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POINTWISE CHAIN RECURRENT MAPS OF THE TREE

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Let T be a tree, $f : T \to T$ be a continuous map. We show that if f is pointwise chain recurrent (that is, every point of T is chain recurrent under f), then either f^{a_n} is identity or f^{a_n} is turbulent if $\mathrm{Fix}(f) \cap \mathrm{End}(T) = \emptyset$; or else $f^{a_{n-1}}$ is identity or $f^{a_{n-1}}$ is turbulent if $\mathrm{Fix}(f) \cap \mathrm{End}(T) \neq \emptyset$. Here n denotes the number of endpoints of T and, a_n denotes the minimal common multiple of $2, 3, \ldots, n$.

1. Introduction

Firstly some notation and definitions are established. Let (X, d) be a compact metric space and $g : X \to X$ be a continuous map. If $g^n(x) = x \neq g^k(x), k = 1, 2, \ldots, n-1$, for some $x \in X$ and some positive integer n, then the point x is called a *periodic point* of period n, where $g^0 = id, g^i = g \circ g^{i-1} (i \geqslant 1)$. In particular, if $g(x) = x$, then x is called a *fixed point* of g, the set of all fixed points of g is denoted by $\mathrm{Fix}(g)$. For $x, y \in X$ and $\varepsilon > 0$, an ε-chain from x to y is a finite sequence $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ with $d(g(x_i), x_{i+1}) < \varepsilon$ for $0 \leqslant i \leqslant n-1$. We say x chains to y under g, if for each $\varepsilon > 0$, there is an ε-chain from x to y. A point x is said to be *chain recurrent* if x chains to itself. The map g is said to be *pointwise chain recurrent* if every point of X is chain recurrent under g. The following facts about chain recurrent are standard observations:

(a) If g is pointwise chain recurrent, then g maps X onto X.

(b) g is pointwise chain recurrent if and only if g^n is pointwise chain recurrent for every $n > 0$.

(c) [**3**, Theorem A] If X is connected and $g : X \to X$ is pointwise chain recurrent, then there is no nonempty open set $U \neq X$ such that $g(\overline{U}) \subset U$.

Being chain recurrent is an important dynamical property of a system and has been studied intensively in recent years. For more details see [**1, 2, 3, 4, 5, 7, 8**].

A tree is any space which is uniquely arcwise connected and homeomorphic to the union of finitely many copies of the unit interval, that is, a graph(see [**6**])containing no

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cycles. Let T be a tree, the points of T which have no neighbourhood homeomorphic to an open interval of the real line are called *vertices*, written by $V(T)$. Let $z \in T$, the number of connected components of $T \setminus \{z\}$ is called the *valence* of z. A vertex of valence 1 is called an *endpoint* of T and a vertex of valence larger than 2 is called a *branching point* of T. Denote by $\mathrm{End}(T)$ and $\mathrm{Br}(T)$ the sets of endpoints and branching points of T, respectively, and let $\mathrm{NE}(T)$ be the number of endpoints of T. For a point $p \in \mathrm{Fix}(f)$, the connected component of $\mathrm{Fix}(f)$ which contains p is represented by C_p. Let N be the set of positive integers. For any $n \in N$ with $n \geqslant 2$, let a_n denote the minimal common multiple of $2, 3, \ldots, n$.

For $a, b \in T$, we use $[a, b]$, to denote the smallest closed connected subset containing a and b. We define $(a, b] = [a, b] \setminus \{a\}$ and similarly define (a, b) and $[a, b)$. We also use $T_b(a)$ to denote the connected component of $T \setminus \{b\}$ which contains a. For a subset A of T, we use $\mathrm{int}(A), \overline{A}$ and $\partial(A)$ to denote the interior, the closure and the boundary of A, respectively.

A map $g : T \to T$ is called turbulent if there are closed non-degenerate connected subsets J and K with disjoint interiors such that $g(J) \cap g(K) \supset J \cup K$.

The following are obviously:

(1) If g is turbulent then g^n is turbulent for any $n > 1$.

(2) If there exist $p \in \mathrm{Fix}(g), y \in T$ such that $y \in \big((g(y), p\big)$ and $p = g^2(y)$, then g is turbulent.

In [**2**], it is proved that a pointwise chain recurrent map h of the interval must satisfy that either h^2 is the identity or h^2 is turbulent. In [**4**], it is shown that a pointwise chain recurrent map h of the space Y satisfy that either h^{12} is identity or h^{12} is turbulent.

In this paper, we prove the following:

MAIN THEOREM. *Let T be a tree with n endpoints, $f : T \to T$ be a continuous map. If f is pointwise chain recurrent, then*

(1) *If $\mathrm{Fix}(f) \cap \mathrm{End}(T) = \emptyset$, then either f^{a_n} is the identity or f^{a_n} is turbulent;*

(2) *If $\mathrm{Fix}(f) \cap \mathrm{End}(T) \neq \emptyset$, then either $f^{a_{n-1}}$ is the identity or $f^{a_{n-1}}$ is turbulent.*

From the Main Theorem, we obtain the following Corollary, which sharpens the result of [**4**].

COROLLARY. *Let f be a pointwise chain recurrent map of the space Y, then*

(1) *If $\mathrm{Fix}(f) \cap \mathrm{End}(T) \neq \emptyset$, then either f^2 is turbulent or f^2 is identity;*

(2) *If $\mathrm{Fix}(f) \cap \mathrm{End}(T) = \emptyset$, then either f^6 is identity or f^6 is turbulent.*

2. Proof of Main Theorem

Before proving the main theorem, some lemmas are established.

LEMMA 2.1. *Suppose $k, m, n \in N$ with $km \leqslant n$, then $ka_m \leqslant a_n$ and $ka_m \mid a_n$.*

PROOF: Let $a_m = \prod_{i=1}^{s+l} q_i^{r_i}$ and $k = \prod_{i=1}^{l} q_i^{t_i}$, where q_i are prime number with $q_i \neq q_j$, for all $1 \leqslant i < j \leqslant s+l$. Then $ka_m = \prod_{i=1}^{l} q_i^{r_i+t_i} \prod_{j=l+1}^{s+l} q_j^{r_j}$. Obviously, $q_i^{r_i+t_i} \leqslant n$ for all $i < l+1$, and $q_i^{r_i} \leqslant m$ for all $l < i < l+s+1$. Thus, $ka_m \mid a_n$, which completes the proof. $\square$

LEMMA 2.2. *Let T be a tree, $f : T \to T$ be a pointwise chain recurrent map. Then $\mathrm{Fix}(f) \cap \mathrm{int}(T) \neq \emptyset$. In particular, if $\mathrm{Fix}(f) = \{p\}$, then $p \notin \mathrm{End}(T)$.*

PROOF Suppose that $\mathrm{Fix}(f) \cap \mathrm{int}(T) = \emptyset$, that is $\mathrm{Fix}(f) \subset \mathrm{End}(T)$. Given $v_1 \in \mathrm{int}(T)$, then $f(v_1) \in (v_1, e_1)$ for some endpoint e_1.

(1) If $(v_1, e_1) \cap \mathrm{Br}(T) = \emptyset$, then $(v_1, e_1] \cap \mathrm{Fix}(f) = \{e_1\}$ and $f([v_1, e_1]) \subset (v_1, e_1]$. But this contradicts (c). Hence there exists some point $v_2 \in (v_1, e_1) \cap \mathrm{Br}(T)$ satisfying $(v_1, v_2) \cap \mathrm{Br}(T) = \emptyset$.

(2) Obviously $v_2 \in (v_1, f(v_2))$. Then there exists some endpoint e_2 such that $f(v_2) \in (v_2, e_2)$. Hence there exists some point $v_3 \in (v_2, e_2) \cap \mathrm{Br}(T)$ satisfying $(v_2, v_3) \cap \mathrm{Br}(T) = \emptyset$.

(3) Thus, we can find infinite points $v_1, v_2, v_3, \dots$ satisfy: $v_{i+1} \in \mathrm{Br}(T), (v_i, v_{i+1}) \cap \mathrm{Br}(T) = \emptyset, v_i \neq v_j$ for all $i, j \in N$ and $i \neq j$. This is contrary to the fact that $\mathrm{Br}(T)$ is finite, which completes the proof. $\square$

LEMMA 2.3. *Let T be a tree, $f : T \to T$ be a pointwise chain recurrent map. $p \in \mathrm{End}(T) \cap \mathrm{Fix}(f)$. If $C_p \cap \mathrm{Br}(T) = \emptyset$, then either $f^{-1}(C_p) \cap T \setminus \mathrm{Fix}(f) \neq \emptyset$ or f is turbulent.*

PROOF: Without loss of generality, let $C_p = \{p\}$. Suppose $f^{-1}(p) = \{p\}$.

CASE 1. There is some point c with $(p, c) \cap (\mathrm{Fix}(f) \cup \mathrm{Br}(T)) = \emptyset$. If $f(x) \in (p, x)$ for all $x \in (p, c)$, then $f([p, x]) \subset [p, x)$ for all $x \in (p, c)$. That is a contradiction. Thus, $x \in (p, f(x))$ for all $x \in (p, c)$, and there exists some point $b \in (p, c)$ with $f(x) \in T_b(c)$ for all $x \notin T_c(b)$. Thus $f(\overline{T_b(c)}) \subset T_b(c)$, also a contradiction.

CASE 2. There are some fixed points $p_0, p_1, \dots$ with $(p_0, p) \cap \mathrm{Br}(T) = \emptyset$, $d(p_i, p) < d(p_{i-1}, p)$ for all $i \in N$ and $\lim_{i\to\infty} d(p_i, p) = 0$. Without loss of generality, we assume $(p_i, p_{i+1}) \cap \mathrm{Fix}(f) = \emptyset$. If there exists some positive integer i_0 such that $f(x) \in [p, x]$ for all $x \in [p, p_{i_0}]$ or $x \in [p, f(x)]$ for all $x \in [p, p_{i_0}]$, we obtain a analogous contradiction as case 1. Then, there exists some $i_0 \in N$ satisfies $f(x) \in (p, x)$ for all $x \in (p_{i_0+1}, p_{i_0})$ and $x \in (p, f(x))$ for all $x \in (p_{i_0+1}, p_{i_0+2})$. If $f(x) \neq p_{i_0+2}$ for all $x \in (p_{i_0+1}, p_{i_0})$ and $f(x) \neq p_{i_0}$ for all $x \in (p_{i_0+1}, p_{i_0+2})$, then there some open set $U \subset (p_{i_0}, p_{i_0+2})$ such that $f(\overline{U}) \subset U$. That is a contradiction. Without loss of generality, we assume $y \in [p_{i_0}, p_{i_0+1}]$ with $f(y) = p_{i_0+2}$ and $f^{-1}(p_{i_0+2}) \cap (p_{i_0+1}, y) = \emptyset$. If $f(x) \in (p_{i_0+2}, b)$ for all $x \in (p_{i_0+2}, p_{i_0+1})$ and some $b \in (p_{i_0+1}, y)$, then $f(\overline{U}) \subset U$ for some open set $U \subset (p_{i_0+2}, b)$. that is a contradiction. Else, $f(x) = y$ for some $x \in (p_{i_0+2}, p_{i_0+1})$, then f is turbulent. This completes the proof. ∎

LEMMA 2.4. *Let $[p,c]$ be an interval, $g \in C^0[p,c]$ be a pointwise chain recurrent map. If $\{p,c\} \cap \mathrm{Fix}(g) \neq \emptyset$, then g is turbulent or g is identity.*

PROOF: Suppose that g is not turbulent and g is not identity and $p \in \mathrm{Fix}(g)$. Without loss of generality, we assume that $C_p = \{p\}$. Then, by Lemma 2.3, $f(y) = p$ for some $y \in (p,c]$. There exists some point $b \in (p,y)$ with $f(x) \in [p,b)$ for all $x \in [p,y]$. Hence $f([p,y]) \subset [p,y)$, a contradiction, which completes the proof. ∎

LEMMA 2.5. *Let T be a tree, $f : T \to T$ be a pointwise chain recurrent map. If $n > 2$ and $f^{-1}(\mathrm{Fix}(f)) \cap (T \setminus \mathrm{Fix}(f)) = \emptyset$, then either $\mathrm{Card}(\mathrm{End}(\overline{H})) < n$ for each connected component H of $T \setminus \mathrm{Fix}(f)$ or f^2 is turbulent.*

Proof Suppose that there exists some connected component H of $T\setminus\mathrm{Fix}(f)$ satisfied that $\overline{H}$ has n endpoints and let $T \setminus H = H_1$.

CASE 1. If $\mathrm{Card}(\mathrm{Fix}(f)) \geqslant 2$, then $\mathrm{Card}(\overline{H} \cap \mathrm{Fix}(f)) \geqslant 2$ or $\mathrm{Card}(H_1 \cap \mathrm{Fix}(f)) \geqslant 2$.

SUBCASE 1.1. $\mathrm{Card}(\overline{H} \cap \mathrm{Fix}(f)) \geqslant 2$, then $f(\overline{H}) = \overline{H}$. But $\mathrm{Fix}(f) \cap \overline{H} \subset \mathrm{End}(\overline{H})$, this contradicts Lemma 2.2.

SUBCASE 1.2. $\mathrm{Card}(H_1 \cap \mathrm{Fix}(f)) \geqslant 2$ and $\mathrm{Card}(\overline{H} \cap \mathrm{Fix}(f)) = 1$, then H_1 is connected set and $\mathrm{int}(H_1) \cap \mathrm{Fix}(f) \neq \emptyset$. Then $f(H_1) = H_1$, hence $f(\overline{H}) = \overline{H}$. Also a contradiction.

CASE 2. If $\overline{H} \cap H_1 = \mathrm{Fix}(f) = \{p\}$, then $f(\overline{H}) = \overline{H}$, $f(H_1) = H_1$ or $f(\overline{H}) = H_1$, $f(H_1) = \overline{H}$.

SUBCASE 2.1. If $f(\overline{H}) = \overline{H}$ and $f(H_1) = H_1$, we have a contradiction as above.

SUBCASE 2.2. if $f(\overline{H}) = H_1$ and $f(H_1) = \overline{H}$, then we have $f^2(H_1) = H_1$. Let $g = f^2|_{H_1}$, it is not difficult to see that g is a not identity. Then, by Lemma 2.4, g is turbulent and f^2 is turbulent. This completes the proof. ∎

LEMMA 2.6. *Let T be a tree, $f : T \to T$ be a pointwise chain recurrent map and, f is not identity. If $f^{-1}(\mathrm{Fix}(f)) \cap (T \setminus \mathrm{Fix}(f)) = \emptyset$, then*

(1) *$\mathrm{Fix}(f)$ is a connected set and;*

(2) *If $\mathrm{Card}(\mathrm{Fix}(f)) > 1$, then $\partial(\mathrm{Fix}(f)) \subset \mathrm{Br}(T) \cup \mathrm{End}(T)$ and;*

(3) *If $\mathrm{Fix}(f) \cap \mathrm{End}(T) \neq \emptyset$, then $\mathrm{Fix}(f) \cap \mathrm{Br}(T) \neq \emptyset$.*

PROOF: (1) If $\mathrm{Fix}(f)$ is not a connected set, then, there is a connected component H of $T \setminus \mathrm{Fix}(f)$ such that $\mathrm{Card}(\partial(H) \cap \mathrm{Fix}(f)) > 1$. We have $f(\overline{H}) = \overline{H}$ and $\mathrm{Fix}(f) \cap H = \emptyset$. This is a contradiction, by Lemma 2.2.

(2) Suppose $\mathrm{Card}(\mathrm{Fix}(f)) > 1$. If there exists some point $p \in \partial(\mathrm{Fix}(f)) \setminus (\mathrm{Br}(T) \cup \mathrm{End}(T))$, let H be the unique connected component of $T \setminus \mathrm{Fix}(f)$ such that $p \in \overline{H}$. Then $f(\overline{H}) = \overline{H}$ and $\mathrm{Fix}(f) \cap \overline{H} \subset \mathrm{End}(\overline{H})$. This is a contradiction, by Lemma 2.2.

(3) If $p \in \mathrm{Fix}(f) \cap \mathrm{End}(T)$ and $\mathrm{Fix}(f) \cap \mathrm{Br}(T) = \emptyset$, then, by (2), $\mathrm{Fix}(f) = \{p\}$. But this contradicts Lemma 2.2. This completes the proof. ∎

PROOF OF MAIN THEOREM: We do argument on induction.

If $n = 2$, by [**3**] and Lemma 2.4, we know that the assertion is true.

Inductively, we assume that the assertion is true for all T with endpoints less than n and $n > 2$. Now suppose f is not the identity and continue argument for T with n endpoints in the following two cases.

CASE 1. $f^{-1}(\mathrm{Fix}(f)) \cap (T \setminus \mathrm{Fix}(f)) \neq \emptyset$. In this case the following two subcases are considered.

SUBCASE 1.1. $\mathrm{Fix}(f) \cap \mathrm{End}(T) = \emptyset$ Without loss of generality, let $f(z_1) = p \in \mathrm{Fix}(f)$ and $z_1 \neq p$. Denotes $C_0 = \{z_1\}$.

(1)$f^{-1}(z_1) \cap T_{z_1}(p) \neq \emptyset$. Otherwise, there is some nonempty open set $U \subset T_{z_1}(p)$ with $f(\overline{U}) \subset U$. A contradiction.

(2) Let $B_1 = \{z_1^1, z_1^2, \dots, z_1^{k_1}\} \subset f^{-1}(z_1) \cap T_{z_1}(p)$ with $(z_1^i, p) \cap f^{-1}(z_1) = \emptyset$, for all $1 \leqslant i \leqslant k_1$ and B_1 is the largest set with this property. Let A_1 denote $T_{z_1}(p) \cap T_{z_1^1}(p) \cap T_{z_1^2}(p) \cap \dots \cap T_{z_1^{k_1}}(p)$. Then $f^{-1}(B_1) \cap A_1 \neq \emptyset$ (Else, there is some connected open subset U with $f(\overline{U}) \subset U$, which is a contradiction). Denote $f(A_1) \cap B_1 = C_1$.

(3) Let $B_2 = \{z_2^1, z_2^2, \dots, z_2^{k_2}\} \subset f^{-1}(B_1) \cap A_1$ with $(z_2^i, p) \cap f^{-1}(B_1) = \emptyset$, for all $1 \leqslant i \leqslant k_2$ and B_2 is the largest set with this property. Let A_2 denote $T_{z_2^1}(p) \cap T_{z_2^2}(p) \cap \dots \cap T_{z_1^{k_2}}(p) \cap A_1$. Then $f^{-1}(B_2) \cap A_2 \neq \emptyset$. Denote $f(A_2) \cap B_2 = C_2$.

(4) By a repetition of this process we can get nonempty sets $\{A_i, B_i, C_i\}_{i=1}^{\infty}$ with $f(A_i) \cap B_i = C_i$ and $C_{i+1} \subset B_{i+1} \subset A_i \cap f^{-1}(B_i) \subset f^{-1}(C_i)$ for all $i \geqslant 0$. Take some point $z_{n+1} \in C_n$, then $z_i = f^{n-i+1}(z_{n+1}) \in C_{i-1}$ for all $1 \leqslant i \leqslant n+1$. Obviously, $z_j \in T_{z_i}(p)$ for all $i < j$ and $T = \bigcup_{i=1}^{n} [p, e_i]$, where $e_1, e_2, \dots e_n \in \mathrm{End}(T)$. Thus, $z_{i_0}, z_{j_0} \in [p, e_i]$ for some $1 \leqslant j_0 < i_0 \leqslant n+1$ and $i \leqslant n$. Then $z_{i_0} \in (p, z_{j_0})$.

Denote k the minimal common multiple of $i_0 - j_0$ and j_0, and Let $g = f^k$. There is a point $w \in (p, z_{j_0})$ with $g(w) = z_{j_0}$, since $z_{i_0} \in (z_{j_0}, p)$ and $f^{i_0 - j_0}(z_{i_0}) = z_{j_0}$. Then $w \in (p, z_{j_0})$, $g(w) = z_{j_0}$ and $g(z_{j_0}) = p \in \mathrm{Fix}(g)$. It follows that g is turbulent. Thus f^{a_n} is turbulent, since $k \mid a_n$.

SUBCASE 1.2. If $\mathrm{Fix}(f) \cap \mathrm{End}(T) \neq \emptyset$, let $p \in \mathrm{Fix}(f) \cap \mathrm{End}(T)$.

If $f(z_1) = z_0$ for some $z_0 \in C_p$ and $z_1 \notin C_p$, then, taken the process as subcase 1.1, we shall get points $z_0, z_1, \dots, z_n, \dots$ with $f(z_{i+1}) = z_i$ for all $i \geqslant 0$, $z_j \in T_{z_i}(z_0)$ for all $1 \leqslant i < j$. Obviously $T = \bigcup_{i=1}^{n} [z_0, e_i]$, where $e_1, e_2, \dots e_n \in \mathrm{End}(T)$ and $e_n = p$. Then, $z_{i_0}, z_{j_0} \in [z_0, e_i]$ for some $1 \leqslant j_0 < i_0 \leqslant n$ and $i \leqslant n-1$, since $[z_0, p] \subset \mathrm{Fix}(f)$. Then $z_{i_0} \in (z_{j_0}, z_0)$, $i_0 - j_o \leqslant n-1$ and $1 \leqslant j_0 < n-1$. Let k be the minimal common multiple of $i_0 - j_0$ and j_0, and Let $g = f^k$. There is a point $w \in (p, z_{j_0})$ with $g(w) = z_{j_0}$, since $z_{i_0} \in (z_{j_0}, z_0)$ and $f^{i_0 - j_0}(z_{i_0}) = z_{j_0}$. Then $w \in (p, z_{j_0})$, $g(w) = z_{j_0}$ and $g(z_{j_0}) = z_0 \in \mathrm{Fix}(g)$. It follows that g is turbulent. Thus $f^{a_{n-1}}$ is turbulent, since $k \mid a_{n-1}$.

If $f^{-1}(z_0) = \{z_0\}$ for all $z_0 \in C_p$, then, there exists some fixed point $q \notin C_p$ with

$f^{-1}(q) \cap T \setminus \mathrm{Fix}(f) \neq \emptyset$. Let H be the connected component of $T \setminus C_p$ which contains point q, then, $f(\overline{H}) = \overline{H}$. Let $\{w\} = C_p \cap \overline{H}$. It is obviously that $\{w\} \cap \mathrm{Br}(\overline{H}) = \emptyset$ and $f^{-1}(w) \cap \overline{H} = \{w\}$, then f is turbulent, by Lemma 2.3. It follows that $f^{a_{n-1}}$ is turbulent.

CASE 2. $f^{-1}(\mathrm{Fix}(f)) \cap (T \setminus \mathrm{Fix}(f)) = \emptyset$.

Let G_1, G_2 be two connected components of $T \setminus \mathrm{Fix}(f)$. Then $f(\overline{G_1}) = \overline{G_2}$ and $\overline{G_1} \cap \overline{G_2} \subset \mathrm{Fix}(f)$, if $f(G_1) \cap G_2 \neq \emptyset$.

If f^2 is turbulent, then $f^{a_{n-1}}$ is turbulent.

Now we suppose f^2 is not turbulent, then, by Lemma 2.5 and Lemma 2.6, $\mathrm{NE}(\overline{H}) < n$ and $\mathrm{Card}(\overline{H} \cap \mathrm{Fix}(f)) = 1, f(\overline{H}) \cap \overline{H} \subset \mathrm{Fix}(f)$ for each connected component H of $T \setminus \mathrm{Fix}(f)$.

Given G a connected component of $T \setminus \mathrm{Fix}(f)$. Without loss of generality, we assume that (1) $f^k(\overline{G}) = \overline{G}$ and k be the minimal positive integer with this property; (2) $m = \mathrm{NE}(\overline{G}) \leqslant \mathrm{NE}(f^j(\overline{G}))$ for all $k \geqslant j \geqslant 1$. Obviously $m < n$. Then either $(f^k|_{\overline{G}})^{a_{m-1}}$ is identity or $(f^k|_{\overline{G}})^{a_{m-1}} = f^{ka_{m-1}}|_{\overline{G}}$ is turbulent, by induction. And then either $(f^k|_{f^j(\overline{G})})^{a_{m-1}}$ is identity or $f^{ka_{m-1}}|_{f^j(\overline{G})}$ is turbulent for all $k \geqslant j \geqslant 0$. Since $\overline{G}$ is arbitrary and $k(m-1) \leqslant n$, it follows that either f^{a_n} is identity or f^{a_n} is turbulent, by Lemma 2.1.

In particular, if $\mathrm{Fix}(f) \cap \mathrm{End}(T) \neq \emptyset$, then, by Lemma 2.6, $k(m-1) \leqslant n-1$. Thus, we have either $f^{a_{n-1}}$ is identity or $f^{a_{n-1}}$ is turbulent, by Lemma 2.1. This completes the proof. ∎

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