THE GROUPS OF REGULAR COMPLEX POLYGONS

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1. Introduction. The two-dimensional unitary space, $\mathbf{U}_2$, is a complex vector space of points $(x, y) = (x_1 + ix_2, y_1 + iy_2)$, for which the distance between (x, y) and (x', y') is defined by $[(x - x')\overline{(x - x')} + (y - y')\overline{(y - y')}]^{\frac{1}{2}}$. A *unitary transformation* is a linear transformation which preserves distance. A *line* is the set of points (x, y) satisfying some complex equation $ax + by = c$. A unitary transformation is a (*unitary*) *reflection* if it is of finite period $n > 1$ and leaves a line pointwise invariant. Thus a unitary matrix represents a reflection if its two characteristic roots are 1 and a complex nth root $(n > 1)$ of 1.

Shephard **(7)** has introduced the notion of *regular complex polygon* as follows. Consider a configuration $\mathbf{P}$ consisting of points ("vertices") and lines ("edges") in $\mathbf{U}_2$. If the group of automorphisms of $\mathbf{P}$ is generated by two reflections, one, say S, which permutes cyclically the vertices on an edge and another, T, which permutes cyclically the edges at one of these vertices, then $\mathbf{P}$ is called a regular complex polygon. Now the finite groups in $\mathbf{U}_2$ generated by S and T can be interpreted as finite groups of orthogonal transformations in four-dimensional Euclidean space, $\mathbf{E}_4$. These groups in $\mathbf{E}_4$ have been enumerated by Seifert and Threlfall **(6)**, using the fact that each such group is homomorphic (either 2:1 or 1:1) to one of the finite groups of displacements in elliptic space of three dimensions enumerated by Goursat **(5)**. The purpose of this paper is to find the groups in Goursat's list corresponding to Shephard's groups.

In §2 we find quaternion transformations $q' = aqb$ corresponding to the generators of Shephard's groups. In §3 these are used to associate groups $\mathfrak{L}$ and $\mathfrak{R}$ of Clifford translations to Shephard's groups. (This discussion closely follows that of **(6)**.) In §4 Goursat's groups are described analogously, leading to the natural homomorphism between Shephard's groups and Goursat's described in §5. The results are tabulated, and summarized in the Theorem.

We write $\mathfrak{C}_n$ and $\mathfrak{E}$ for cyclic groups of order n and 1 respectively. The polyhedral group defined by $A^\mu = B^\nu = (AB)^2 = E$ is denoted $(2, \mu, \nu)$, and the binary polyhedral group $A^\mu = B^\nu = (AB)^2$ is $\langle 2, \mu, \nu\rangle$. With $\mathfrak{C}_n$ the latter are the only finite groups of quaternions. For quaternions the exponential form $\exp s\pi\mathbf{j}/n$ means $\cos s\pi/n + \mathbf{j}\sin s\pi/n$. The order of a group $\mathfrak{G}$ is $|\mathfrak{G}|$.

2. The quaternion representation of a unitary reflection. If the

Received June 29, 1959. This paper is a portion of the author's Ph.D. thesis at the University of Michigan, prepared under the direction of Professor H. S. M. Coxeter.

https://doi.org/10.4153/CJM-1961-011-4 Published online by Cambridge University Press

point $(x, y) = (x_1 + ix_2, y_1 + iy_2)$ in $\mathbf{U}_2$ is represented by the point (x_1, x_2, y_1, y_2) in $\mathbf{E}_4$, then the transformation

$$(2.1) \qquad (x_1' + ix_2', y_1' + iy_2') = (x, y) \begin{pmatrix} r_1 + ir_2 & s_1 + is_2 \\ t_1 + it_2 & u_1 + iu_2 \end{pmatrix}$$

is represented by the transformation

$$(2.2) \qquad (x_1', x_2', y_1', y_2') = (x_1, x_2, y_1, y_2) \begin{pmatrix} r_1 & r_2 & s_1 & s_2 \\ -r_2 & r_1 & -s_2 & s_1 \\ t_1 & t_2 & u_1 & u_2 \\ -t_2 & t_1 & -u_2 & u_1 \end{pmatrix}.$$

In particular, if 2.1 is a unitary reflection then 2.2 is proper orthogonal. The transformation 2.2 can in turn be expressed as a quaternion transformation **(2)**

$$(2.3) \qquad q' = (a_1 + \mathbf{i}a_2 + \mathbf{j}a_3 + \mathbf{k}a_4)\, q\, (b_1 + \mathbf{i}b_2 + \mathbf{j}b_3 + \mathbf{k}b_4),$$

where $q = x_1 + \mathbf{i}x_2 + \mathbf{j}y_1 + \mathbf{k}y_2$, $q' = x_1' + \mathbf{i}x_2' + \mathbf{j}y_1' + \mathbf{k}y_2'$, and $Na = Nb = 1$. Since in our case 2.2 corresponds to a unitary reflection, we have also $a_1 = b_1$ **(2**, p. 141**)**.

The a_i and b_i in 2.3 can be found in terms of the r_i, s_i, t_i, and u_i in 2.2 by applying 2.2 and 2.3 to $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ and $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively, and equating coefficients. For example, applying 2.2 and 2.3 to $(1, 0, 0, 0)$ and $\mathbf{1}$ yields

$$r_1 = a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4,\ r_2 = a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3,$$
$$s_1 = a_1b_3 + a_3b_1 - a_2b_4 + a_4b_2,\ s_2 = a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2.$$

Repeating this in the other three cases yields 12 more equations. Adding and subtracting these 16 equations in pairs containing $r_1, r_2, \ldots, u_1, u_2$ yields 16 equivalent equations which reduce to

$$\begin{aligned} a_3 &= a_4 = 0 \text{ and} \\ 2a_1b_1 &= r_1 + u_1,\ 2a_1b_3 = s_1 - t_1, \\ 2a_1b_2 &= r_2 - u_2,\ 2a_1b_4 = s_2 + t_2, \\ 2a_2b_1 &= r_2 + u_2,\ 2a_2b_3 = s_2 - t_2, \\ 2a_2b_2 &= u_1 - r_1,\ 2a_2b_4 = -s_1 - t_1. \end{aligned}$$

These equations readily give the quaternion transformation 2.3 corresponding to a unitary matrix. For example, if

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \exp 2\pi i/n \end{pmatrix}$$

we get

$$a_1b_3 = a_2b_3 = a_1b_4 = a_2b_4 = 0.$$

Since either a_1 or a_2 is different from zero this implies $b_3 = b_4 = 0$. Moreover, $2a_1b_2 = -u_2 = -2a_2b_1$, that is

$$a_1/a_2 = -b_1/b_2,$$

and since

$$a_1b_1 - a_2b_2 = r_1 > 0$$

we have $b = \bar{a}$. Then

$$2a_1b_1 = 2a_1^2 = r_1 + u_1 = 1 + \cos 2\pi/n,$$

so that

$$a_1 = \pm \cos \pi/n = b_1.$$

We choose the upper sign. Finally

$$2a_2b_1 = 2a_2a_1 = 2a_2 \cos \pi/n = \sin 2\pi/n,$$

which yields

$$a_2 = \sin \pi/n = -b_2.$$

Consequently the quaternion form of T is

$$q' = (\cos \pi/n + \mathbf{i} \sin \pi/n)\ q\ (\cos \pi/n - \mathbf{i} \sin \pi/n). \tag{2.4}$$

In Table I at the end of this paper we list the groups of the regular complex polygons, writing $p_1[t]p_2$ for the group of the polygon $p_1\{t\}p_2$ in the notation of (**4**, p. 80). The generators S and T are taken from (**7**) except for the group $2[n]2$ for which the given S is not a reflection. For this group we let

$$S = S^{-1} = \begin{pmatrix} \cos 2\pi/n & \sin 2\pi/n \\ \sin 2\pi/n & -\cos 2\pi/n \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Professor Coxeter has pointed out that the defining relations for Shephard's groups are particularly simple in terms of S^{-1} and T. Consequently we use these generators in preference to S and T. The quaternion form of S^{-1} appears in the table. By 2.4, T is always $(\exp \pi\mathbf{i}/p_2)\ q\ (\exp -\pi\mathbf{i}/p_2)$.

3. The groups $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{l}$, $\mathfrak{r}$. If S^{-1} is represented by $q' = aqb$ and T by $q' = cqd$ we designate by $\mathfrak{L}^*$ the group generated by a and c, and by $\mathfrak{R}^*$ the group generated by b and d. These groups are either cyclic groups or binary polyhedral groups $\langle 2, \mu, \nu\rangle$ (**1**), and can thus be readily determined. We remark that in all cases $\mathfrak{L}^*$ is cyclic.

We turn now to a more detailed discussion of Shephard's groups in terms of the groups $\mathfrak{L}^*$ and $\mathfrak{R}^*$. Let $\mathfrak{G}$ be such a group. Then $\mathfrak{G}$ is a group of transformations $q' = aqb$, $a \in \mathfrak{L}^*$, $b \in \mathfrak{R}^*$. But not every transformation of the given form is in the group, and furthermore there are certain redundancies. The latter occur because the transformation $q' = aqb$ is identical to the transformation $q' = (-a)q(-b)$. There are no other redundancies of this nature, for if $aqb = cqd$ for all q then $c^{-1}a = db^{-1}$ is a real number, say s, so that $c = as^{-1}$ and $d = sb$. But $Na = Nb = Nc = Nd = 1$, so $s = \pm 1$. We remove these redundancies by identifying the elements (a, b) and $(-a, -b)$ of $\mathfrak{L}^* \times \mathfrak{R}^*$. (Observe that multiplication in $\mathfrak{L}^* \times \mathfrak{R}^*$ is defined by (a, b) $(c, d) = (ac, bd)$, which does indeed correspond to composition of the corresponding transformations since the commutativity in $\mathfrak{L}^*$ implies $ac = ca$.) For finite groups of quaternions the only case in which this identification is trivial is if either $\mathfrak{L}^*$ or $\mathfrak{R}^*$ is a cyclic group of odd order. For our groups this is never the case. Essentially we thus form $(\mathfrak{L}^* \times \mathfrak{R}^*)/\mathfrak{C}_2$, whose elements are the classes $\{(a, b)\}$. We let

$$\mathfrak{L} = \{\{(a, 1)\} : a \in \mathfrak{L}^*\} \text{ and } \mathfrak{R} = \{\{(1, b)\} : b \in \mathfrak{R}^*\}.$$

The elements of $\mathfrak{L}$ and $\mathfrak{R}$ can be multiplied in the obvious manner:

$$\{(a, 1)\}\,\{(1, b)\} = \{(a, b)\}.$$

Clearly $\mathfrak{R} \cong \mathfrak{R}^*$ and $\mathfrak{L} \cong \mathfrak{L}^*$, but $\mathfrak{L}\mathfrak{R} \cong (\mathfrak{L}^* \times \mathfrak{R}^*)/\mathfrak{C}_2$. Every element of $\mathfrak{L}$ commutes with every element of $\mathfrak{R}$. The group $\mathfrak{G}$ is isomorphic to a subgroup of $\mathfrak{L}\mathfrak{R}$ and will be treated as if it were itself a subgroup.

Let $\mathfrak{l} = \mathfrak{L} \cap \mathfrak{G}$ and $\mathfrak{r} = \mathfrak{R} \cap \mathfrak{G}$. Then $\mathfrak{l}$ is a normal subgroup of $\mathfrak{L}$. For let $L \in \mathfrak{L}$ and $l \in \mathfrak{l}$. Certainly $L^{-1}lL \in \mathfrak{L}$. There is some $R \in \mathfrak{R}$ such that $LR \in \mathfrak{G}$, for $\mathfrak{L}$ consists of exactly such elements L. Consequently

$$L^{-1}l\,L = (LR)^{-1}l\,(LR) \in \mathfrak{G}.$$

That is, $L^{-1}\,lL \in \mathfrak{L} \cap \mathfrak{G} = \mathfrak{l}$. Similarly, $\mathfrak{r}$ is a normal subgroup of $\mathfrak{R}$. Furthermore $\mathfrak{l}\mathfrak{r}$ is a normal subgroup in $\mathfrak{G}$ of order $\frac{1}{2}|\mathfrak{l}||\mathfrak{r}|$. For if $l \in \mathfrak{l}$ and $r \in \mathfrak{r}$ then $(LR)^{-1}lr\,(LR) = (L^{-1}l\,L)\,(R^{-1}r\,R) \in \mathfrak{l}\mathfrak{r}$.

We say that an element $L \in \mathfrak{L}$ is *paired* with an element $R \in \mathfrak{R}$ if $LR \in \mathfrak{G}$. The cosets of $\mathfrak{l}$ in $\mathfrak{L}$ are in 1:1 correspondence (given by the pairing) with the cosets of $\mathfrak{r}$ in $\mathfrak{R}$. For if L and L' are paired with R then LR, $(LR)^{-1} = L^{-1}R^{-1}$ and $L'R$ are in $\mathfrak{G}$. Therefore $L^{-1}R^{-1}L'R = L^{-1}L'R^{-1}R = L^{-1}L' \in \mathfrak{G}$. That is, $L^{-1}L' \in \mathfrak{l}$, and consequently L and L' are in the same coset of $\mathfrak{l}$. Conversely, if L and L' are in the same coset and if L is paired with R, that is, $LR \in \mathfrak{G}$, we have $L'L^{-1}LR = L'R \in \mathfrak{G}$. That is, L' is also paired with R. This correspondence is an isomorphism. For let L be in the coset corresponding to the coset containing R, that is, $LR \in \mathfrak{G}$, and let $L'R' \in \mathfrak{G}$. Then

$$LRL'R' = LL'RR' \in \mathfrak{G},$$

that is LL' is in the coset corresponding to the coset containing RR'.

This isomorphism

$$\mathfrak{L}/\mathfrak{l} \cong \mathfrak{R}/\mathfrak{r} \tag{3.1}$$

enables us to determine $\mathfrak{l}$ and $\mathfrak{r}$. For each of the $|\mathfrak{L}|$ elements of $\mathfrak{L}$ is paired with the $|\mathfrak{r}|$ elements of a coset of $\mathfrak{r}$ in $\mathfrak{R}$, and each of the $|\mathfrak{R}|$ elements of $\mathfrak{R}$ is paired with each of the $|\mathfrak{l}|$ elements of a coset of $\mathfrak{l}$ in $\mathfrak{L}$. These pairings give all the elements of $\mathfrak{G}$, but each element appears twice, for if $\{(a, 1)\}$ is paired with $\{(1, b)\}$ then $\{(-a, 1)\}$ is paired with $\{(1, -b)\}$ and $\{(a, 1)\}\,\{(1, b)\} = \{(-a, 1)\}\,\{(1, -b)\} = \{(a, b)\}$. That is, $|\mathfrak{G}| = \frac{1}{2}|\mathfrak{L}||\mathfrak{r}| = \frac{1}{2}|\mathfrak{R}||\mathfrak{l}|$. This gives $|\mathfrak{r}|$ and $|\mathfrak{l}|$ and consequently $\mathfrak{r}$ and $\mathfrak{l}$, since in all but two cases the normal subgroups of these orders are unique. We discuss these two cases separately.

If $\mathfrak{G} = 2[4]n$, n even, we have $\mathfrak{L} = \mathfrak{C}_{2n}$, $\mathfrak{R} = \langle 2, 2, n\rangle$ and $\mathfrak{l} = \mathfrak{C}_n$. In fact the $2n$ elements of $\mathfrak{L}$ are

$$\exp s\pi\mathbf{i}/n,\ s = 1, 2, \ldots, 2n.$$

(Strictly speaking, these are the elements of $\mathfrak{L}^*$. But $\mathfrak{L} \cong \mathfrak{L}^*$ and it is simpler to write $\pm a$ than $\{(\pm a, 1)\}$. The same applies to $\mathfrak{R}$ and $\mathfrak{R}^*$.) The $4n$ elements of $\mathfrak{R}$ are

$$\exp s\pi\mathbf{i}/n \text{ and } \mathbf{k} \exp s\pi\mathbf{i}/n,\ s = 1, 2, \ldots, 2n.$$

The possible choices of $\mathfrak{r}$, that is, of normal subgroups of index 2 in $\mathfrak{R}$, are $\mathfrak{C}_{2n}$ with elements $\exp s\pi\mathbf{i}/n$, $s = 1, 2, \ldots, 2n$, and $\langle 2, 2, n/2\rangle$ with elements

$$\exp 2s\pi\mathbf{i}/n \text{ and } \mathbf{k} \exp 2s\pi\mathbf{i}/n,\ s = 1, 2, \ldots, n.$$

We know $\mathfrak{G}$ contains the element $T = \{(\exp \pi\mathbf{i}/n,\ \exp -\pi\mathbf{i}/n)\}$. Since $\exp \pi\mathbf{i}/n$ is not in $\mathfrak{l}$, but in the other coset of $\mathfrak{l}$ in $\mathfrak{L}$ we know $\exp -\pi\mathbf{i}/n$ is not in $\mathfrak{r}$. But $\exp -\pi\mathbf{i}/n$ is in $\mathfrak{C}_{2n}$. Therefore $\mathfrak{r} = \langle 2, 2, n/2\rangle$.

In the case of groups $2[n]2$, n divisible by 4, where $\mathfrak{L} = \mathfrak{C}_4$, $\mathfrak{R} = \langle 2, 2, n/2\rangle$ and $\mathfrak{l} = \mathfrak{C}_2$, it can be verified in a similar manner that $\mathfrak{r} = \mathfrak{C}_n$ and not $\langle 2, 2, n/4\rangle$. For $T = \{(\mathbf{i}, -\mathbf{i})\}$ is in $\mathfrak{G}$ and $\mathbf{i}$ is not in $\mathfrak{l}$. But $-\mathbf{i}$ is in $\langle 2, 2, n/4\rangle$, so $\mathfrak{r} = \mathfrak{C}_n$. (In this case the elements of $\mathfrak{R}$ are $\exp 2s\pi\mathbf{j}/n$ and $\mathbf{i} \exp 2s\pi\mathbf{j}/n$, $s = 1, 2, \ldots, n$.)

Conversely, given the groups $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{l}$, $\mathfrak{r}$ from our list we can always determine $\mathfrak{G}$ uniquely. To show this we need only show that if distinct isomorphisms 3.1 are chosen, the groups $\mathfrak{G}$ arising from the corresponding pairings of cosets are isomorphic. In most cases $\mathfrak{R}/\mathfrak{r}$ is $\mathfrak{C}_2$ or $\mathfrak{C}$, and there is thus only one isomorphism. However, there are cases where

$$\mathfrak{R}/\mathfrak{r} = \langle 2, 2, n\rangle/\mathfrak{C}_n \cong \mathfrak{C}_4$$

or

$$\mathfrak{R}/\mathfrak{r} = \langle 2, 3, 3\rangle/\langle 2, 2, 2\rangle \cong \mathfrak{C}_3.$$

There is only one non-trivial automorphism of $\mathfrak{C}_4$, and it is induced by an automorphism α of $\langle 2, 2, n\rangle$, namely by $\alpha b = (\mathbf{i})b(-\mathbf{i})$, where the elements b of $\langle 2, 2, n\rangle$ are again $\exp s\pi\mathbf{i}/n$ and $\mathbf{k} \exp s\pi\mathbf{i}/n$, $s = 1, 2, \ldots, 2n$. This establishes an isomorphism between the two possible groups $\mathfrak{G}$ by the correspondence $\{(a, b)\} : \{(a, \alpha b)\}$. Similarly, the only non-trivial automorphism of $\langle 2, 3, 3\rangle/\langle 2, 2, 2\rangle$ is induced by the automorphism

$$\beta b = \left(\frac{1 + \mathbf{i}}{\sqrt{2}}\right) b \left(\frac{1 - \mathbf{i}}{\sqrt{2}}\right)$$

of the group $\langle 2, 3, 3\rangle$ whose elements are $\pm\mathbf{1}$, $\pm\mathbf{i}$, $\pm\mathbf{j}$, $\pm\mathbf{k}$, $(\pm\mathbf{1} \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$. That is, an isomorphism between the two groups obtained from the two possible pairings of cosets $\mathfrak{L}/\mathfrak{l}$ with those of $\langle 2, 3, 3\rangle/\langle 2, 2, 2\rangle$ is determined by the correspondence $\{(a, b)\} : \{(a, \beta\, b)\}$.

4. The relevant Goursat groups. Goursat **(5)** has shown that the finite groups of motions in elliptic 3-space can all be obtained in an analogous fashion from pairings of corresponding cosets of isomorphic quotient groups of polyhedral or cyclic groups. Explicitly, consider a polyhedral or cyclic group $\mathfrak{L}'$ with a normal subgroup $\mathfrak{l}'$ and a polyhedral or cyclic group $\mathfrak{R}'$ with a normal subgroup $\mathfrak{r}'$ such that

$$\mathfrak{L}'/\mathfrak{l}' \cong \mathfrak{R}'/\mathfrak{r}'. \tag{4.1}$$

Then $|\mathfrak{L}'||\mathfrak{r}'| = |\mathfrak{R}'||\mathfrak{l}'|$ is the order of a group $\mathfrak{G}'$ whose elements are the pairs (a, b) for which b is an element of the coset of $\mathfrak{r}'$ in $\mathfrak{R}'$ which corresponds to the coset containing a in the isomorphism 4.1. The multiplication of elements of $\mathfrak{G}'$ is defined by $(a, b)\ (c, d) = (ac, bd)$, and $\mathfrak{G}'$ is a subgroup of $\mathfrak{L}' \times \mathfrak{R}'$.

In all but one of the cases which concern us the quotient groups 4.1 are either $\mathfrak{C}_2$ or $\mathfrak{E}$. Consequently the given construction for $\mathfrak{G}'$ is unambiguous. In the remaining case, where $\mathfrak{R}' = (2, 3, 3)$ and $\mathfrak{r}' = (2, 2, 2)$, there are two distinct ways of pairing the cosets of $\mathfrak{L}'/\mathfrak{l}'$ with those of $\mathfrak{R}'/\mathfrak{r}'$. But these two pairings again lead to isomorphic groups, for there is an automorphism of $(2, 3, 3)$ which induces the non-trivial automorphism of $(2, 3, 3)/(2, 2, 2)$ as follows. Let $(2, 3, 3)$ be the group of a regular tetrahedron, and let γ be a rotation by angle π about a line joining the midpoints of opposite edges of a cube whose vertices are those of the tetrahedron and its dual. Then the transformation $\gamma b \gamma^{-1}$ induces the non-trivial automorphism of $(2, 3, 3)/(2, 2, 2)$. Consequently an isomorphism between the two possible pair groups is given by the correspondence $(a, b) : (a, \gamma b \gamma^{-1})$.

5. The homomorphism from Shephard's groups to Goursat's. Let a group $\mathfrak{G}$ of our list be given by 3.1. Consider the natural homomorphism from the cyclic or binary polyhedral group $\mathfrak{L}$ to the corresponding cyclic or polyhedral group $\mathfrak{L}'$ obtained by identifying the elements $\pm a$ of $\mathfrak{L}$. Let a' be the image of $\pm a$ under this homomorphism. Similarly let b' be the image of $\pm b$ under the natural homomorphism of $\mathfrak{R}$ onto $\mathfrak{R}'$. Clearly this induces a homomorphism from $\mathfrak{G}$ onto some group $\mathfrak{G}'$ whose elements are of the form (a', b'). Thus $\mathfrak{G}'$ is one of Goursat's groups, for its elements are pairs from cyclic or polyhedral groups, and Goursat's list includes all such. We distinguish two cases. (i) The elements $\pm a$ are not in the same coset of $\mathfrak{l}$ in $\mathfrak{L}$, and the elements $\pm b$ are not in the same coset of $\mathfrak{r}$ in $\mathfrak{R}$. The only groups for which this occurs are $2[n]2$, n odd, and $2[4]n$, n odd, that is, the groups for which $\mathfrak{l}$ and $\mathfrak{r}$ are cyclic groups of odd order. (ii) The elements $\pm a$ are in the same coset of $\mathfrak{l}$ in $\mathfrak{L}$ and the elements $\pm b$ are in the same coset of $\mathfrak{r}$ in $\mathfrak{R}$. This is the situation for all other groups in our list.

Now in case (i) the homomorphism described from $\mathfrak{G}$ to $\mathfrak{G}'$ is actually an isomorphism. For the only elements of $\mathfrak{G}$ whose images are (a', b') are $\{(a, b)\}$ and $\{(-a, -b)\}$, since if $\{(a, b)\}$ is in $\mathfrak{G}$ there is no element $\{(a, -b)\}$ in $\mathfrak{G}$. But $\{(a, b)\} = \{(-a, -b)\}$, so the correspondence between the elements of $\mathfrak{G}'$ and those of $\mathfrak{G}$ is 1:1. This determines the order of $\mathfrak{G}'$, as well as $\mathfrak{L}'$ and $\mathfrak{R}'$, so that we can find $\mathfrak{l}'$ and $\mathfrak{r}'$ immediately. In fact, $\mathfrak{l}' = \mathfrak{l}$ and $\mathfrak{r}' = \mathfrak{r}$.

https://doi.org/10.4153/CJM-1961-011-4 Published online by Cambridge University Press

In case (ii) the homomorphism of $\mathfrak{G}$ to $\mathfrak{G}'$ is 2:1 since the distinct elements $\{(a, \pm b)\}$ of $\mathfrak{G}$ both have (a', b') as image. In particular the image of $(1, \pm b)$ is $(1', b')$, so if $b \in \mathfrak{r}$ then $b' \in \mathfrak{r}'$. That is, $\mathfrak{r}'$ is the image of $\mathfrak{r}$ under the homo-

TABLE I

Group $p_1[t]p_2$	Quaternion transformation corresponding to S^{-1} (§ 2)	$\mathfrak{L}/\mathfrak{l}$ (§ 3)	$\mathfrak{R}/\mathfrak{r}$ (§ 3)
$2[4]n$	$(\mathbf{i})q(-\mathbf{k})$	$\mathfrak{C}_{2n}/\mathfrak{C}_n$ (n even)	$\langle 2, 2, n\rangle/\langle 2, 2, n/2\rangle$
		$\mathfrak{C}_{4n}/\mathfrak{C}_n$ (n odd)	$\langle 2, 2, n\rangle/\mathfrak{C}_n$
$2[n]2$	$(\mathbf{i})q(-\mathbf{i}\exp 2\pi\mathbf{j}/n)$	$\mathfrak{C}_4/\mathfrak{C}_2$ (n even)	$\langle 2, 2, n/2\rangle/\mathfrak{C}_n$
		$\mathfrak{C}_4/\mathfrak{C}$ (n odd)	$\langle 2, 2, n\rangle/\mathfrak{C}_n$
$3[6]2$	$\left(\frac{1}{2}+\mathbf{i}\frac{\sqrt{3}}{2}\right)q\left(\frac{1}{2}-\mathbf{i}\frac{1}{2}+\mathbf{j}\frac{\sqrt{2}}{4}-\mathbf{k}\frac{\sqrt{6}}{4}\right)$	$\mathfrak{C}_{12}/\mathfrak{C}$[illegible]	$\langle 2, 3, 3\rangle/\langle 2, 2, 2\rangle$
$3[4]3$	$\left(\frac{1}{2}+\mathbf{i}\frac{\sqrt{3}}{2}\right)q\left(\frac{1}{2}-\mathbf{i}\frac{\sqrt{3}}{6}+\mathbf{j}\frac{\sqrt{6}}{6}-\mathbf{k}\frac{\sqrt{2}}{2}\right)$	$\mathfrak{C}_6/\mathfrak{C}_6$	$\langle 2, 3, 3\rangle/\langle 2, 3, 3\rangle$
$3[3]3$	$\left(\frac{1}{2}+\mathbf{i}\frac{\sqrt{3}}{2}\right)q\left(\frac{1}{2}+\mathbf{i}\frac{\sqrt{3}}{6}+\mathbf{j}\frac{\sqrt{6}}{6}-\mathbf{k}\frac{\sqrt{2}}{2}\right)$	$\mathfrak{C}_6/\mathfrak{C}_2$	$\langle 2, 3, 3\rangle/\langle 2, 2, 2\rangle$
$3[8]2$	$\left(\frac{1}{2}+\mathbf{i}\frac{\sqrt{3}}{2}\right)q\left(\frac{1}{2}-\mathbf{i}\frac{\sqrt{2}}{2}+\mathbf{j}\frac{1}{4}-\mathbf{k}\frac{\sqrt{3}}{4}\right)$	$\mathfrak{C}_{12}/\mathfrak{C}_6$	$\langle 2, 3, 4\rangle/\langle 2, 3, 3\rangle$
$4[6]2$	$\left(\frac{\sqrt{2}}{2}+\mathbf{i}\frac{\sqrt{2}}{2}\right)q\left(\frac{\sqrt{2}}{2}-\mathbf{i}\frac{1}{2}+\mathbf{j}\frac{\sqrt{2}}{4}-\mathbf{k}\frac{\sqrt{2}}{4}\right)$	$\mathfrak{C}_8/\mathfrak{C}_8$	$\langle 2, 3, 4\rangle/\langle 2, 3, 4\rangle$
$4[4]3$	$\left(\frac{\sqrt{2}}{2}+\mathbf{i}\frac{\sqrt{2}}{2}\right)q\left(\frac{\sqrt{2}}{2}-\mathbf{i}\frac{\sqrt{6}}{6}+\mathbf{j}\frac{\sqrt{6}}{6}-\mathbf{k}\frac{\sqrt{6}}{6}\right)$	$\mathfrak{C}_{24}/\mathfrak{C}_{12}$	$\langle 2, 3, 4\rangle/\langle 2, 3, 3\rangle$
$4[3]4$	$\left(\frac{\sqrt{2}}{2}+\mathbf{i}\frac{\sqrt{2}}{2}\right)q\left(\frac{\sqrt{2}}{2}+\mathbf{j}\frac{\sqrt{2}}{2}\right)$	$\mathfrak{C}_8/\mathfrak{C}_4$	$\langle 2, 3, 4\rangle/\langle 2, 3, 3\rangle$
$3[10]2$	$\left(\frac{1}{2}+\mathbf{i}\frac{\sqrt{3}}{2}\right)q\left(\frac{1}{2}-\mathbf{i}\frac{\tau^*}{2}+\mathbf{j}\frac{1}{2\tau}-\mathbf{k}\frac{\sqrt{3}}{4\tau}\right)$	$\mathfrak{C}_{12}/\mathfrak{C}_{12}$	$\langle 2, 3, 5\rangle/\langle 2, 3, 5\rangle$
$5[6]2$	$\left(\frac{\tau}{2}+\mathbf{i}\frac{\sigma}{2}\right)^{\dagger}q\left(\frac{\tau}{2}-\mathbf{i}\frac{1}{2}+\mathbf{j}\frac{1}{4}-\mathbf{k}\frac{\sigma^3\sqrt{5}}{20}\right)$	$\mathfrak{C}_{20}/\mathfrak{C}_{20}$	$\langle 2, 3, 5\rangle/\langle 2, 3, 5\rangle$
$5[4]3$	$\left(\frac{\tau}{2}+\mathbf{i}\frac{\sigma}{2}\right)q\left(\frac{\tau}{2}-\mathbf{i}\frac{\tau\sqrt{3}}{6}+\mathbf{j}\frac{\sqrt{3}}{6}+\mathbf{k}\frac{\sigma^3\sqrt{15}}{30}\right)$	$\mathfrak{C}_{30}/\mathfrak{C}_{30}$	$\langle 2, 3, 5\rangle/\langle 2, 3, 5\rangle$
$3[5]3$	$\left(\frac{1}{2}+\mathbf{i}\frac{\sqrt{3}}{2}\right)q\left(\frac{1}{2}-\mathbf{i}\frac{\sqrt{15}}{6}+\mathbf{j}\frac{\sqrt{3}}{6}-\mathbf{k}\frac{1}{2}\right)$	$\mathfrak{C}_6/\mathfrak{C}_6$	$\langle 2, 3, 5\rangle/\langle 2, 3, 5\rangle$
$5[3]5$	$\left(\frac{\tau}{2}+\mathbf{i}\frac{\sigma}{2}\right)q\left(\frac{\tau}{2}-\mathbf{i}\frac{\sigma\sqrt{5}}{10}+\mathbf{j}\frac{1}{2\sigma}-\mathbf{k}\frac{1}{2\tau}\right)$	$\mathfrak{C}_{10}/\mathfrak{C}_{10}$	$\langle 2, 3, 5\rangle/\langle 2, 3, 5\rangle$

$^*\tau = 2\cos\pi/5 = (1+\sqrt{5})/2$. $\quad ^\dagger\sigma = 2\sin\pi/5 = (3-\tau)^{\frac{1}{2}}$.

https://doi.org/10.4153/CJM-1961-011-4 Published online by Cambridge University Press

morphism taking $\pm b$ to b'. This determines the group $\mathfrak{r}'$. Order considerations alone determine the cyclic group $\mathfrak{l}'$ for which $2|\mathfrak{l}'| = |\mathfrak{l}|$.

The groups $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{l}$, $\mathfrak{r}$ appear in Table I. These tabulations then readily yield the Goursat groups in the form 4.1. Except for the cases $2[4]n$ and $2[n]2$ the results are summarized in the Theorem, for which the following notation is convenient:

Let $p_1[t]p_2$ be the group generated by reflections S^{-1} and T, having the defining relations

$$(S^{-1})^{p_1} = T^{p_2} = E, \; S^{-1}TS^{-1} \ldots = TS^{-1}T \ldots \text{ (t factors on each side).}$$

The centre of this group is the cyclic group $\mathfrak{Z}$ generated by $(S^{-1}T)^{t/2}$ if t is even or by $(S^{-1}T)^t$ if t is odd. The period of $(S^{-1}T)^{t/2}$ is

$$2p_1p_2/k, \text{ where } 2k = 2p_1p_2 + p_1t + p_2t - p_1p_2t$$

(**4**, pp. 76, 77, 79). The quotient group $p_1[t]p_2/\mathfrak{Z}$ is a polyhedral group $(2, \mu, \nu)$. (**7**, p. 84).

THEOREM. *The group $p_1[t]p_2$ ($p_1 \neq 2$) with centre $\mathfrak{Z}$ is* 2:1 *homomorphic to the group of motions in elliptic* 3-*space defined by the isomorphism*

$$\mathfrak{L}'/\mathfrak{l}' \cong \mathfrak{R}'/\mathfrak{r}',$$

where

(a) $\mathfrak{L}'$ *and* $\mathfrak{l}'$ *are cyclic groups.*
(b) $|\mathfrak{L}'| = \text{l.c.m.}\{p_1, p_2\}$.
(c) $2|\mathfrak{l}'| = |\mathfrak{Z}|$.
(d) $\mathfrak{R}' = p_1[t]p_2/\mathfrak{Z}$.
(e) $\mathfrak{r}'$ *is the unique normal subgroup of* $\mathfrak{R}'$ *such that* $|\mathfrak{L}'||\mathfrak{r}'| = |\mathfrak{R}'||\mathfrak{l}'|$.

((a), (b), and (d) also hold in case $p_1 = 2$.)

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