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ON DOMAINS OF PARTIAL ATTRACTION

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Abstract

A new necessary and sufficient condition for a distribution of unbounded support to be in a domain of partial attraction is given. This relates the recent work of [5] and [6].

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1. Introduction

Suppose X_i, $i \geqslant 1$, are independently distributed random variables with common non-degenerate probability distribution function $F(x) = P(X < x)$, and put $S_n = \sum_{i=1}^{n} X_i$. F is said to be in a domain of partial attraction, written $F \in D_p$, if there is a sequence n_i of integers, and constants A_i and B_i, $B_i > 0$, such that $n_i \to \infty$, $B_i \to \infty$ as $i \to \infty$ and $(S_{n_i}/B_i) - A_i$ converges in law to a non-degenerate distribution (which must be infinitely divisible). The domain of partial attraction of a non-degenerate normal distribution is denoted by $D_p(2)$.

Write $H(x) = P(|X| > x)$, $x > 0$ and define $Q(\lambda)$, $\lambda \geqslant 1$: for $\lambda = 1$ by $Q(1) = 1$; and for $\lambda > 1$ by

$$Q(\lambda) = \begin{cases} \liminf\limits_{x \to \infty} H(x\lambda)/H(x), & \text{if } H(x) > 0 \ \forall x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$Q(\lambda)$ is a non-increasing non-negative function. If $Q(\lambda) > 0$ for all $\lambda \geqslant 1$, then for λ, $\mu \geqslant 1$, $Q(\lambda\mu) \geqslant Q(\lambda)Q(\mu)$, so $-\log Q(e^{\tau})$ is subadditive for $\tau > 0$, and hence ([4], p. 244)

$$\lim_{\lambda\to\infty} -\log Q(\lambda)/\log\lambda \tag{1}$$

exists, and is finite (and non-negative). If $Q(\lambda) = 0$ for some $\lambda_0 > 1$ then this is evidently true for all $\lambda \geqslant \lambda_0$, and in fact for all $\lambda > 1$. For let $\lambda_1 = \inf\{\lambda; \lambda > 1, Q(\lambda) = 0\}$, and suppose $\lambda_1 > 1$. Choose λ_2 and λ_3 to satisfy: $\lambda_2 > \lambda_1$, $1 < \lambda_3 < \lambda_1$, and $\lambda_2/\lambda_3 < \lambda_1$. Then $0 = Q(\lambda_2) \geqslant Q(\lambda_2/\lambda_3)Q(\lambda_3) > 0$, which is a contradiction. In the case $Q(\lambda) = 0$, $\lambda > 1$, we formally define the value of $-\log Q(\lambda)$ and of (1) as ∞.

The behaviour of the function $Q(\lambda)$ has recently been used in the study of $F \in D_p$ by Maller [6]. He proves

THEOREM 1. *$F \in D_p(2)$ if and only if $-\log Q(\lambda)/\log\lambda \geqslant 2$ for $\lambda > 1$.*

THEOREM 2. *If $F \in D_p$ then $\lim_{\lambda\to\infty} Q(\lambda) = 0$. If $\lim_{\lambda\to\infty} -\log Q(\lambda)/\log\lambda > 0$, then $F \in D_p$.*

The sufficiency of the condition for $F \in D_p(2)$ in Theorem 1 is implicit in Feller ([3], Theorem 1); while the necessity condition for $F \in D_p$ in Theorem 2 is due to Doeblin ([1], Theorem VII). Maller does not give a necessary and sufficient condition for $F \in D_p$.

In a paper apparently written without knowledge of [6], Jain and Orey [5] assume throughout that $H(x) > 0$, $x > 0$; and define a subset S of $[0, \infty)$ to be of *uniform decrease* for H if S is unbounded and

$$\lim_{\lambda\to\infty} H(\lambda x)/H(x) = 0 \text{ uniformly in } x \in S.$$

They effectively prove in their Theorem 2.1 that $F \in D_p$ if and only if H has a set of uniform decrease.

By a small extension of Maller's methods we may obtain the following alternative necessary and sufficient condition to Jain and Orey's:

THEOREM A. *Assume $H(x) > 0$ for all $x > 0$. Then $F \in D_p$ if and only if there exists a sequence $\{x_i\}$, $i \geqslant 1$, $x_i \to \infty$ as $i \to \infty$, such that*

$$\left\{\limsup_{i\to\infty} H(x_i\lambda)/H(x_i)\right\} \to 0 \quad \text{as } \lambda \to \infty.$$

The condition of uniform decrease is easily seen to imply the last-mentioned necessary and sufficient condition. Conversely:

LEMMA. *If $H(x) > 0$, $x > 0$ and the condition of Theorem A holds then the set $\{x_i\}$, $i \geqslant 1$, is a set of uniform decrease.*

Thus while Theorem A is obtainable, via the last result, from [5], it also follows from the arguments in [6]. The proofs of Theorem A, and of the lemma, which therefore unify the approaches of [5] and [6], follow.

2. Proofs

Proof of Theorem A. Suppose $F \in D_p$. If $Q(2) = 0$ then there exists $\{x_i\}$, $x_i \to \infty$, such that $\lim_{i\to\infty} H(2x_i)/H(x_i) = 0$, whence, since H is non-increasing, $\lim_{i\to\infty} H(\lambda x_i)/H(x_i) = 0$ for $\lambda \geqslant 2$, so that the condition of Theorem A holds. If $Q(2) > 0$ then $Q(\lambda) > 0$ for all $\lambda \geqslant 1$, according to the remarks in Section 1. Now consider the two cases $F \in D_p$, $F \notin D_p(2)$, and $F \in D_p(2)$. If $F \in D_p$, $F \notin D_p(2)$, the necessity proof of Theorem 2 [6] shows that there are sequences $\{n_i\}$ (of integers) and $\{t_i\}$, $n_i \to \infty$ $t_i \to \infty$, and a function $T(\lambda)$, $\lambda > 0$, which is non-increasing on $(0, \infty)$, with a point of continuity λ_0 such that $T(\lambda_0) > 0$, and $T(+\infty) = 0$, such that $n_i H(t_i\lambda) \to T(\lambda)$ as $i \to \infty$ at points of continuity $\lambda > 0$ of T. Hence $H(t_i\lambda_0\mu)/H(t_i\lambda_0) \to T(\mu\lambda_0)/T(\lambda_0)$, where $\mu = \lambda/\lambda_0$, at continuity points λ of $T(\lambda)$. Letting $\mu \to \infty$, we see that the condition of Theorem A holds with the sequence $\{x_i\}$, where $x_i = t_i\lambda_0$.

If on the other hand $F \in D_p(2)$, then from Theorem 1

$$0 < (2 \leqslant) \lim_{\lambda\to\infty} -\log Q(\lambda)/\log\lambda < \infty. \tag{2}$$

This has the form of the sufficient condition of [6] Theorem 2 for the case $Q(\lambda) > 0$, $\lambda \geqslant 1$; and by examining the last part of the proof of sufficiency in that theorem, we find that (2) is there invoked (in essence) to show that there exist Pólya peaks of strictly positive finite order q [2] of second kind for the positive non-decreasing function $g(x) = 1/H(x)$. In particular, this means that there is a sequence $\{s_i\}$, $s_i \to \infty$, and some $a_i \to \infty$, $\delta_i \to 0$, as $i \to \infty$, such that

$$g(\lambda s_i)/g(s_i) \geqslant \lambda^q\{1 - \delta_i\}, \qquad 1 \leqslant \lambda \leqslant a_i,$$

so the sequence $\{s_i\}$ will do for the sequence $\{x_i\}$ required to complete the proof of necessity in Theorem A.

Conversely supposing a sequence $\{x_i\}$ as specified exists, then for every $\lambda > 1$,

$$0 \leqslant Q(\lambda) = \liminf_{x\to\infty} H(\lambda x)/H(x) \leqslant \limsup_{i\to\infty} H(\lambda x_i)/H(x_i),$$

so $\lim_{\lambda\to\infty} Q(\lambda) = 0$, which with the existence of such an $\{x_i\}$, implies $F \in D_p$ if it is assumed $F \notin D_p(2)$, as is shown in the proof of sufficiency of Theorem 2 in Maller [6]. Of course $F \in D_p(2) \Rightarrow F \in D_p$. Thus the sufficiency in Theorem A is proved.

Proof of Lemma. Choose $\varepsilon > 0$. There exists $\lambda_0 > 1$ such that $\limsup_{i\to\infty} H(\lambda_0 x_i)/H(x_i) < \varepsilon$, whence there exists an i_0 such that $H(\lambda_0 x_i)/H(x_i) \leqslant \varepsilon$, $i \geqslant i_0$. Because $\lim_{x\to\infty} H(x) = 0$, there is a λ_1, $\lambda_1 \geqslant \lambda_0$, such that $H(\lambda_1 x_i)/H(x_i) \leqslant \varepsilon$, $i = 1, 2, \ldots, i_0 - 1$. Since H is non-increasing, $H(\lambda x_i)/H(x_i) \leqslant \varepsilon$, $\lambda \geqslant \lambda_1$, $i \geqslant 1$. Thus $H(\lambda x_i)/H(x_i) \to 0$ as $\lambda \to \infty$ uniformly for all $i \geqslant 1$, so $S = \{x_i\}$, $i \geqslant 1$, is a set of uniform decrease for H.

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