\ h \in \mathbb{N}$, we have

$$\begin{aligned}
\pi_{r_0'}\left(\left(\left\langle T_\psi\left((x_{jk})_{j=1}^n\right), y_k\right\rangle\right)_{k=1}^h\right) &= \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}} \left|\sum_{k=1}^h \alpha_k\ \psi\left(\left(\otimes_{j=1}^n x_{jk}\right) \otimes y_k\right)\right| \\
&= \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}} \left|\psi\left(\sum_{k=1}^h \alpha_k \left(\otimes_{j=1}^n x_{jk}\right) \otimes y_k\right)\right| \\
&\leq \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}} \|\psi\|\ \pi_{r_0}\left((\alpha_k)_{k=1}^h\right)\left(\prod_{j=1}^n \varepsilon_{r_j'}\left((x_{jk})_{k=1}^h\right)\right)\varepsilon_{r_{n+1}'}\left((y_k)_{k=1}^h\right) \\
&\leq \|\psi\|\left(\prod_{j=1}^n \varepsilon_{r_j'}\left((x_{jk})_{k=1}^h\right)\right)\varepsilon_{r_{n+1}'}\left((y_k)_{k=1}^h\right).
\end{aligned}$$

By $\sigma(F'', F')$-density of F in F'' the latter inequality also holds when $y_k \in F''$, $1 \leq k \leq h$. Hence, $\mathbf{P}_{\mathbf{r}}(T_\psi) \leq \|\psi\|$ and clearly $\varphi_{T_\psi} = \psi$, giving by (1) $\mathbf{P}_{\mathbf{r}}(T_\psi) = \|\psi\|$. □

The name of **r**-dominated multi-linear maps is suggested by the following characterisation.

THEOREM 3. *Given Banach spaces E_j, $1 \leq j \leq n$ and F and $T \in \mathcal{L}^n(\prod_{j=1}^n E_j, F)$, the following assertions are equivalent:*

(1) $T \in \mathfrak{P}_{\mathbf{r}}(\prod_{j=1}^n E_j, F)$.

(2) *(Pietsch–Grothendieck's domination theorem) There are Radon probability measures μ_j, $1 \leq j \leq n$ (resp. ν) in the unit balls $B_{E_j'}$, (resp. in $B_{F''}$) and $C \geq 0$ such that, $\mathcal{B}_j$ (resp. $\mathcal{B}_{n+1}$) being the σ-algebra of Borel sets in $B_{E_j'}$ (resp. $B_{F''}$) for every $(x_j)_{j=1}^n \in \prod_{j=1}^n E_j$ and every $y' \in F'$ one has*

$$\left|\left\langle T\left((x_j)_{j=1}^n\right), y'\right\rangle\right| \leq C\ \left\|f_{y'}\right\|_{L^{r_{n+1}'}(B_{F''},\ \mathcal{B}_{n+1},\ \nu)} \prod_{j=1}^n \left\|f_{x_j}\right\|_{L^{r_j'}(B_{E_j'},\ \mathcal{B}_j,\ \mu_j)}. \tag{13}$$

Moreover, $\mathbf{P}_{\mathbf{r}}(T) = \inf C$ taking the infimum over all $C \geq 0$ and μ_j, $1 \leq j \leq n$ and ν verifying (13).

(3) *(Generalised Kwapien's factorisation theorem) There exist Banach spaces M_j and linear maps $A_j \in \mathfrak{P}_{r_j'}(E_j, M_j)$, $1 \leq j \leq n$ and an n-linear map $S : \prod_{j=1}^n M_j \longrightarrow F$ such that $T = S \circ ((A_1, A_2, \dots, A_n))$ and the adjoint map $S' \in \mathfrak{P}_{r_{n+1}'}\left(F', \mathcal{L}^n\left(\prod_{j=1}^n M_j, \mathbb{R}\right)\right)$.*

Proof. (1) $\Longrightarrow$ (2). Clearly, the restriction to $\mathcal{C}\left(\left(B_{E'}, \sigma(E', E)\right)\right)$ of each $\Psi \in \left(L^\infty(B_{E'})\right)'$ is a Radon measure. Then condition (2) follows from (1) directly by definition of **r**-dominated maps and the very general result of Defant ([**2**], Theorem 1).

Moreover, the proof of that result allows us to obtain

$$\inf\left\{C \geq 0 \,\middle|\, (13)\ \text{holds}\right\} \leq \mathbf{P_r}(T). \tag{14}$$

(2) $\Longrightarrow$ (3). Let μ_j, $1 \leq j \leq n$ and ν be probability Radon measures in the unit balls $B_{E'_j}$ and $B_{F''}$, respectively (with corresponding σ-algebras $\mathcal{B}_j$ and $\mathcal{B}_{n+1}$ of measurable sets) such that (13) holds.

Put $\Omega := \prod_{j=1}^n B_{E'_j}$ provided with the product measure $\mu := \otimes_{j=1}^n \mu_j$ and its corresponding σ-algebra $\mathcal{B}$ of measurable sets. For every $x_j \in E_j$, $1 \leq j \leq n$, we define the map $G_{x_j} : \Omega \longrightarrow \mathbb{R}$ given by $G_{x_j}(\mathbf{x}') = \langle x_j, x'_j \rangle$ for every $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_n) \in \Omega$. Clearly, as a consequence of Fubini's theorem, we have $G_{x_j} \in L^{r'_j}\big(\Omega, \mathcal{B}, \mu\big)$ and, moreover, for each $y' \in F'$ the inequality

$$\left|\left\langle T\Big((x_j)_{j=1}^n\Big), y'\right\rangle\right| \leq C \left\|f_{y'}\right\|_{L^{r'_{n+1}}(B_{F''},\, \mathcal{B}_{n+1},\, \nu)} \prod_{j=1}^n \left\|G_{x_j}\right\|_{L^{r'_j}(\Omega,\, \mathcal{B},\, \mu)} \tag{15}$$

holds still.

Define $A_j \in \mathcal{L}\big(E_j, L^{r'_j}(\Omega, \mathcal{B}, \mu)\big)$, as $A_j(x_j) = G_{x_j}$ for every $x_j \in E_j$ and $M_j := \overline{A_j(E_j)}$, taking the closure in $L^{r'_j}(\Omega, \mathcal{B}, \mu)$ and providing it with the induced topology. It is easy to check (classical Pietsch–Grothendieck's domination theorem) that

$$\forall\, 1 \leq j \leq n \quad A_j \in \mathfrak{P}_{r'_j}(E_j, M_j) \quad \text{and} \quad \mathbf{P}_{r'_j}(A_j) \leq 1. \tag{16}$$

Now we define the *multi-linear* map $S : \prod_{j=1}^n A_j(E_j) \longrightarrow F$ as

$$\forall\, (x_j)_{j=1}^n \in \prod_{j=1}^n E_j \qquad S\big((G_{x_j})_{j=1}^n\big) = T\big((x_j)_{j=1}^n\big).$$

S is well defined because $\big(G_{x_j}\big)_{j=1}^n = \big(G_{\overline{x}_j}\big)_{j=1}^n$ implies $G_{x_j} = G_{\overline{x}_j} \in L^{r'_j}(\Omega, \mathcal{B}, \mu)$, $1 \leq j \leq n$ and

$$T\big((x_j)_{j=1}^n\big) - T\big((\overline{x}_j)_{j=1}^n\big) = \sum_{j=1}^n T\big(\overline{x}_1, \ldots, \overline{x}_{j-1}, x_j - \overline{x}_j, x_{j+1}, \ldots, x_n\big)$$

and by (15) we obtain $\left\|T\big((x_j)_{j=1}^n\big) - T\big((\overline{x}_j)_{j=1}^n\big)\right\| = 0$. Equation (15) also gives the continuity of S and hence it can be *continuously* extended to a map (still denoted by S) in $\mathcal{L}^n\Big(\prod_{j=1}^n M_j, F\Big)$. To finish the proof, we only need to see that $S' \in \mathfrak{P}_{r'_{n+1}}\big(F', \mathcal{L}^n\big(\prod_{j=1}^n M_j, \mathbb{R}\big)\big)$.

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Given $\{y'_k\}_{k=1}^h \subset F'$, $h \in \mathbb{N}$, fix a finite sequence $\{\alpha_k\}_{k=1}^h$ verifying $\left\|(\alpha_k)_{k=1}^h\right\|_{\ell_h^{r'_{n+1}}} = 1$. For every $\varepsilon > 0$, there are $G_{x_{jk}} \in B_{M_j}$, $1 \leq k \leq h$, $1 \leq j \leq n$ such that

$$\forall\, 1 \leq k \leq h \quad \left\|S'(y'_k)\right\|_{\mathcal{L}^n\left(\prod_{j=1}^n M_j, \mathbb{R}\right)} \leq \left|\left\langle S'(y'_k), (G_{x_{jk}})_{j=1}^n \right\rangle\right| + \varepsilon\, |\alpha_k|.$$

Hence, from Hölder's inequality and (13) we obtain

$$\begin{aligned}
\pi_{r'_{n+1}}\left(\left(S'(y'_k)\right)_{k=1}^h\right) &= \sup_{(\beta_k)\in B_{\ell_h^{r_{n+1}}}} \left|\sum_{k=1}^h \beta_k \left\|S'(y'_k)\right\|_{\mathcal{L}^n\left(\prod_{j=1}^n M_j, \mathbb{R}\right)}\right| \\
&\leq \sup_{(\beta_k)\in B_{\ell_h^{r_{n+1}}}} \left|\sum_{k=1}^h \beta_k \left(\left|\left\langle S'(y'_k), (G_{x_{jk}})_{k=1}^n\right\rangle\right| + \varepsilon|\alpha_k|\right)\right| \\
&\leq \sup_{(\beta_k)\in B_{\ell_h^{r_{n+1}}}} \left\|(\beta_k)\right\|_{\ell_h^{r_{n+1}}} \left(\sum_{k=1}^h \left|\left\langle y'_k, T\left((x_{jk})_{j=1}^n\right)\right\rangle\right|^{r'_{n+1}}\right)^{\frac{1}{r'_{n+1}}} \\
&\quad + \varepsilon \sup_{(\beta_k)\in B_{\ell_h^{r_{n+1}}}} \left\|(\beta_k)_{k=1}^h\right\|_{\ell_h^{r_{n+1}}} \left\|(\alpha_k)_{k=1}^h\right\|_{\ell_h^{r'_{n+1}}} \\
&\leq C \left(\sum_{k=1}^h \left(\left\|f_{y'_k}\right\|^{r'_{n+1}}_{L^{r'_{n+1}}(B_{F''},\, \mathcal{B}_{n+1},\, \nu)} \prod_{j=1}^n \left\|G_{x_{jk}}\right\|^{r'_{n+1}}_{L^{r'_j}(\Omega,\, \mathcal{B},\, \mu)}\right)\right)^{\frac{1}{r'_{n+1}}} + \varepsilon \\
&\leq C \left(\sum_{k=1}^h \left(\int_{B_{F''}} \left|\left\langle y'_k, y''\right\rangle\right|^{r'_{n+1}} d\nu(y'')\right)\right)^{\frac{1}{r'_{n+1}}} + \varepsilon \\
&= C \left(\int_{B_{F''}} \sum_{k=1}^h \left|\left\langle y'_k, y''\right\rangle\right|^{r'_{n+1}} d\nu(y'')\right)^{\frac{1}{r'_{n+1}}} + \varepsilon \\
&\leq C\, \varepsilon_{r'_{n+1}}\left(\left(y'_k\right)_{k=1}^h\right) \nu\left(B_{F''}\right)^{\frac{1}{r'_{n+1}}} + \varepsilon = C\, \varepsilon_{r'_{n+1}}\left(\left(y'_k\right)_{k=1}^h\right) + \varepsilon
\end{aligned}$$

and $\varepsilon > 0$ being arbitrary, the result follows. Moreover, by (16) and the definition of $\mathbf{P}_{r'_{n+1}}(S')$ we obtain

$$\mathbf{P}_{r'_{n+1}}(S') \prod_{j=1}^n \mathbf{P}_{r'_j}(A_j) \leq C. \tag{17}$$

(3) $\Longrightarrow$ (1). Assume there are Banach spaces M_j and maps $A_j \in \mathfrak{P}_{r'_j}(E_j, M_j)$, $1 \leq j \leq n$ and $S \in \mathcal{L}^n\left(\prod_{j=1}^n M_j, F\right)$ such that $S' \in \mathfrak{P}_{r'_{n+1}}\left(F', \mathcal{L}^n\left(\prod_{j=1}^n M_j,\ \mathbb{R}\right)\right)$ and $T = S \circ \left((A_j)_{j=1}^n\right)$. Given finite sequences $\{x_{jk}\}_{k=1}^h \subset E_j$ and $\{y'_k\}_{k=1}^h \subset F'$, $h \in \mathbb{N}$, using (2)

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and Hölder's inequality we have

$$\begin{aligned}\pi_{r_0'}\left(\left(\left\langle T\left((x_{jk})_{j=1}^n\right), y_k'\right\rangle\right)_{k=1}^n\right) &= \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}}\left|\sum_{k=1}^h \alpha_k\left\langle\left(A_j(x_{jk})\right)_{j=1}^n\right), S'(y_k')\right\rangle\right| \\ &\leq \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}}\sum_{k=1}^h |\alpha_k|\,\left\|S'(y_k')\right\|_{\mathcal{L}^n(\prod_{j=1}^n M_j,\mathbb{R})}\prod_{j=1}^n\left\|A_j(x_{jk})\right\| \\ &\leq \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}}\left\|(\alpha_k)_{k=1}^h\right\|_{\ell_h^{r_0}}\left(\prod_{j=1}^n \pi_{r_j'}\left(\left(A_j(x_{jk})\right)_{k=1}^h\right)\right)\pi_{r_{n+1}'}\left(\left(S'(y_k')\right)_{k=1}^h\right) \\ &\leq \mathbf{P}_{r_{n+1}'}(S')\left(\prod_{j=1}^n \mathbf{P}_{r_j'}(A_j)\right)\varepsilon_{r_{n+1}'}\left((y_k')_{k=1}^h\right)\left(\prod_{j=1}^n \varepsilon_{r_j'}\left((x_{jk})_{k=1}^h\right)\right)\end{aligned}$$

and hence $T \in \mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, F\right)$ and

$$\mathbf{P}_{\mathbf{r}}(T) \leq \mathbf{P}_{r_{n+1}'}(S')\prod_{j=1}^n \mathbf{P}_{r_j'}(A_j). \tag{18}$$

The assertions about $\mathbf{P}_{\mathbf{r}}(T)$ follow from (14), (17) and (18). □

Theorem 3 can be used to find some equivalences between some tensor norms $\alpha_{\mathbf{r}}$ and $\alpha_{\mathbf{s}}$ derived from different admissible $(n+2)$-plas $\mathbf{r}$ and $\mathbf{s}$ on certain classes of Banach spaces. We present some results of this type, which will be of fundamental importance in the final section of the paper.

COROLLARY 4. *Let $\mathbf{r} = (r_j)_{j=0}^{n+1}$ be such that $r_{n+1}' \leq 2$ and let $\mathbf{s} = (s_j)_{j=0}^{n+1}$ be an admissible $(n+2)$-pla such that $s_{n+1}' \leq 2$, and $s_j' = r_j'$, $1 \leq j \leq n$. If E_j, $1 \leq j \leq n+1$ are Banach spaces and E_{n+1}'' has cotype 2, one has $\left(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\right) \approx \left(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{s}}\right)$.*

Proof. By Theorem 2 and the open mapping theorem it is enough to see that $\mathfrak{P}_{\mathbf{s}}\left(\prod_{j=1}^n E_j, E_{n+1}'\right) = \mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, E_{n+1}'\right)$. Given $T \in \mathfrak{P}_{\mathbf{s}}\left(\prod_{j=1}^n E_j, E_{n+1}'\right)$ and using Kwapien's generalised theorem, we choose a factorisation $T = C \circ (A_j)_{j=1}^n$ throughout some product $\prod_{j=1}^n M_j$ of Banach spaces in such a way that $A_j \in \mathfrak{P}_{s_j'}\left(E_j, M_j\right)$, $1 \leq j \leq n$ and $C' \in \mathfrak{P}_{s_{n+1}'}\left(E_{n+1}'', \mathcal{L}^n(\prod_{j=1}^n M_j', \mathbb{R})\right)$. Being E_{n+1}'' of cotype 2 and $r_{n+1}' \leq 2$, Maurey's theorem ([**3**], Corollary 3, Section 31.6) and Pietsch's inclusion theorem for absolutely p-summing maps give $C' \in \mathfrak{P}_1\left(E_{n+1}'', \mathcal{L}^n(\prod_{j=1}^n M_j', \mathbb{R})\right) \subset \mathfrak{P}_{r_{n+1}'}\left(E_{n+1}'', \mathcal{L}^n(\prod_{j=1}^n M_j', \mathbb{R})\right)$. As $r_j' = s_j'$, $1 \leq j \leq n$, by the sufficient part of Kwapien's generalised theorem we obtain $T \in \mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, E_{n+1}'\right)$. In the same way we show $\mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, E_{n+1}'\right) \subset \mathfrak{P}_{\mathbf{s}}\left(\prod_{j=1}^n E_j, E_{n+1}'\right)$ and the proof is complete. □

COROLLARY 5. *Let E_j, $1 \leq j \leq n+1$ be Banach spaces and let $\mathbf{r} = (r_j)_{j=0}^{n+1}$ be an admissible $(n+2)$-pla such that $r_j' \geq 2$ for every $1 \leq j \leq n+1$. Let $\mathbf{s} = (s_j)_{j=0}^{n+1}$ be another admissible $(n+2)$-pla such that $2 \leq s_j'$ for every $1 \leq j \leq n$ and $s_{n+1} = r_{n+1}$. Then $\left(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\right) \approx \left(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{s}}\right)$.*

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Proof. Arguing as above, we only need to show that $\mathfrak{P}_{\mathbf{s}}(\prod_{j=1}^{n} E_j, E'_{n+1}) = \mathfrak{P}_{\mathbf{r}}(\prod_{j=1}^{n} E_j, E'_{n+1})$. The crucial step is the proof of the inclusion $\mathfrak{P}_{\mathbf{r}}(\prod_{j=1}^{n} E_j, E'_{n+1}) \subset \mathfrak{P}_{\mathbf{s}}(\prod_{j=1}^{n} E_j, E'_{n+1})$ since the proof of the converse inclusion can be made exactly in the same way.

Let $T \in \mathfrak{P}_{\mathbf{r}}(\prod_{j=1}^{n} E_j, E'_{n+1})$. By the proof of (2) $\Longrightarrow$ (3) in Theorem 3, there are a probability space $(\Omega, \mathcal{B}, \mu)$, maps $A_j \in \mathfrak{P}_{r'_j}(E_j, L^{r'_j}(\Omega;\mu))$, $1 \le j \le n$ and a map $S \in \mathcal{L}^n(\prod_{j=1}^{n} \overline{A_j(E_j)}, E'_{n+1})$ such that $S' \in \mathfrak{P}_{r'_{n+1}}(E''_{n+1}, \mathcal{L}^n(\prod_{j=1} \overline{A_j(E_j)}, \mathbb{R}))$ and $T = S \circ ((A_j)_{j=1}^{n})$. Consider the tensor products $\mathfrak{T}_\pi := (\widehat{\bigotimes}_{j=1}^{n} L^{r'_j}(\Omega, \mu)), \pi)$ and $\mathfrak{H}_\pi := L^{r_0}(\Omega, \mu)\widehat{\bigotimes}_\pi \mathfrak{T}_\pi$. The canonical linear map $\widetilde{\mathfrak{M}}_\mu$ from $\mathfrak{H}_\pi$ onto $L^{r_{n+1}}(\Omega, \mu)$, (recall the notation of the introductory section) induces an isomorphism $\widehat{\mathfrak{M}}_\mu$ from the quotient space $K_1 := \mathfrak{H}_\pi / Ker(\widetilde{\mathfrak{M}}_\mu)$ onto $L^{r_{n+1}}(\Omega, \mu)$. As $r_{n+1} \le 2$, K_1 has cotype 2.

Let $\Psi_1 : \mathfrak{H}_\pi \longrightarrow K_1$ be the canonical quotient map. For every $1 \le j \le n$ we consider the map $\psi_j \in \mathcal{L}(L^{r'_j}(\Omega), \mathfrak{H}_\pi)$ defined by

$$\psi_j : z \in L^{r'_j}(\Omega) \longrightarrow [\chi_\Omega] \otimes [\chi_\Omega] \otimes \cdots \otimes [\chi_\Omega] \otimes z \otimes [\chi_\Omega] \otimes \cdots \otimes [\chi_\Omega]$$

(z in the position $j+1$) and define $\mathfrak{T}_j := \psi_j(L^{r'_j}(\Omega))$. $[\chi_\Omega]$ being of dimension 1 is complemented in each $L^p(\Omega, \mu), p \ge 1$. It follows that $\mathfrak{T}_j$ is a complemented (and hence closed) subspace of $\mathfrak{H}_\pi$. Define $F_j := \overline{A_j(E_j)}$. Clearly $H_j := \psi_j(F_j)$ is a closed subspace of $\mathfrak{T}_j$. □

Claim. For every $1 \le j \le n$, $\Psi_1(\mathfrak{T}_j)$ is closed in K_1.

Proof of the Claim. Fix $1 \le j \le n$. Let $P_j \in \mathcal{L}(\mathfrak{H}_\pi, \mathfrak{T}_j)$ be a projection and let $W_j := Ker(P_j) \oplus (Ker(\widetilde{\mathfrak{M}}_\mu) \cap \mathfrak{T}_j)$. The quotient space $K_{2j} := \mathfrak{H}_\pi / W_j$ is well defined. Let $\Psi_{2j} \in \mathcal{L}(\mathfrak{H}_\pi, K_{2j})$ be the canonical quotient map. The map

$$\forall\, z \in \mathfrak{H}_\pi \qquad L_j : \Psi_{2j}(z) \in K_{2j} \longrightarrow \Psi_1 \circ P_j(z) \in \Psi_1(\mathfrak{T}_j) \subset K_1$$

is well defined and continuous. In fact, given $z_1 = P_j(z_1) + (I_\pi - P_j)(z_1) \in \mathfrak{H}_\pi$ and $z_2 = P_j(z_2) + (I_\pi - P_j)(z_2) \in \mathfrak{H}_\pi$ (I_π denotes the identity map on $\mathfrak{H}_\pi$) such that $\Psi_{2j}(z_1) = \Psi_{2j}(z_2)$, as $(I_\pi - P_j)(z_1) \in Ker(P_j) \subset W$ and $(I_\pi - P_j)(z_2) \in Ker(P_j) \subset W$, we obtain $\Psi_{2j} \circ P_j(z_1) = \Psi_{2j} \circ P_j(z_2)$, i.e.

$$P_j(z_1) - P_j(z_2) \in W \quad \Longrightarrow \quad P_j(z_1) - P_j(z_2) \in Ker(\widetilde{\mathfrak{M}}_\mu) \cap \mathfrak{T}_j \subset Ker(\widetilde{\mathfrak{M}}_\mu),$$

and hence $L_j(z_1) = \Psi_1 \circ P_j(z_1) = \Psi_1 \circ P_j(z_2) = L_j(z_2)$ and L_j is well defined. On the other hand, given $\Psi_{2j}(z) \in K_{2j}$, there is $w \in \mathfrak{T}_\pi$ such that $\Psi_{2j}(w) = \Psi_{2j}(z)$ and $\|w\|_{\mathfrak{T}_\pi} \le 2\,\|\Psi_{2j}(z)\|_{K_{2j}}$. Then

$$\begin{aligned} \|L_j \circ \Psi_{2j}(z)\|_{K_1} &= \|L_j \circ \Psi_{2j}(w)\|_{K_1} = \|\Psi_1 \circ P_j(w)\|_{K_1} \\ &\le \|\Psi_1\|\,\|P_j\|\,\|w\|_{\mathfrak{H}_\pi} \le 2\,\|P_j\|\,\|\Psi_{2j}(z)\|_{K_{2j}} \end{aligned}$$

and L_j turns out to be continuous. But, clearly, L_j is surjective. Then the canonical induced map $\widetilde{L}_j \in \mathcal{L}(K_{3j}, K_1)$ from the quotient space $K_{3j} := K_{2j}/Ker(L_j)$ onto K_1 is an isomorphism. Let $\Psi_{3j} \in \mathcal{L}(K_{2j}, K_{3j})$ be the canonical quotient map. Note that we have

$$\Psi_1 \circ P_j = L_j \circ \Psi_{2j} = \widetilde{L}_j \circ \Psi_{3j} \circ \Psi_{2j}. \tag{19}$$

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Next, take $z \in \overline{\Psi_1(\mathfrak{T}_j)}$. There is a sequence $\{z_m\}_{m=1}^{\infty} \subset \mathfrak{T}_j$ such that $z = \lim_{m\to\infty} \Psi_1(z_m)$ in K_1. Then $\{\widetilde{L}_j^{-1}(z_m)\}_{m=1}^{\infty}$ is a Cauchy sequence in K_{3j}. By a standard procedure (see ([**8**], Section 14.4. (3)), for instance) and switching to a suitable subsequence, if necessary, we can assume that there is a sequence $\{w_m\}_{m=1}^{\infty} \subset \mathfrak{H}_\pi$ such that

$$\forall\, m \in \mathbb{N} \quad \Psi_{3j} \circ \Psi_{2j}(w_m) = \widetilde{L}_j^{-1}(z_m) = \Psi_{3j} \circ \Psi_{2j}(z_m) \tag{20}$$

and

$$\forall\, m, k \in \mathbb{N} \ \left\| w_m - w_k \right\|_{\mathfrak{H}_\pi} \leq 2 \left\| \Psi_{2j}(w_m) - \Psi_{2j}(w_k) \right\|_{K_{2j}} \leq 4 \left\| \widetilde{L}_j^{-1}(z_m) - \widetilde{L}_j^{-1}(z_k) \right\|_{K_{3j}}.$$

Then $\{w_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{T}_\pi$ and there exists $w = \lim_{m\to\infty} w_m \in \mathfrak{H}_\pi$. By (20) we obtain

$$\Psi_{3j} \circ \Psi_{2j}(z_m) = \Psi_{3j} \circ \Psi_{2j}(w_m) = \Psi_{3j} \circ \Psi_{2j}\big(P_j(w_m) - (I_\pi - P_j)(w_m)\big) = \Psi_{3j} \circ \Psi_{2j} \circ P_j(w_m)$$

and since P_j is a projection and $P_j(z_m) = z_m$, by the definitions of Ψ_{3j} and L_j

$$\Psi_1(z_m) = \Psi_1 \circ P_j(z_m) = L_j \circ \Psi_{2j}(z_m) = L_j \circ \Psi_{2j} \circ P_j(w_m) = \Psi_1 \circ P_j(w_m)$$

and $\Psi_1 \circ P_j(w) = \lim_{m\to\infty} \Psi_1 \circ P_j(w_m) = \lim_{m\to\infty} \Psi_1(z_m) = z$. As $P_j(w) \in \mathfrak{T}_j$, we obtain $z \in \Psi_1(\mathfrak{T}_j)$, and $\Psi_1(\mathfrak{T}_j)$ is closed. □

End of the proof of Corollary 5. Let Φ_j be the restriction to $\mathfrak{T}_j$ of Ψ_1. Let Ψ_{4j} be the canonical quotient map from $\mathfrak{T}_j$ onto the quotient space $K_{4j} := \mathfrak{T}_j / \big(\mathfrak{T}_j \cap Ker(\mathfrak{M}_\mu)\big)$. The map $\widetilde{\Phi}_j : \Psi_{4j} \circ \psi_j(z_j) \in K_{4j} \longrightarrow \Phi_j \circ \psi_j(z_j) \in \Phi_j(\mathfrak{T}_j)$, $z_j \in F_j$ is well defined. In fact, if $\overline{z}_j \in F_j$ and $\Psi_{4j} \circ \psi_j(z_j - \overline{z}_j) = 0$, we will have $\psi_j(z_j - \overline{z}_j) \in Ker(\widetilde{\mathfrak{M}}_\mu)$ and hence, by definition of $\widetilde{\mathfrak{M}}_\mu$ and ψ_j, one has $z_j = \overline{z}_j$ and $\Phi_j \circ \psi_j(z_j) = \Phi_j \circ \psi_j(\overline{z}_j)$, turning $\widetilde{\Phi}_j$ well defined. The same argument shows that $\widetilde{\Phi}_j$ is injective. By the claim $\Phi_j(\mathfrak{T}_j)$ is closed in K_1. As $\widetilde{\Phi}_j$ is clearly surjective by the open map theorem it turns out that $\widetilde{\Phi}_j$ is an isomorphism from K_{4j} onto $\Phi_j(\mathfrak{T}_j)$.

Next, remark that given $z_j \in L^{r'_j}(\Omega, \mu)$ and $\varepsilon > 0$, there is $\overline{z}_j \in L^{r'_j}(\Omega, \mu)$ such that $\Psi_{4j} \circ \psi_j(z_j) = \Psi_{4j} \circ \psi_j(\overline{z}_j)$ and

$$\begin{aligned} \left\| \psi_j(\overline{z}_j) \right\|_{\mathfrak{T}_j} &\leq \left\| \Psi_{4j} \circ \psi_j(z_j) \right\|_{K_{4j}} + \varepsilon \leq \left\| \widetilde{\Phi}_j^{-1} \right\| \left\| \widetilde{\Phi}_j \circ \Psi_{4j} \circ \psi_j(z_j) \right\|_{K_1} + \varepsilon \\ &= \left\| \widetilde{\Phi}_j^{-1} \right\| \left\| \Phi_j \circ \psi_j(z_j) \right\|_{K_1} + \varepsilon \leq \left\| \widetilde{\Phi}_j^{-1} \right\| \left\| \psi_j(z_j) \right\|_{\mathfrak{T}_j} + \varepsilon. \end{aligned}$$

But, as we have shown previously, $\Psi_{4j} \circ \psi_j(z_j) = \Psi_{4j} \circ \psi_j(\overline{z}_j)$ implies $z_j = \overline{z}_j$ and so $\psi_j(z_j) = \psi_j(\overline{z}_j)$. Then $\varepsilon > 0$ being arbitrary, we obtain

$$\left\| \psi_j(z_j) \right\|_{\mathfrak{T}_j} \leq \left\| \widetilde{\Phi}_j^{-1} \right\| \left\| \Phi_j \circ \psi_j(z_j) \right\|_{K_1} \leq \left\| \widetilde{\Phi}_j^{-1} \right\| \left\| \psi_j(z_j) \right\|_{\mathfrak{T}_j},$$

which means that Φ_j is an isomorphism from $\mathfrak{T}_j$ onto $\Phi_j(\mathfrak{T}_j)$.

As a consequence the isomorphisms $F_j \approx H_j \approx \Phi_j(H_j)$ hold and F_j has cotype 2 because $\Phi_j(H_j)$ is a closed subspace of K_1, which has cotype 2. As $A_j \in \mathfrak{P}_{r'_j}(E_j, F_j)$, by Maurey's theorem ([**3**], Corollary 3, Section 31.6) and Pietsch's inclusion theorem for p-absolutely summing maps, we obtain $A_j \in \mathfrak{P}_2(E_j, F_j) \subset \mathfrak{P}_{s'_j}(E_j, F_j)$. It follow from the properties of S and from Kwapien's generalised theorem that $T \in \mathfrak{P}_{\mathbf{s}}\big(\prod_{j=1}^n E_j, E'_{n+1}\big)$ as desired. □

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COROLLARY 6. *Let $E_j, 1 \leq j \leq n+1$ be Banach spaces and let $\mathbf{r} = (r_j)_{j=0}^{n+1}$ be an admissible $(n+2)$-pla such that $r_{j_0} \leq 2$ for some $1 \leq j_0 \leq n+1$ and $r'_{j_1} \geq 2$ for some $1 \leq j_1 \neq j_0 \leq n+1$. Choose $s_{j_0} < r_{j_0}$ and define $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r'_{j_0}} - \frac{1}{s'_{j_0}}$ and $s_j := r_j, 1 \leq j \neq j_0 \leq n+1$. Then $\mathbf{s} = (s_j)_{j=0}^{n+1}$ is an admissible $(n+2)$-pla such that $s_0 < \infty$ and $\left(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\right) \approx \left(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{s}}\right)$.*

Proof. After the eventual transposition $j_1 \to n+1, n+1 \to j_1$ we can assume that $j_1 = n+1$. Then the proof is essentially the same one of Corollary 5 because we have $r_{n+1} \leq 2$ and Maurey's theorem will be still applicable in the 'axis' j_0. □

Another application of Theorem 3 concerns to the approximation of $\mathbf{r}$-dominated maps by finite rank maps.

THEOREM 7. *Let $E_j, 1 \leq j \leq n+1$, be Banach spaces with duals E'_j having the metric approximation property and such that each E'_j, $1 \leq j \leq n$ has the Radon–Nikodym property. Then $\mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, E'_{n+1}\right) = \left(\widehat{\bigotimes}_{j=1}^{n+1} E'_j, \alpha'_{\mathbf{r}}\right)$.*

Proof. Let $T \in \mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, E'_{n+1}\right)$. By Kwapien's theorem (Theorem 3) there are Banach spaces M_j and operators $A_j \in \mathfrak{P}_{r'_j}(E_j, M_j)$, $1 \leq j \leq n$ and $S \in \mathcal{L}^n\left(\prod_{j=1}^n M_j, E'_{n+1}\right)$ such that $T = S \circ (A_1, A_2, \ldots, A_n)$. Since every E'_j has the Radon–Nikodym property, by the result ([**10**], p. 228) of Makarov and Samarskii, each A_j is a quasi r'_j-nuclear operator. By ([**12**], Theorems 26 and 43) there is a sequence

$$\left\{B_{jh} = \sum_{s_j=1}^{t_{jh}} x'_{jhs_j} \otimes m_{jhs_j}\right\}_{h=1}^{\infty} \subset E'_j \otimes M_j$$

of finite rank operators such that

$$\forall\, 1 \leq j \leq n \qquad \lim_{h\to\infty} \mathbf{P}_{r'_j}(A_j - B_{jh}) = 0. \tag{21}$$

In particular, every sequence $\{B_{jh}\}_{h=1}^{\infty}$ is a Cauchy sequence (and so bounded) in $\mathfrak{P}_{r'_j}(E_j, M_j)$, $1 \leq j \leq n$.

Since for every $(x_j)_{j=1}^n \in \prod_{j=1}^n E_j$ and $h \in \mathbb{N}$, we have

$$\begin{aligned}\left(S \circ \left((B_{jh})_{j=1}^n\right)\right)\left((x_j)_{j=1}^n\right) &= S\left(\left(\sum_{s_j=1}^{t_{jh}} \langle x'_{jhs_j}, x_j\rangle m_{jhs_j}\right)_{j=1}^n\right)\\ &= \sum_{s_1=1}^{t_{1h}} \cdots \sum_{s_n=1}^{t_{nh}} \left(\prod_{j=1}^n \langle x'_{jhs_j}, x_j\rangle\right) S\left((m_{jhs_j})_{j=1}^n\right),\end{aligned}$$

it turns out that $S \circ \left((B_{jh})_{j=1}^n\right) \in \mathcal{L}^n\left(\prod_{j=1}^n E_j, E'_{n+1}\right)$ has finite dimensional range and

$$S \circ \left((B_{jh})_{j=1}^n\right) = \sum_{s_1=1}^{t_{1h}} \cdots \sum_{s_n=1}^{t_{nh}} \left(\otimes_{j=1}^n x'_{jhs_j}\right) \otimes S\left((m_{jhs_j})_{j=1}^n\right) \in \bigotimes_{j=1}^{n+1} E'_j.$$

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With a similar proof to the one given in [3] it can be seen that $(\widehat{\bigotimes}_{j=1}^{n+1} E_j', \alpha_{\mathbf{r}}')$ is a topological subspace of $\mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^{n} E_j, E_{n+1}'\right)$. Hence, by Theorem 3, (18) and (21)

$$\begin{aligned}
&\alpha_{\mathbf{r}}'\Big(S \circ \big(B_{1h}, B_{2h}, \ldots, B_{nh}\big) - S \circ \big(B_{1k}, B_{2k}, \ldots, B_{nk}\big)\Big) \\
&= \mathbf{P}_{\mathbf{r}}\left(\sum_{j=1}^{n}\Big(S \circ B_{1k}, \ldots, B_{j-1,k}, B_{jh} - B_{jk}, B_{j+1,h}, \ldots, B_{nh}\Big)\right) \\
&\leq \mathbf{P}_{r_{n+1}'}(S') \sum_{j=1}^{n} \mathbf{P}_{r_j'}(B_{jh} - B_{jk})\Big(\prod_{1\leq s<j} \mathbf{P}_{r_s'}(B_{sk})\Big)\Big(\prod_{j<s\leq n} \mathbf{P}_{r_s'}(B_{sh})\Big)
\end{aligned}$$

is arbitrarily small when h and k let to infinity and so there exists $z := \lim_{h\to\infty} S \circ \big(B_{1h}, B_{2h}, \ldots, B_{nh}\big) \in (\widehat{\bigotimes}_{j=1}^{n+1} E_j', \alpha_{\mathbf{r}}')$. On the other hand, it can be shown in an analogous way that

$$\lim_{h\to\infty} \mathbf{P}_{\mathbf{r}}\Big(T - S \circ \big((B_{jh})_{j=1}^{n}\big)\Big) = \lim_{h\to\infty} \mathbf{P}_{\mathbf{r}}\Big(S \circ \big((A_j)_{j=1}^{n}\big) - S \circ \big((B_{jh})_{j=1}^{n}\big)\Big) = 0$$

and hence $T = z$. □

3. r-nuclear multi-linear maps. With the same methods used in the classical case of Lapresté's tensor topologies, it can be shown that every element $z \in \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, \ldots, E_n, F\big)$ can be represented as a convergent series

$$z = \sum_{m=1}^{\infty} \lambda_m \Big(\otimes_{j=1}^{n} x_{jm}\Big) \otimes z_m \tag{22}$$

where $(\lambda_m) \in \ell^{r_0}$, $(x_{jm})_{m=1}^{\infty} \in \ell^{r_j'}(E_j)$, $j = 1, 2, \ldots, n$ and $(z_m)_{m=1}^{\infty} \in \ell^{r_{n+1}'}(F)$. Moreover, the norm of such elements z can be computed as in (10) but using representations (22) and $h = \infty$.

If F is a Banach space, every $z \in \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, \ldots, E_n, F\big)$ defines canonically a multi-linear map $T_z \in \mathcal{L}^n\left(\prod_{j=1}^{n} E_j', F\right)$ by the rule

$$\forall\ (x_j')_{j=1}^{n} \in \prod_{j=1}^{n} E_j' \quad T_z\big((x_j')_{j=1}^{n}\big) = \sum_{m=1}^{\infty} \lambda_m \left(\prod_{j=1}^{n} \left\langle x_m^j, x_m'\right\rangle\right) z_m. \tag{23}$$

Remark that T_z is independent on the representing series (22) for z as a consequence of Theorem 2 and the easy fact that $\big(\bigotimes_{j=1}^{n} E_j'\big) \bigotimes F' \subset \mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^{n} E_j,\ F'\Big)$ canonically. In this way we have defined a canonical *linear map*

$$\Phi : z \in \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, \ldots, E_n, F\big) \longrightarrow T_z \in \mathcal{L}^n\Big(\prod_{j=1}^{n} E_j', F\Big), \tag{24}$$

which suggest the next definition.

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Definition 8. A multi-linear map $A \in \mathcal{L}^n\Big(\prod_{j=1}^n E_j, F\Big)$ is said to be **r**-nuclear if it is the restriction $R(T_z)$ to $\prod_{j=1}^n E_j$ of a map T_z for some $z \in \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1', E_2', \ldots, E_n', F\big)$.

It can be shown that the set $\mathfrak{N}_{\mathbf{r}}\Big(\prod_{j=1}^n E_j, F\Big)$ of all n-linear **r**-nuclear maps from $\prod_{j=1}^n E_j$ into F becomes a Banach space under the **r**-nuclear norm

$$\mathbf{N}_{\mathbf{r}}(A) = \inf\Big\{\alpha_{\mathbf{r}}(z) \;\Big|\; A = R(T_z),\; z \in \widehat{\otimes}_{\alpha_{\mathbf{r}}}\big(E_1', E_2', \ldots, E_n', F\big)\Big\}$$

if all E_j, $1 \le j \le n$ and F are Banach spaces. **r**-nuclear maps can be characterised by means of suitable factorisations as follows.

Theorem 9. *Let F and E_j, $1 \le j \le n$ be Banach spaces and $T \in \mathcal{L}^n\big(\prod_{i=1}^n E_j, F\big)$. T is* **r**-*nuclear if and only if there are maps $A_j \in \mathcal{L}(E_j, \ell^{r_j'})$, $1 \le j \le n$, $C \in \mathcal{L}(\ell^{r_{n+1}}, F)$ and $\lambda := (\lambda_m) \in \ell^{r_0}$ such that T factorises in the way*

$$\begin{array}{ccc} \prod_{j=1}^n E_j & \xrightarrow{\quad T \quad} & F \\ {\scriptstyle (A_j)_{j=1}^n}\downarrow & & \uparrow{\scriptstyle C} \\ \prod_{j=1}^n \ell^{r_j'} & \xrightarrow[\quad D_\lambda \quad]{} & \ell^{r_{n+1}} \end{array} .$$

Moreover, $\mathbf{N}_{\mathbf{r}}(T) = \inf\Big(\prod_{j=1}^n \|A_j\|\Big)\,\|D_\lambda\|\,\|C\|$ *taking the infimum over all factorisations as above.*

Proof. The proof being quite standard (compare with [**9**]) is omitted.

Remark. By Theorem 9, (2) and the compactness result of Alencar and Floret ([**1**], Theorem 4.2), if $r_0 < \infty$, every **r**-nuclear mapping is compact.

As an application of Theorem 7 we can obtain a sufficient condition in order that the map Φ be injective. Although the formulation of this condition is far to be optimal, it will be enough for our applications in the sequel.

Corollary 10. *Let $E_j, 1 \le j \le n$ be reflexive Banach spaces having the approximation property. Then, for every Banach space E_{n+1} such that E_{n+1}' has the metric approximation property, the map Φ in (24) is injective and so* $\big(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\big) = \mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^n E_j', E_{n+1}\big)$.

Proof. Since we have actually $\Phi \in \mathcal{L}\Big(\big(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\big), \mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^n E_j', E_{n+1}\big)\Big)$, it is enough to show that this map is injective. It is easy to see that $\bigotimes_{j=1}^{n+1} E_j' \subset \Big(\mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^n E_j', E_{n+1}\big)\Big)'$. Now Theorem 7 implies that the transposed map

$$\Phi' : \Big(\mathfrak{N}_{\mathbf{r}}\Big(\prod_{j=1}^n E_j', E_{n+1}\Big)\Big)' \longrightarrow \mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^n E_j, E_{n+1}'\Big)$$

has dense range, getting the injectivity of Φ. $\square$

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4. r-integral multi-linear maps.

DEFINITION 11. Let E_j, $1 \leq j \leq n$, and F be Banach spaces. A continuous n-linear map T from $\prod_{j=1}^n E_j$ into F is called **r**-integral if $J_F T \in \left(\widehat{\bigotimes}_{\alpha'_{\mathbf{r}}}(E_1, E_2, \ldots, E_n, F')\right)'$.

The norm of $J_F T$ in that dual space is taken as definition of the **r**-integral norm $\mathbf{I_r}(T)$ of a map $T \in \mathfrak{I}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, F\right)$, the set of **r**-integral multi-linear maps from $\prod_{j=1}^n E_j$ into F. $(\mathfrak{I}_{\mathbf{r}}, \mathbf{I_r})$ turns out to be the maximal ideal of multi-linear maps associated to the $(n+1)$-tensor norm $\alpha_{\mathbf{r}}$ in the sense of Defant and Floret (see [3], and Theorem 4.5 in [5]). The next theorem gives the prototype of **r**-integral maps.

THEOREM 12. *Given a measure space* $(\Omega, \mathcal{A}, \mu)$ *and* $g \in L^{r_0}(\Omega, \mathcal{A}, \mu)$, *the canonical multi-linear map* $D_g : \prod_{j=1}^n L^{r'_j}(\Omega, \mathcal{A}, \mu) \longrightarrow L^{r_{n+1}}(\Omega, \mathcal{A}, \mu)$ *is* **r**-*integral.*

Proof. Let $\mathcal{S}_j$, $1 \leq j \leq n$ be the subspace of $L^{r'_j}(\Omega, \mu)$ of simple functions with support of finite measure. Every $\mathcal{S}_j$ being dense in $L^{r'_j}(\Omega, \mu)$, it is enough to see that $D_g \in \left(\bigotimes_{\alpha'_{\mathbf{r}}}(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n, L^{r'_{n+1}}(\Omega, \mu))\right)'$ (density lemma for $(n+1)$-tensor norms).

Fix $z \in \bigotimes_{\alpha'_{\mathbf{r}}}\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n, L^{r'_{n+1}}(\Omega, \mu)\right)$. There exist finite dimensional subspaces $M_j \subset \mathcal{S}_j$, $1 \leq j \leq n$ generated by the characteristic functions $\{\chi_{B_k}\}_{k=1}^h$ of a finite family of pairwise disjoints sets of finite measure $\{B_k\}_{k=1}^h \subset \mathcal{A}$, and there exists a finite dimensional subspace $N \subset L^{r'_{n+1}}(\Omega, \mu)$ such that $z \in \otimes\left(M_1, M_2, \ldots, M_n, N\right)$. Then for every $f_j \in M_j$, $1 \leq j \leq n$ and $f_{n+1} \in N$, using (4)

$$\left\langle \otimes_{j=1}^{n+1} f_j, D_g \right\rangle = \left\langle \left(\otimes_{j=1}^n \sum_{k=1}^h \alpha_{jk} \chi_{B_k}\right) \otimes f_{n+1}, D_g \right\rangle = \sum_{k=1}^h \left(\prod_{j=1}^n \alpha_{jk}\right) \left\langle \chi_{B_k} g, f_{n+1} \right\rangle$$

$$= \sum_{k=1}^h \frac{1}{\mu(B_k)^n} \left(\prod_{j=1}^n \left(\int_{B_k} f_j \, d\mu\right)\right) \left\langle \chi_{B_k} g, f_{n+1} \right\rangle$$

$$= \sum_{k=1}^h \left(\int_{B_k} |g|^{r_0} \, d\mu\right)^{\frac{1}{r_0}} \left(\prod_{j=1}^n \left(\frac{1}{\mu(B_k)^{\frac{1}{r_j}}} \int_{B_k} f_j \, d\mu\right)\right) \left\langle \frac{\left(\int_{B_k} |g|^{r_0} \, d\mu\right)^{-\frac{1}{r_0}}}{\mu(B_k)^{\frac{1}{w}}} \chi_{B_k} g, f_{n+1} \right\rangle.$$

As a consequence

$$\forall\, z \in \bigotimes\left(M_1, M_2, \ldots, M_n, N\right) \quad \left\langle z, D_g \right\rangle = \left\langle z, V \right\rangle \tag{25}$$

where we have defined

$$V := \sum_{k=1}^h \left(\int_{B_k} |g|^{r_0} \, d\mu\right)^{\frac{1}{r_0}} \left(\otimes_{j=1}^n \varphi_{jk}\right) \otimes \frac{\left(\int_{B_k} |g|^{r_0} \, d\mu\right)^{-\frac{1}{r_0}}}{\mu(B_k)^{\frac{1}{w}}} \chi_{B_k} g$$

and where φ_{jk} is the class in $L^{r_j}(\Omega, \mu)/M_j^{\perp} = M'_j$ of the function $\mu(B_k)^{-\frac{1}{r_j}} \chi_{B_k}$ for every $\forall\, 1 \leq j \leq n$, $1 \leq k \leq h$. Moreover, (the class of) $\chi_{B_k} g \in N'$ for every $1 \leq k \leq h$ since $\chi_{B_k} g \in L^{r_0}(\Omega, \mu)$, and by (8) we obtain $\chi_{B_k} g \in L^{r_{n+1}}(\Omega, \mu)$, B_k being of finite measure.

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Note that by finite dimensionality

$$V \in \bigotimes_{\alpha_{\mathbf{r}}}(M_1', M_2', \dots, M_n', N') = \Big(\bigotimes_{\alpha_{\mathbf{r}}'}(M_1, M_2, \dots, M_m, N)\Big)'. \tag{26}$$

Now we perform some computations. The first one is

$$\pi_{r_0}\left(\left(\left(\int_{B_k} |g|^{r_0}\, d\mu\right)^{\frac{1}{r_0}}\right)_{k=1}^{h}\right) = \left(\sum_{k=1}^{h}\int_{B_k} |g|^{r_0}\, d\mu\right)^{\frac{1}{r_0}} = \|g\|_{L^{r_0}(\Omega)}. \tag{27}$$

In second time, for every $1 \le j \le n$, using (4) and Hölder's inequality, we obtain

$$\begin{aligned}
\varepsilon_{r_j'}\left(\left(\varphi_{j,k}\right)_{k=1}^{h}\right) &= \sup_{\|f\|_{L^{r_j'}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\frac{1}{\mu(B_k)^{\frac{r_j'}{r_j}}}\left(\int_{B_k} f\, d\mu\right)^{r_j'}\right)^{\frac{1}{r_j'}}\\
&\le \sup_{\|f\|_{L^{r_j'}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\frac{1}{\mu(B_k)^{\frac{r_j'}{r_j}}}\left(\int_{B_k} |f|^{r_j'}\, d\mu\right)\mu(B_k)^{\frac{r_j'}{r_j}}\right)^{\frac{1}{r_j'}}\\
&\le \sup_{\|f\|_{L^{r_j'}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\int_{B_k} |f|^{r_j'}\, d\mu\right)^{\frac{1}{r_j'}} = \sup_{\|f\|_{L^{r_j'}(\Omega)}\le 1}\|f\|_{L^{r_j'}(\Omega)} = 1.
\end{aligned} \tag{28}$$

Finally, by Hölder's inequality and (6) we have

$$\begin{aligned}
&\varepsilon_{r_{n+1}'}\left(\left(\mu(B_k)^{-\frac{1}{w}}\left(\int_{B_k}|g|^{r_0}\, d\mu\right)^{-\frac{1}{r_0}}\chi_{B_k}\, g\right)_{k=1}^{h}\right)\\
&= \sup_{\|f\|_{L^{r_{n+1}'}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\mu(B_k)^{-\frac{r_{n+1}'}{w}}\left(\int_{B_k}|g|^{r_0}\, d\mu\right)^{-\frac{r_{n+1}'}{r_0}}\left(\int_{B_k} g f\, d\mu\right)^{r_{n+1}'}\right)^{\frac{1}{r_{n+1}'}}\\
&\le \sup_{\|f\|_{L^{r_{n+1}'}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\int_{B_k}|f|^{r_{n+1}'}\, d\mu\right)^{\frac{1}{r_{n+1}'}} = \sup_{\|f\|_{L^{r_{n+1}'}(\Omega)}\le 1}\left(\int_{\Omega}|f|^{r_{n+1}'}\, d\mu\right)^{\frac{1}{r_{n+1}'}} = 1.
\end{aligned} \tag{29}$$

Then, by (25), (26), (27), (28) and (29),

$$\begin{aligned}
\left|\langle z, D_g\rangle\right| &\le \alpha_{\mathbf{r}}'\Big(z; \bigotimes(M_1, M_2, \dots, M_n, N)\Big)\,\alpha_{\mathbf{r}}\Big(V; \bigotimes(M_1', M_2', \dots, M_n', N')\Big)\\
&\le \alpha_{\mathbf{r}}'\Big(z; \bigotimes(M_1, M_2, \dots, M_n, N)\Big)\,\|g\|_{L^{r_0}(\Omega)}
\end{aligned}$$

and, $\alpha_{\mathbf{r}}'$ being a finite generated $(n+1)$-tensor norm,

$$\left|\langle z, D_g\rangle\right| \le \alpha_{\mathbf{r}}'\Big(z; \bigotimes(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n, L^{r_{n+1}}(\Omega, \mu)\Big)\,\|g\|_{L^{r_0}(\Omega)},$$

which means $\mathbf{I}_{\mathbf{r}}(D_g) \le \|g\|_{L^{r_0}(\Omega)}$. □

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To find a characterisation of $\mathbf{r}$-integral maps we need to use ultraproducts $(E_\gamma)_{\mathcal{U}}$ of a given family $\{E_\gamma, \gamma \in \mathfrak{G}\}$ of Banach spaces over an ultrafilter $\mathcal{U}$ on the index set $\mathfrak{G}$. For this topic our main reference is [**16**]. We use the natural notation $(x_\gamma)_{\mathcal{U}}$ for every element in $(E_\gamma)_{\mathcal{U}}$.

Given a family $\{T_\gamma \in \mathcal{L}^n(\prod_{j=1}^n E_\gamma^j, F_\gamma),\ |\ \gamma \in \mathfrak{G}\}$ of maps between the cartesian product $\prod_{j=1}^n E_\gamma^j$ of Banach spaces E_γ^j and $F_\gamma, 1 \leq j \leq n,\ \gamma \in \mathfrak{G}$, such that $\sup_{\gamma\in\mathfrak{G}} \|T_\gamma\| < \infty$, there is a canonical n-linear continuous ultraproduct map $(T_\gamma)_{\mathcal{U}}$ from the ultraproduct $(\prod_{j=1}^n E_\gamma^j)_{\mathcal{U}}$ into the ultraproduct $(F_\gamma)_{\mathcal{U}}$ such that for every $\mathbf{x} := ((x_\gamma^j)_{j=1}^n)_{\mathcal{U}} \in (\prod_{j=1}^n E_\gamma^j)_{\mathcal{U}}$ we have $(T_\gamma)_{\mathcal{U}}(\mathbf{x}) = (T_\gamma((x_\gamma^j)_{j=1}^n))_{\mathcal{U}}$. The main result we shall need is the following factorisation theorem.

LEMMA 13. *Consider a family of canonical maps* $D_{g_\gamma} : \prod_{j=1}^n \ell^{r_j'} \longrightarrow \ell^{r_{n+1}},\ \gamma \in \mathfrak{G} \neq \emptyset$ *defined by a family of elements* $\{g_\gamma |\ \gamma \in \mathfrak{G}\} \subset \ell^{r_0}$ *such that* $0 < \sup_{\gamma\in\mathfrak{G}} \|D_{g_\gamma}\| < \infty$. *There exists a decomposable measure space* $(\Omega, \mathcal{M}, \mu)$, *a function* $g \in L^{r_0}(\Omega, \mathcal{M}, \mu)$ *and order onto isometries* $\mathfrak{X}_j : (\ell^{r_j'})_{\mathcal{U}} \longrightarrow L^{r_j'}(\Omega, \mathcal{M}, \mu),\ 1 \leq j \leq n,\ \mathfrak{X}_0 : (\ell^{r_0})_{\mathcal{U}} \longrightarrow L^{r_0}(\Omega, \mathcal{M}, \mu)$ *and* $\mathfrak{X}_{n+1} : (\ell^{r_{n+1}})_{\mathcal{U}} \longrightarrow L^{r_{n+1}}(\Omega, \mathcal{M}, \mu)$ *such that the diagram*

$$\begin{array}{ccc}
\left(\prod_{j=1}^n \ell^{r_j'}\right)_{\mathcal{U}} & \xrightarrow{\quad (D_{g_\gamma})_{\mathcal{U}} \quad} & (\ell^{r_{n+1}})_{\mathcal{U}} \\
{\scriptstyle (\mathfrak{X}_j)_{j=1}^n}\Big\downarrow & & \Big\uparrow{\scriptstyle \mathfrak{X}_{n+1}^{-1}} \\
\prod_{j=1}^n L^{r_j'}(\Omega) & \xrightarrow{\quad D_g \quad} & L^{r_{n+1}}(\Omega).
\end{array}$$

is commutative. Moreover, $\|D_g\| = \|(D_{g_\gamma})_{\mathcal{U}}\|$.

Proof. By (5) and a factorisation result of Raynaud ([**14**], Theorem 5.1), there are decomposable measure space $(\Omega, \mathcal{M}, \mu)$ and isometric order isomorphisms

$$\mathfrak{X}_0 : (\ell^{r_0})_{\mathcal{U}} \longrightarrow L^{r_0}(\Omega, \mathcal{M}, \mu), \quad \mathfrak{X}_j : (\ell^{r_j'})_{\mathcal{U}} \longrightarrow L^{r_j'}(\Omega, \mathcal{M}, \mu),\ 1 \leq j \leq n,$$

and $\mathfrak{X}_{n+1} : (\ell^{r_{n+1}})_{\mathcal{U}} \longrightarrow L^{r_{n+1}}(\Omega, \mathcal{M}, \mu)$ such that, $\mathfrak{M}_\gamma$ being the map corresponding to $\gamma \in \mathfrak{G}$ (recall the notations introduced in Section 1), we have $(\mathfrak{M}_\gamma)_{\mathcal{U}} = \mathfrak{X}_{n+1}^{-1} \circ \mathfrak{M}_\mu \circ ((\mathfrak{X}_j)_{j=1}^n)$. The lemma follows taking $g = \mathfrak{X}_0((g_\gamma)_{\mathcal{U}})$. □

Now we can obtain the following characterisation.

THEOREM 14. *Let* $E_j, 1 \leq j \leq n$ *and* F *be Banach spaces and* $T \in \mathcal{L}^n(\prod_{j=1}^n E_j, F)$. *The following are equivalent:*

(1) T *is* $\mathbf{r}$*-integral.*

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(2) *$J_F T$ can be factorised as*

$$\begin{array}{ccccc} \prod_{j=1}^{n} E_j & \xrightarrow{\quad T \quad} & F & \xrightarrow{\quad J_F \quad} & F'' \\ \Big\downarrow (A_j)_{j=1}^{n} & & & & \Big\uparrow C \\ \prod_{j=1}^{n} L^{r_j'}(\Omega, \mathcal{M}, \mu) & & \xrightarrow[\quad D_g \quad]{} & & L^{r_{n+1}}(\Omega, \mathcal{M}, \mu) \end{array} \tag{30}$$

where $A_j \in \mathcal{L}\big(E_j, L^{r_j'}(\Omega, \mathcal{M}, \mu)\big)$, $1 \leq j \leq n$, $C \in \mathcal{L}\big(L^{r_{n+1}}(\Omega, \mathcal{M}, \mu), F''\big)$ and D_g is the multi-linear diagonal operator corresponding to some $g \in L^{r_0}(\Omega, \mathcal{M}, \mu)$. Moreover,

$$\mathbf{I_r}(T) = \inf \|D_g\| \, \|C\| \prod_{j=1}^{n} \|A_j\| \tag{31}$$

taking the infimum over all factorisations as in the previous diagram.

(3) *$J_F T$ can be factorised as above but $(\Omega, \mathcal{M}, \mu)$ being a finite measure space and $g = \chi_\Omega$. Formula (31) holds too taking the infimum over the factorisations of that type.*

Proof. (1) $\Longrightarrow$ (2). This can be done using standard methods with the help of Theorem 9 and Lemma 13 (see, for instance, **[9]** for a detailed development of the method used in a similar framework).

(2) $\Longrightarrow$ (3). Given $\varepsilon > 0$, select a factorisation of type (30) with $g \in L^{r_0}(\Omega, \mathcal{M}, \mu)$ and such that

$$\|g\|_{L^{r_0}(\Omega,\mu)} \|C\| \prod_{j=1}^{n} \|A_j\| \leq \mathbf{I_r}(T) + \varepsilon. \tag{32}$$

After projection onto the sectional subspaces $L^{r_j'}(Supp(g))$, $1 \leq j \leq n$ if necessary, we can assume that $\Omega = Supp(g)$. Consider the new *finite measure* ν on $(\Omega, \mathcal{M})$ defined by

$$\forall\, M \in \mathcal{M} \quad \nu(M) = \int_M |g|^{r_0} \, d\mu$$

and the mappings

$$\forall\, 1 \leq j \leq n \quad H_j : f_j \in L^{r_j'}(\Omega, \mu) \longrightarrow H_j(f_j) = f_j \, |g|^{-\frac{r_0}{r_j'}} \in L^{r_j'}(\Omega, \nu)$$

and

$$H_{n+1} : f \in L^{r_{n+1}}(\Omega, \mu) \longrightarrow H_{n+1}(f) = f \, |g|^{-\frac{r_0}{r_{n+1}}} \in L^{r_{n+1}}(\Omega, \nu).$$

By Radon–Nikodym's theorem

$$\big\|H_{n+1}(f)\big\|_{L^{r_{n+1}}(\Omega,\nu)} = \big\|f\big\|_{L^{r_{n+1}}(\Omega,\mu)}, \quad \big\|H_j(f_j)\big\|_{L^{r_j'}(\Omega,\nu)} = \big\|f_j\big\|_{L^{r_j'}(\Omega,\mu)}, \; 1 \leq j \leq n \tag{33}$$

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and for every $(f_j)_{j=1}^n \in \prod_{j=1}^n L^{r'_j}(\Omega, \mu)$, using (2)

$$\left(H_{n+1}^{-1} \circ D_{\chi_\Omega} \circ (H_j)_{j=1}^n\right)\left((f_j)_{j=1}^n\right) = |g|^{\frac{r_0}{r_{n+1}}} \prod_{j=1}^n f_j \, |g|^{-\frac{r_0}{r'_j}} = |g|^{r_0\left(\frac{1}{r_{n+1}} - \sum_{j=1}^n \frac{1}{r'_j}\right)} \prod_{j=1}^n f_j$$

$$= |g|^{r_0\left(\frac{1}{r_{n+1}} - 1 + \frac{1}{r_0} + \frac{1}{r'_{n+1}}\right)} \prod_{j=1}^n f_j = g \prod_{j=1}^n f_j = D_g\left((f_j)_{j=1}^n\right). \tag{34}$$

As $\chi_\Omega \in L^{r_0}(\Omega, \nu)$, joining the factorisation (34) with the initial one we get our goal and, moreover, by (33) and (32)

$$\mathbf{I_r}(T) \leq \left\| C \circ H_{n+1}^{-1} \right\| \left\| D_{\chi_\Omega} \right\| \prod_{j=1}^n \left\| H_j \circ A_j \right\|$$

$$\leq \|C\| \left\| H_{n+1} \circ D_g \circ H_j^{-1} \right\| \prod_{j=1}^n \|A_j\| \leq \mathbf{I_r}(T) + \varepsilon. \tag{35}$$

(3) $\Longrightarrow$ (1). It is immediate by Theorem 12 and the ideal properties of multi-linear **r**-integral operators. □

5. Applications to reflexivity. Previous results allow us to obtain some information about the reflexivity of completed tensor products of type $\alpha_{\mathbf{r}}$.

THEOREM 15. *Let E_j, $1 \leq j \leq n \in \mathbb{N}$ and F be reflexive Banach spaces such that E'_j, $1 \leq j \leq n$ and F' have the metric approximation property. Given an admissible $(n+2)$-pla $\mathbf{r}$, the space $\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\left(E_1, E_2, \ldots, E_n, F\right)$ is reflexive if and only if*

$$\mathfrak{N}_{\mathbf{r}}\left(\prod_{j=1}^n E'_j,\ F\right) = \mathfrak{I}_{\mathbf{r}}\left(\prod_{j=1}^n E'_j,\ F\right). \tag{36}$$

Proof. If (36) holds, by Theorem 7 and Corollary 10 we obtain

$$\left(\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}(E_1, E_2, \ldots, E_n, F)\right)'' = \left(\mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j,\ F'\right)\right)' = \left(\widehat{\bigotimes}_{\alpha'_{\mathbf{r}}}(E'_1, E'_2, \ldots, E'_n, F')\right)'$$

$$= \mathfrak{I}_{\mathbf{r}}\left(\prod_{j=1}^n E'_j,\ F\right) = \mathfrak{N}_{\mathbf{r}}\left(\prod_{j=1}^n E'_j,\ F\right) = \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}(E_1, E_2, \ldots, E_n, F).$$

Conversely, if $\widehat{\otimes}_{\alpha_{\mathbf{r}}}\left(E_1, E_2, \ldots, E_n, F\right)$ is reflexive, by definition of **r**-integral maps, Theorem 7 and Corollary 10 we obtain

$$\mathfrak{I}_{\mathbf{r}}\left(\prod_{j=1}^n E'_j,\ F\right) = \left(\mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j,\ F'\right)\right)' = \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}(E_1, E_2, \ldots, E_n, F) = \mathfrak{N}_{\mathbf{r}}\left(\prod_{j=1}^n E'_j,\ F\right).$$

□

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We apply Theorem 15 to characterise the reflexivity of $\big(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j},\alpha_{\mathbf{r}}\big)$. First, we need a lemma.

LEMMA 16. *Let* $\mathbf{r}=(r_j)_{j=0}^{n+1}$ *an admissible* $(n+2)$*-pla verifying* $r_0=\infty$ *and let* $1<u_j'\le r_j'$ *for every* $1\le j\le n+1$. *Then there exists a non compact map* $T\in\mathfrak{I}_{\mathbf{r}}\big(\prod_{j=1}^{n}\ell^{u_j'},\ell^{u_{n+1}}\big)$.

Proof. Let $I_1:=\big[0,\frac{1}{2}\big[$ and $I_m:=\big[\sum_{i=1}^{m}\frac{1}{2^i},\sum_{i=1}^{m+1}\frac{1}{2^i}\big[$ if $m>1$. The map $A_j:(\beta_i)\in\ell^{u_j'}\longrightarrow\sum_{m=1}^{\infty}\beta_m\,\mu(I_m)^{-\frac{1}{r_j'}}\,\chi_{I_m}\in L^{r_j'}([0,1],\mu)$, $1\le j\le n$ (μ is the Lebesgue measure on $[0,1]$), is well defined and continuous since

$$\big\|A_j\big((\beta_m)\big)\big\|=\left(\sum_{m=1}^{\infty}\frac{|\beta_m|^{r_j'}}{\mu(I_m)}\,\mu(I_m)\right)^{\frac{1}{r_j'}}\le\big\|(\beta_m)\big\|_{\ell^{u_j'}}.$$

Take $g=\chi_{[0,1]}\in L^{\infty}([0,1],\mu)$. Consider now the closed linear subspace F generated by the set $\{\chi_{I_m},m\in\mathbb{N}\}$ in $L^{r_{n+1}}([0,1])$. The map

$$Q:f\in L^{r_{n+1}}([0,1])\longrightarrow\sum_{m=1}^{\infty}\frac{1}{\mu(I_m)}\left(\int_{I_m}f\,d\mu\right)\chi_{I_m}\in F$$

is continuous, since by Hölder's inequality

$$\begin{aligned}\big\|Q(f)\big\|_F&=\left(\sum_{m=1}^{\infty}\left(\int_{I_m}f\,d\mu\right)^{r_{n+1}}\mu(I_m)^{1-r_{n+1}}\right)^{\frac{1}{r_{n+1}}}\\&\le\left(\sum_{m=1}^{\infty}\left(\int_{I_m}|f|^{r_{n+1}}\,d\mu\right)\mu(I_m)^{\frac{r_{n+1}}{r_{n+1}'}+1-r_{n+1}}\right)^{\frac{1}{r_{n+1}}}=\|f\|_{L^{r_{n+1}}([0,1])}.\end{aligned}$$

It is immediate that Q is a projection from $L^{r_{n+1}}([0,1])$ onto F. Finally, consider the map

$$C:f=\sum_{m=1}^{\infty}\beta_m\,\chi_{I_m}\in F\longrightarrow\Big(\beta_m\,\mu(I_m)^{\frac{1}{r_{n+1}}}\Big)\in\ell^{u_{n+1}}$$

is continuous since $r_{n+1}\le u_{n+1}$ and

$$\|C(f)\|_{\ell^{u_{n+1}}}=\left(\sum_{m=1}^{\infty}|\beta_m|^{u_{n+1}}\mu(I_m)^{\frac{u_{n+1}}{r_{n+1}}}\right)^{\frac{1}{u_{n+1}}}\le\left(\sum_{m=1}^{\infty}|\beta_m|^{r_{n+1}}\mu(I_m)\right)^{\frac{1}{r_{n+1}}}=\|f\|_F.$$

Hence, $T:=C\circ Q\circ D_g\circ\big((A_j)_{j=1}^{n}\big)\in\mathfrak{I}_{\mathbf{r}}\big(\prod_{j=1}^{n}\ell^{u_j'},\ell^{u_{n+1}}\big)$ but T is not compact, since using (2)

$$\forall\,m\in\mathbb{N}\quad T\big((\mathbf{e}_m,\mathbf{e}_m,\ldots,\mathbf{e}_m)\big)=\frac{1}{\mu(I_m)^{\frac{1}{r_{n+1}}}}\,\mu(I_m)^{\frac{1}{r_{n+1}}}\,\mathbf{e}_m=\mathbf{e}_m.$$

□

We can now state the main result of this section.

THEOREM 17. *If* $1 < u_j < \infty$ *for every* $1 \le j \le n+1$, $\left(\widehat{\bigotimes}_{i=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ *is reflexive if and only if at least one of the following set of conditions holds:*

(S1) *There is* $1 \le j_0 \le n+1$ *such that* $u_j' > 2$ *and* $u_j' > r_j'$ *for all* $1 \le j \ne j_0 \le n+1$.

(S2) *There exists* $1 \le j_0 \le n+1$ *such that* $u_j' > 2$ *for every* $1 \le j \ne j_0 \le n+1$ *and*

$$\frac{1}{r_{j_0}} > \sum_{1 \le j \ne j_0}^{n+1} \frac{1}{u_j'}, \tag{37}$$

and moreover, there exists $1 \le j_1 \ne j_0 \le n+1$ *such that* $r_j' \ge 2$ *for every* $1 \le j \ne j_1 \le n+1$.

(S3) *We have* $u_j' > 2$ *for every* $1 \le j \le n+1$, *and there exists* $1 \le j_0 \le n+1$ *such that* $r_{j_0}' \le 2$ *and*

$$\frac{1}{2} > \sum_{1 \le j \ne j_0}^{n+1} \frac{1}{u_j'}. \tag{38}$$

(S4) *There is* $1 \le j_0 \le n+1$ *such that* $u_{j_0}' = 2$, $r_{j_0}' \le 2$, $u_j' > 2$ *for every* $1 \le j \ne j_0 \le n+1$ *and*

$$\frac{1}{2} > \sum_{1 \le j \ne j_0}^{n+1} \frac{1}{u_j'}. \tag{39}$$

Proof. (Sufficient conditions. **Case (S1)**). After the transposition $j_0 \longrightarrow n+1$, $n+1 \longrightarrow j_0$ if necessary, we can assume $j_0 = n+1$ and so $u_j' > 2$ and $u_j' > r_j'$ for every $1 \le j \le n$.

By Theorem 14, given $T \in \mathfrak{I}_{\mathbf{r}}\left(\prod_{j=1}^n \ell^{u_j'}, \ell^{u_{n+1}}\right)$, there are a finite measure space $(\Omega, \mathcal{M}, \mu)$ and mappings $A_j \in \mathcal{L}(\ell^{u_j'}, L^{r_j'}(\Omega, \mu))$, $1 \le j \le n$ and $C \in \mathcal{L}(L^{r_{n+1}}(\Omega, \mu), \ell^v)$ such that $T = C \circ D_{\chi_\Omega} \circ (A_j)_{j=1}^n$. By Rosenthal's result ([**15**], Theorem A.2) every A_j is compact, and by the metric approximation property of ℓ^{u_j}, there is a bounded sequence

$$\left\{ A_{jm} = \sum_{k=1}^{k_{jm}} \mathbf{x}_{jk} \otimes f_{jm}^k \right\}_{m=1}^{\infty} \subset \ell^{u_j} \otimes L^{r_j'}(\Omega, \mu) \tag{40}$$

such that

$$\forall\, 1 \le j \le n \quad \lim_{m \to \infty} \left\| A_j - A_{jm} \right\|_{\mathcal{L}(\ell^{u_j'},\, L^{r_j'}(\Omega,\mu))} = 0. \tag{41}$$

Define $T_m := C \circ D_{\chi_\Omega} \circ \left((A_{jm})_{j=1}^n\right)$ for every $m \in \mathbb{N}$. Arguing as in Theorem 7 and using Theorem 14 we obtain for every $1 \le j \le n$ and $m \in \mathbb{N}$

$$\left\{ C \circ D_{\chi_\Omega} \circ \left(A_{1m}, \ldots, A_{j-1,m}, A_j - A_{jm}, A_{j+1,m}, \ldots, A_{nm}\right)\right\}_{m=1}^{\infty} \subset \mathfrak{I}_{\mathbf{r}}\left(\prod_{j=1}^n \ell^{u_j'}, \ell^{u_{n+1}}\right)$$

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and by (41)

$$\mathbf{I}_{\mathbf{r}}(T-T_m) \leq \sum_{j=1}^{n} \mathbf{I}_{\mathbf{r}}\big(C \circ D_{\chi_\Omega} \circ \big(A_{1m}, \ldots, A_{j-1,m}, A_j - A_{jm}, A_{j+1}, \ldots, A_n\big)\big)$$

$$\leq \mu(\Omega)^{\frac{1}{r_0}} \; \|C\| \; \sum_{j=1}^{n} \|A_j - A_{jm}\| \left(\prod_{1\leq s<j} \|A_{sm}\|\right) \left(\prod_{j<s\leq n} \|A_s\|\right), \tag{42}$$

which approach to 0 if $m \longrightarrow \infty$. But actually we have

$$T_m = \sum_{k=1}^{k_{jm}} \big(\otimes_{j=1}^{n} \mathbf{x}_{jk}\big) \otimes \big(C \circ D_{\chi_\Omega} \circ \big((f_{jm}^{k})\big)\big) \in \mathfrak{N}_{\mathbf{r}}\Big(\prod_{j=1}^{n} \ell^{u_j'}, \ell^{u_{n+1}}\Big).$$

It follows from Theorem 7 that $\mathbf{N}_{\mathbf{r}}(T_m - T_s) = \mathbf{I}_{\mathbf{r}}(T_m - T_s)$ for $m, s \in \mathbb{N}$ and using (42), it turns out that $\{T_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^{n} \ell^{u_j'}, \ell^{u_{n+1}}\big)$. Then $T \in \mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^{n} \ell^{u_j'}, \ell^{u_{n+1}}\big)$ and by Theorem 15 $(\widehat{\bigotimes}_{i=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}})$ is reflexive.

Case (S2). Let $1 \leq j_0 \neq j_1 \leq n+1$ such that $u_j' > 2$, $1 \leq j \neq j_0 \leq n+1$, $r_j' \geq 2$, $1 \leq j \neq j_1 \leq n+1$ and (37) holds. In a first step we are going to see that we can assume $r_{j_1}' \geq 2$ too.

Consider the case that $r_{j_1}' < 2$. In such a case we have $u_{j_1}' > 2$ because $j_0 \neq j_1$. If $j_1 = n+1$, defining $s_{n+1}' = 2$, $s_j' := r_j'$, $1 \leq j \leq n$ and $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r_{n+1}'} - \frac{1}{2}$ we obtain an admissible $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ verifying (37) still and such that, $\ell^{u_{n+1}}$ having cotype 2, by Corollary 4, we have $(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}) \approx (\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}})$. If $1 \leq j_1 \leq n$, a transposition $j_1 \to n+1, n+1 \to j_1$ would reduce the situation to the just considered case. So, in the formulation of case (S1) we can assume that $r_j' \geq 2$, $1 \leq j \leq n+1$.

After the eventual transposition $j_0 \longrightarrow n+1, n+1 \longrightarrow j_0$ we can assume that $u_j' > 2$ for every $1 \leq j \leq n$, $r_j' \geq 2$ for every $1 \leq j \leq n+1$ and (37) holds for $j_0 = n+1$. Using (5) this last condition can be written in the way

$$\frac{1}{r_0} + \sum_{\{j \,|r_j'<u_j'\}}^{n} \left(\frac{1}{r_j'} - \frac{1}{u_j'}\right) > \sum_{\{j \,|r_j'\geq u_j'\}}^{n} \left(\frac{1}{u_j'} - \frac{1}{r_j'}\right). \tag{43}$$

For every $1 \leq j \leq n$ such that $r_j' \geq u_j'$, choose $2 \leq t_j' < u_j'$ close enough to u_j' in order that

$$\frac{1}{t_0} := \frac{1}{r_0} + \sum_{\{j \,|r_j'<u_j'\}}^{n} \left(\frac{1}{r_j'} - \frac{1}{u_j'}\right) - \sum_{\{j \,|r_j'\geq u_j'\}}^{n} \left(\frac{1}{t_j'} - \frac{1}{r_j'}\right) > 0. \tag{44}$$

Now define $t_j' := r_j'$ if $r_j' < u_j'$, $1 \leq j \leq n$ and $t_{n+1} := r_{n+1}$. By (2) we have

$$\frac{1}{t_{n+1}} = \sum_{j=1}^{n} \frac{1}{t_j'} + \sum_{\{j \,|r_j'<u_j'\}}^{n} \left(\frac{1}{r_j'} - \frac{1}{t_j'}\right) + \sum_{\{j \,|r_j'\geq u_j'\}}^{n} \left(\frac{1}{r_j'} - \frac{1}{t_j'}\right) + \frac{1}{r_0}$$

and it turns out that $\mathbf{t} = (t_j)_{j=0}^{n+1}$ is an admissible $(n+2)$-pla such that $2 \le t'_j < u'_j$ and $t'_j \le r'_j$ for every $1 \le j \le n$, and moreover, by Corollary 5 we have $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{t}}\big)$. Hence, by case (S1), $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$ is reflexive.

Case (S3). Once again, after the transposition $j_0 \longrightarrow n+1, n+1 \longrightarrow j_0$ we can assume that $r'_{n+1} \le 2$, $u'_j > 2$ for every $1 \le j \le n+1$ and (38) holds for $j_0 = n+1$, or in an equivalent way (by (2)),

$$\frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2} + \sum_{\{j\ |r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{u'_j}\right) > \sum_{\{j\ |r'_j \ge u'_j\}}^{n} \left(\frac{1}{u'_j} - \frac{1}{r'_j}\right).$$

Remark that by (2) we have necessarily $r'_j \ge 2$, $1 \le j \le n$. Since $\ell^{u_{n+1}}$ has cotype 2, by Corollary 4 there exists an $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ such that $s'_{n+1} := 2$, $s'_j := r'_j$, $1 \le j \le n$ and $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2}$ and $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$. Then $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ is reflexive by the case (S2) and so $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$ does.

Case (S4). Assume the existence of $1 \le j_0 \le n+1$ such that $u'_{j_0} = 2$, $r'_{j_0} \le 2$, $u'_j > 2$ for every $1 \le j \ne j_0 \le n+1$ and (39) holds. Consider the admissible $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ such that $s_{j_0} := 2$, $s_j := r_j$ for every $1 \le j \ne j_0 \le n+1$ and $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r'_{j_0}} - \frac{1}{2}$. We obtain from Kwapien's generalised theorem and Pietsch's inclusion theorem that $\mathfrak{P}_{\mathbf{r}}\big(\prod_{j=1}^{n} \ell^{u_j}, \ell^{u'_{n+1}}\big) \subset \mathfrak{P}_{\mathbf{s}}\big(\prod_{j=1}^{n} \ell^{u_j}, \ell^{u'_{n+1}}\big)$. The reverse inclusion is true by Kwapien's factorisation theorem and Maurey's theorem ([**3**], Corollary 3, Section 31.6) because $\ell^{u_{j_0}} = \ell^2$ has cotype 2 and $r'_{j_0} < 2$ give $\mathfrak{P}_2(\ell^2, M) = \mathfrak{P}_{r'_{j_0}}(\ell^2, M)$ for every Banach space M. Then $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ and $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ is reflexive by (39) and the case (S2).

Necessary conditions. We are going to see that $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$ is not reflexive if none of the previous conditions holds. It is enough to consider the following cases.

Case (N1). Assume there exists $1 \le j_0 \le n$ such that $u'_{j_0} \le 2$ and $1 \le j_0 \ne j_1 \le n+1$ such that $u_{j_1} \ge 2$. After the transposition $j_1 \longrightarrow n+1, n+1 \longrightarrow j_1$ on $\{1, 2, \ldots, n+1\}$ if necessary, we can assume that $j_1 = n+1$, i.e. $u_{n+1} \ge 2$.

For every $1 < p < \infty$, let $\{R_{p,h}\}_{h=1}^{\infty}$ be the sequence of Rademacher functions in $L^p([0,1])$. It is well known that the sequence $\{R_{p,h}\}_{h=1}^{\infty}$ is equivalent to the standard unit basis of ℓ^2 and its closed linear span X_p is complemented in $L^p([0,1])$ (Khintchine's inequality and ([**11**], Proposition 5)).

Let $P_{n+1} \in \mathcal{L}\big(L^{r_{n+1}}([0,1]), X_{r_{n+1}}\big)$ be a projection. Let $S_{j_0} : \ell^{u'_{j_0}} \longrightarrow X_{r'_{j_0}}$ be the continuous linear map such that $S_{j_0}(\mathbf{e}_h) = R_{r'_{j_0},h}$. On the other hand, for every $1 \le j \ne j_0 \le n$ fix a sequence $(\alpha_{jh})_{h=1}^{\infty} \in \ell^2$ such that $\alpha_{j1} = 1$ and denote by $S_j : \ell^{u'_j} \longrightarrow X_{r'_j}$ the continuous linear map such that $S_j(\mathbf{e}_h) = \alpha_{jh}\, R_{r'_j,h}$ for every $h \in \mathbb{N}$ (remark that

$$\big\|S_j\big((\beta_h)\big)\big\| \le C_j \,\big\|(\alpha_{jh}\beta_h)\big\|_{\ell^2} \le C_j \,\big\|(\alpha_{jh})\big\|_{\ell^2} \,\big\|(\beta_h)\big\|_{\ell^\infty} \le C_j \,\big\|(\alpha_{jh})\big\|_{\ell^2} \,\big\|(\beta_h)\big\|_{\ell^{u'_j}}$$

for some $C_j > 0$ by Khintchine's inequality).

Take $g := \prod_{j=1, j\ne j_0}^{n} R_{r'_j,1} \in L^{r_0}([0,1])$, and consider the well-defined map $T_{n+1} \in \mathcal{L}(X_{r_{n+1}}, \ell^{u_{n+1}})$ such that $T_{n+1}(R_{r_{n+1},h}) = \mathbf{e}_h$ for $h \in \mathbb{N}$. Then

$$T := T_{n+1} \circ P_{n+1} \circ D_g \circ \big(S_j\big)_{j=1}^{n}$$

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is **r**-integral by Theorem 14. Let $\{z_{j_0,h}\}_{h=1}^{\infty} := \{(a_{1h}, a_{2h}, \ldots, a_{nh})\}_{h=1}^{\infty} \subset \prod_{j=1}^{n} \ell^{u'_j}$ such that $a_{jh} = \mathbf{e}_1$ if $j \neq j_0$ and $a_{j_0 h} = \mathbf{e}_h$, for every $h \in \mathbb{N}$. We obtain $T(z_{j_0,h}) = \mathbf{e}_h$ for every $h \in \mathbb{N}$ and so T is not compact. If $r_0 \neq \infty$, by the remark after Theorem 9 we have $T \notin \mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$ and by Theorem 15, $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$ is not reflexive.

In the case $r_0 = \infty$, we need to consider several possibilities. First assume that there are $1 \leq j_2 \neq j_0 \leq n+1$ and $1 \leq j_3 \neq j_2 \leq n+1$ such that $r'_{j_2} \geq 2$ and $r'_{j_3} \geq 2$. By Corollary 6 there is an admissible $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ such that $s_0 \neq \infty$ and $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$. Then by the previous case with $r_0 \neq \infty$, we see that $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$ is not reflexive.

Finally, having (2) in mind, it remains to consider the case that $r'_{j_0} \leq 2$ and $n = 1$. We are dealing with $\ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ where $u'_1 \leq 2$, $r'_1 \leq 2$ and $u_2 \geq 2$. By Theorems 2 and 7 we have $\big(\ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}\big)' = \ell^{u'_1} \widehat{\bigotimes}_{\alpha'_{\mathbf{r}}} \ell^{u'_2}$. The set $K := \big\{\mathbf{e}_i \otimes \mathbf{e}_i,\ i \in \mathbb{N}\big\} \subset \ell^{u'_1} \bigotimes_{\alpha'_{\mathbf{r}}} \ell^{u'_2}$ is bounded. If $\ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ was reflexive, $\ell^{u'_1} \widehat{\bigotimes}_{\alpha'_{\mathbf{r}}} \ell^{u'_2}$ would be reflexive too and by Smul'yan's theorem, switching to a suitable subsequence if necessary, we would assume that $\{\mathbf{e}_i \otimes \mathbf{e}_i\}_{i=1}^{\infty}$ is weakly convergent to some $z \in \ell^{u'_1} \widehat{\bigotimes}_{\alpha'_{\mathbf{r}}} \ell^{u'_2}$. It follows from boundedness of K and the density of $\big[\mathbf{e}_h\big]_{h=1}^{\infty} \bigotimes \big[\mathbf{e}_h\big]_{h=1}^{\infty}$ in $\ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ that given $T \in \ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ and $\rho > 0$, there exist $w \in \bigcup_{k=1}^{\infty} \big[\mathbf{e}_h\big]_{h=1}^{k} \bigotimes \big[\mathbf{e}_h\big]_{h=1}^{k}$ and $m_0 \in \mathbb{N}$ such that

$$\forall\, m \geq m_0 \quad \big|\big\langle T, z\big\rangle\big| \leq \big|\big\langle T, z - \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big| + \big|\big\langle T - w, \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big| + \big|\big\langle w, \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big|$$

$$\leq \big|\big\langle T, z - \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big| + \sup_{k \in \mathbb{N}} \big|\big\langle T - w, \mathbf{e}_k \otimes \mathbf{e}_k\big\rangle\big| + \big|\big\langle w, \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big| \leq \rho$$

because $\langle w, \mathbf{e}_m \otimes \mathbf{e}_m\rangle = 0$ if m is large enough. Then $z = 0$. But we are assuming that $\mathfrak{I}_{\mathbf{r}}(\ell^{u'_1}, \ell^{u_2}) = \big(\ell^{u'_1} \widehat{\bigotimes}_{\alpha'_{\mathbf{r}}} \ell^{u'_2}\big)' = \ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ and so by the construction made in the case $r_0 \neq \infty$ there is $T \in \ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ such that $\big\langle T(\mathbf{e}_i), \mathbf{e}_i\big\rangle = \big\langle \mathbf{e}_i, \mathbf{e}_i\big\rangle = 1$ for every $i \in \mathbb{N}$, a contradiction. Then $\ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ is not reflexive.

Case (N2). Assume that $u'_j \geq 2$ for every $1 \leq j \leq n$, $r'_j \geq 2$ for every $1 \leq j \leq n+1$, $u'_{n+1} \leq r'_{n+1}$ and $\frac{1}{r_{n+1}} \leq \sum_{j=1}^{n} \frac{1}{u'_j}$, or equivalently (by (5))

$$\frac{1}{r_0} + \sum_{\{j \,|r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{u'_j}\right) \leq \sum_{\{j \,|r'_j \geq u'_j\}}^{n} \left(\frac{1}{u'_j} - \frac{1}{r'_j}\right). \tag{45}$$

Given $1 \leq j \leq n$, if $r'_j < u'_j$ and $t'_j \in \big[u'_j, \infty\big[$ it turns out that we have

$$\frac{1}{r_0} + \sum_{\{j \,|r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{t'_j}\right) \in \left[\frac{1}{r_0} + \sum_{\{j \,|r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{u'_j}\right),\ \frac{1}{r_0} + \sum_{\{j \,|r'_j < u'_j\}}^{n} \frac{1}{r'_j}\right[$$

On the other hand, if $r'_j \geq u'_j$ and $t'_j \in \big[u'_j, r'_j\big]$, we have

$$\sum_{\{j \,|r'_j \geq u'_j\}}^{n} \left(\frac{1}{t'_j} - \frac{1}{r'_j}\right) \in \left[0, \sum_{\{j \,|r'_j \geq u'_j\}}^{n} \left(\frac{1}{u'_j} - \frac{1}{r'_j}\right)\right].$$

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Then it follows from (45) that we can choose $t'_j \geq u'_j$ for every $1 \leq j \leq n$ such that $r'_j < u'_j$ and $u'_j \leq t'_j \leq r'_j$ for every $1 \leq j \leq n$, which verifies $u'_j \leq r'_j$ in order that

$$\frac{1}{r_0} + \sum_{\{j \mid r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{t'_j}\right) = \sum_{\{j \mid r'_j \geq u'_j\}}^{n} \left(\frac{1}{t'_j} - \frac{1}{r'_j}\right).$$

By (2) we have

$$\frac{1}{r_{n+1}} = \sum_{j=1}^{n} \frac{1}{t'_j} + \sum_{\{j \mid r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{t'_j}\right) + \sum_{\{j \mid r'_j \geq u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{t'_j}\right) + \frac{1}{r_0} = \sum_{j=1}^{n} \frac{1}{t'_j}.$$

Taking $t_0 = \infty$ and $t_{n+1} = r_{n+1}$ we obtain an admissible $(n+2)$-pla $\mathbf{t} = (t_j)_{j=0}^{n+2}$ such that $t'_j \geq u'_j \geq 2$ for every $1 \leq j \leq n$. By Corollary 5 we have $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{t}}\big)$ and so $\mathfrak{I}_{\mathbf{t}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big) = \mathfrak{I}_{\mathbf{r}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$. But by Lemma 16 there is a non-compact map $S \in \mathfrak{I}_{\mathbf{t}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$. Now we take $s'_j = t'_j$ if $1 \leq j \leq n$, $s'_{n+1} > t'_{n+1}$ and define $s_0 < \infty$ such that $\frac{1}{s_0} := \frac{1}{t'_{n+1}} - \frac{1}{s'_{n+1}}$. Then $\mathbf{s} = (s_j)_{j=0}^{n+1}$ is another admissible $(n+2)$-pla verifying $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{t}}\big)$ by Corollary 6 and $S \in \mathfrak{I}_{\mathbf{s}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$. By remark after Theorem 9 we have $S \notin \mathfrak{N}_{\mathbf{s}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$ and by Theorem 15 $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ turns out to be not reflexive.

Case (N3). Assume that $u'_j \geq 2$ for every $1 \leq j \leq n+1$, $r'_{n+1} \leq 2$ and $\frac{1}{2} \leq \sum_{j=1}^{n} \frac{1}{u'_j}$, or, in an equivalent form (by (2))

$$\frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2} + \sum_{\{j \mid r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{u'_j}\right) \leq \sum_{\{j \mid r'_j \geq u'_j\}}^{n} \left(\frac{1}{u'_j} - \frac{1}{r'_j}\right).$$

By (2) we have $r'_j \geq 2$, $1 \leq j \leq n$. Defining $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2}$, $s'_j := r'_j$, $1 \leq j \leq n$ and $s_{n+1} := 2$ we obtain an admissible $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ such that, $\ell^{u_{n+1}}$ having cotype 2, by Corollary 4 one has $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$. Then $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ is not reflexive by the case (N2) obtaining the desired conclusion by isomorphism.

Case (N4). Assume there are $1 \leq j_0 \leq n$ and $1 \leq j_1 \neq j_0 \leq n+1$ such that $u'_{j_0} < 2$, $r'_{j_0} < 2$ and $r_{j_1} \leq u_{j_1}$.

(a) First we consider the case that $n \geq 2$. By (2) necessarily exists $1 \leq j_2 \neq j_3 \leq n+1$ such that $r'_{j_2} \geq 2$ and $r'_{j_3} \geq 2$, and so by Corollary 6 and eventually switching to an isomorphic tensor product $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ we can suppose moreover that $r_0 < \infty$. After the transposition $j_1 \longrightarrow n+1, n+1 \longrightarrow j_1$ if necessary we can assume that $j_1 = n+1$, i.e. $r_{n+1} \leq u_{n+1}$ indeed. If there exists $1 \leq j_4 \neq j_0 \leq n+1$ such that $u'_{j_4} \leq 2$, the result follows from case (N1). Hence, we can assume $u'_j > 2$ for every $1 \leq j \neq j_0 \leq n+1$.

Fix $t < 2$ such that $r'_{j_0} < t$, $u'_{j_0} < t$ and $u_{n+1} < t$. Let $\{\varphi_k\}_{k=1}^{\infty}$ be a sequence of standard independent identically distributed t-stable random variables in $[0, 1]$. It is known that the norm $K_{t,p} := \|\varphi_k\|_{L^p([0,1])}$, $k \in \mathbb{N}$ is only dependent on t and p for every $1 \leq p < 2$ and that $\big\{\Phi_{k,p} := \frac{\varphi_k}{K_{t,p}}\big\}_{k=1}^{\infty}$ is isometrically equivalent in $L^p([0, 1])$, $1 \leq p < t$ to the canonical basis of ℓ^t (see **[6]**, Proposition IV.4.10 for example). Then $\{\Phi_{k,r_{n+1}}\}_{k=1}^{\infty}$

is a normalised basis in the reflexive subspace $\left[\Phi_{k,r_{n+1}}\right]_{k=1}^{\infty} \approx \ell^t$ of $L^{r_{n+1}}([0,1])$ and thus it is weakly convergent to 0 in $L^{r_{n+1}}([0,1])$ (see, for instance, [**7**], FN p. 169). Switching to a suitable subsequence if necessary, by ([**17**], Chapter III, Theorem 1.8) the sequence $\{\Phi_k, r_{n+1}\}_{k=1}^{\infty}$ can be enlarged to obtain a normalised basis $\mathcal{B} := \{\Phi_{k,r_{n+1}}\}_{k=1}^{\infty} \cup \{\Psi_m\}_{m=1}^{\infty}$ in $L^{r_{n+1}}([0,1])$. By reflexivity the sequence $\{\Phi^*_{k,r_{n+1}}\}_{k=1}^{\infty} \cup \{\Psi^*_m\}_{m=1}^{\infty}$ of associated coefficient functionals to $\mathcal{B}$ is a basis in $L^{r'_{n+1}}([0,1])$. From ([**17**], Chapter I, Theorem 3.1) we find $1 \le M \in \mathbb{R}$ such that $1 \le \left\|\Phi^*_{k,r_{n+1}}\right\| \le M$ and $1 \le \left\|\Psi^*_k\right\| \le M$ for every $k \in \mathbb{N}$. As above, we obtain that $\{\Phi^*_{k,r_{n+1}}\}_{k=1}^{\infty}$ must be weakly convergent to 0. As $r'_{n+1} > 2$, by the result of Kadec and Pełcinśky ([**7**], Corollary 5), switching to a subsequence again, it can be assumed that $\{\Phi^*_{k,r_{n+1}}\}_{k=1}^{\infty}$ is equivalent to the standard unit basis in $\ell^{r'_{n+1}}$ or to the standard unit basis in ℓ^2. By ([**7**], Corollary 1), the latter possibility would imply that $\left[\Phi^*_{k,r_{n+1}}\right]_{k=1}^{\infty}$ would be complemented in $L^{r'_{n+1}}([0,1])$ and by reflexivity and duality, we would have the isomorphisms $\left(\left[\Phi^*_{k,r_{n+1}}\right]_{k=1}^{\infty}\right)' \approx \left[\Phi_{k,r_{n+1}}\right]_{k=1}^{\infty} \approx \ell^t \approx \ell^2$, which is not possible. Then $\left\{\Phi^*_{k,r_{n+1}}\right\}_{k=1}^{\infty}$ is equivalent to the standard basis of $\ell^{r'_{n+1}}$ and so the map $V \in \mathcal{L}\left(\ell^{u'_{n+1}}, L^{r'_{n+1}}([0,1])\right)$ such that $V(\mathbf{e}_h)) = \Phi^*_{h,r_{n+1}},\ h \in \mathbb{N}$ is well defined.

Let $S_j \in \mathcal{L}\left(\ell^{u'_j}, L^{r'_j}([0,1])\right),\ 1 \le j \ne j_0 \le n$ be defined as in previous case (N1) and consider $S_{j_0} \in \mathcal{L}\left(\ell^{u'_{j_0}}, L^{r'_{j_0}}([0,1])\right)$ such that $S_{j_0}(\mathbf{e}_k) = \Phi_{k,r'_{j_0}}$ for every $k \in \mathbb{N}$. Taking g as in case (N1), the map $T := V' \circ D_g \circ \left((S_j)\right)_{j=1}^{n}$ is $\mathbf{r}$-integral. However, for every $k \in \mathbb{N}$ and every $(\gamma_h) \in \ell^{u'_{n+1}}$ we have

$$\left\langle T(z_{j_0,k}), (\gamma_h)\right\rangle = \left\langle \frac{K_{t,r_{n+1}}}{K_{t,r'_{j_0}}}\ \Phi_{k,r_{n+1}}, \sum_{h=1}^{\infty} \gamma_h \Phi^*_{h,r_{n+1}}\right\rangle = \frac{K_{t,r_{n+1}}}{K_{t,r'_{j_0}}}\ \gamma_k,$$

and so $T(z_{j_0,k}) = \frac{K_{t,r_{n+1}}}{K_{t,r'_{j_0}}}\ \mathbf{e}_k$ and T is not compact. By remark after Theorem 9 we obtain $T \notin \mathfrak{N}_{\mathbf{r}}\left(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\right)$ and by Theorem 15 $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive.

(b) Now we consider the case $n = 1$. If $r_0 \ne \infty$, the previous argumentation can be used still and $\ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ is not reflexive. If $r_0 = \infty$, after an eventual transposition, we will be dealing with the case $u'_1 \le 2$, $r'_1 < 2$ and $r_2 \le u_2$. If $u_2 \ge 2$, the result follows from (N1). If $u_2 < 2$ and $u'_1 = 2$ we repeat the proof given in this case for $n \ge 2$, and $\ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ turns out to be non-reflexive. If $u_2 < 2$ and $u'_1 < 2$, the same construction just used in the case $n \ge 2$ shows the existence of a map $T \in \mathfrak{I}_{\mathbf{r}}(\ell^{u'_1}, \ell^{u_2})$ such that $T(\mathbf{e}_i) = \frac{K_{t,r_2}}{K_{t,r'_1}}\ \mathbf{e}_i$ for every $i \in \mathbb{N}$. Then we can repeat the argumentation used in the last part of (N1) with the set $K := \left\{\mathbf{e}_i \otimes \mathbf{e}_i,\ i \in \mathbb{N}\right\} \subset \ell^{u'_1} \bigotimes \ell^{u'_2}$ to conclude that $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive.

Finally, we check that the proof of Theorem 17 is complete. Assume that neither condition (S1), (S2), (S3) and (S4) holds.

(a) *First case*: Assume there is $1 \le j_0 \le n+1$ such that $u'_{j_0} \le 2$. After an eventual transposition with any $1 \le k \ne j_0 \le n+1$, we can take $j_0 \le n$. If there is some $1 \le j_1 \ne j_0 \le n+1$ such that $u'_{j_1} \le 2$, by (N1), $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive. Then we can assume $u'_j > 2,\ 1 \le j \ne j_0 \le n+1$. As (S1) does not hold, there exists $j_1 \ne j_0$ such that $r_{j_1} \le u_{j_1}$. If it would be $u'_{j_0} < 2$ and $r'_{j_0} < 2$, then $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ would be not reflexive by (N4). If $u'_{j_0} = 2$ and $r'_{j_0} < 2$, as (S4) does not hold, after the transposition $j_0 \to n+1, n+1 \to j_0$, by (N3) $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive.

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In the case $r'_{j_0} \geq 2$, by (2) there is at most a unique $1 \leq j_2 \leq n+1$ such that $r'_{j_2} < 2$. Necessarily, $j_2 \neq j_0$. As (S2) does not hold, after an eventual transposition $j_0 \to n+1, n+1 \to j_0$, we see that $u'_{n+1} \leq 2 \leq r'_{n+1}$ and by (N2) $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive.

(b) *Second case*: Assume that $u'_j > 2$, $1 \leq j \leq n+1$. As (S1) does not hold, after an eventual transposition, it turns out that $u'_{n+1} \leq r'_{n+1}$. But (S3) is not verified. Then for every $1 \leq j_0 \leq n+1$ we have $r'_{j_0} > 2$, or (38) does not hold. If it would be $r'_j > 2$ for every $1 \leq j \leq n+1$, as (S2) is not verified, $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ would be not reflexive by (N3). If it would exist $1 \leq j_1 \leq n+1$ such that $r'_{j_1} \leq 2$, then (38) would fail for this index j_1. After an evident transposition, by (N3) $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ would be not reflexive. ■

The application of Theorem 17 to the case $n = 1$ gives the following characterisation of reflexivity of classical Lapresté's tensor products.

Corollary 18. *Let $n = 1$ and let $\mathbf{r} = (r_0, r_1, r_2)$ be an admissible triple. If $1 < u_1, u_2 < \infty$, $\ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ is reflexive if and only if one of the following sets of conditions holds:*

(1) $u'_1 > 2, u'_1 > r'_1$.
(2) $u'_2 > 2, u'_2 > r'_2$.
(3) $u'_1 > 2, r_2 \leq 2$.
(4) $u'_2 > 2, r_1 \leq 2$.
(5) $u'_1 \geq 2, u'_2 > 2$.
(6) $u'_1 > 2, u'_2 \geq 2$.

Proof. By Theorem 17, $\ell^{u_1} \widehat{\bigotimes}_{\alpha_{\mathbf{r}}} \ell^{u_2}$ is reflexive if and only if one of the following sets of conditions holds:

(a) $u'_1 > 2, u'_1 > r'_1$.
(b) $u'_2 > 2, u'_2 > r'_2$.
(c) $u'_1 > 2, u'_1 > r_2, r'_1 \geq 2$.
(d) $u'_2 > 2, u'_2 > r_1, r'_2 \geq 2$.
(e) $u'_1 > 2, u'_2 > 2, r'_1 \leq 2$.
(f) $u'_1 > 2, u'_2 > 2, r'_2 \leq 2$.
(g) $u'_1 = 2, u'_2 > 2, r'_1 \leq 2$.
(h) $u'_2 = 2, u'_1 > 2, r'_2 \leq 2$.

Clearly, (c) and (3) (resp. (*d*) and (4)) are equivalent. On the other hand, if (5) holds and $r'_1 \leq 2$, then (e) or (g) holds. If (5) and $r'_1 > 2$ are true, we have $r_1 < 2 < u'_2$ and (d) is verified. The remaining of the proof is similar or trivial. □

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