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THE DIMENSION OF GRAPHS WITH RESPECT TO THE DIRECT POWERS OF A TWO-ELEMENT GRAPH

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Every finite loopless undirected graph G is isomorphic to an induced subgraph of a suitable finite direct power G_o^m of the undirected graph G_o with two adjacent vertices $0,1$ and one loop at vertex 1. The least natural number m such that G can be represented in this way is called its G_o-dimension. We give some upper and lower bounds of this dimension depending on certain other graph invariants and determine its exact values for some special classes of graphs. Some methods to determine a concrete G_o-representation, that is an embedding of G into G_o^m, are presented. Moreover we show that the problem of determining the G_o-dimension of a graph is NP-complete.

1. Introduction

Let G_o denote the undirected graph with two adjacent vertices $0,1$ and one loop at vertex 1, see figure 1.

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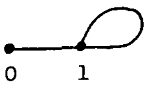

Figure 1

PROPOSITION 1.1. *Every finite loopless undirected graph is isomorphic to an induced subgraph of a suitable finite direct power of* G_o .

This result is implicitly or explicitly mentioned in several papers, for example in [7], [5] and [10] (in connection with investigations of graph algebras) or in [11]. The graph G_o also appears in connection with investigations of subdirectly irreducibles of so-called productive classes of graphs (see for example [13], [14], [12]).

We recall some notions and notation. Let $G = (V,E)$ be a graph (without multiple edges) with vertex set $V = V(G)$ and edge set $E = E(G) \subseteqq V \times V$. For $V' \subseteqq V$ the graph G' with $V(G') = V'$ and $E(G') = E(G) \cap V' \times V'$ is the induced subgraph of G (induced by V'), which will be denoted by $G(V')$. In the following we are mainly concerned with undirected and loopless graphs $G = (V,E)$, that is we have $(a,b) \in E \Longrightarrow (b,a) \in E$ for all $a,b \in V$, and $(c,c) \notin E$ for $c \in V$. Then the two directed edges $(a,b),(b,a)$ are considered often as one undirected edge ab . The set of all finite undirected loopless graphs is denoted by G^o . The graphs (V,E) with $E = V \times V \setminus \{(a,a) \mid a \in V\}$ are called (loopless) cliques or complete graphs.

The m-th direct power G_o^m of G_o is the graph with vertex set $V(G_o^m) = \{0,1\}^m$, and there is an edge between two vertices $(a_1,\ldots,a_m)$, $(b_1,\ldots, b_m) \in V(G_o^m)$ if and only if there is an edge (in G_o) between a_i and b_i (that is, these are not both 0) for all components $i = 1,\ldots, m$. Considering the elements of some subset $W \subseteq V(G_o^m)$ as rows of a matrix one can represent the induced subgraph $G_o^m(W)$ by an $(n \times m)$-matrix $(n = |W|)$ where two zeros in a column indicate that the vertices represented by the corresponding rows are not adjacent. This leads to the following definition.

DEFINITION 1.2. *Let* $G = (V,E)$ *be a graph with* $|V| = n$. *An* $(n \times m)$*-matrix* $M = (a_{ij})_{n \times m}$ *with* $a_{ij} \in \{0,1\} (1 \leq i \leq n\ ,\ 1 \leq j \leq m)$ *and with pairwise different rows is called a* G_o*-representation of* G *if there is an isomorphism* $\rho\colon V \longrightarrow W$ *from* G *onto* $G_o^{\,m}(W)$ *where* $W = \{(a_{i1}, \ldots, a_{im}) \mid i = 1, \ldots, n\}$ *is the set of all rows of* M . *We say the row* $\rho(v)$ *represents the vertex* $v \in V$. *The least natural number* m *such that* G *has a* G_o*-representation* $(a_{ij})_{n \times m}$ *is called the* G_o*-dimension of* G *and it is denoted by* $\dim_{G_o} G$ *(for short* $\dim G$*). A corresponding* G_o*-representation is called minimal.*

Note that by definition $\dim_{G_o} G$ is the least m such that G is isomorphic to an induced subgraph of $G_o^{\,m}$. Proposition 1.1 ensures that every $G \in \mathcal{G}^o$ has a finite G_o-dimension. This can be proved directly by induction on the number of vertices constructing a concrete G_o-representation. In fact, start with the (1×1)-matrix $M = (0)$ for the first vertex. Then, for the $(i+1)$th vertex, say w , add a new row containing only 1's , and then, for each vertex, say v , among the i former ones not adjacent with w , add a new column containing exactly two zeros, namely in the row representing v and in the last row (representing w from now on). If w is adjacent with all i former vertices v then add a single new column with exactly one zero in the last row. Since there are added at most i new columns we get:

$$\text{(1.3)} \qquad \dim_{G_o} G \leq \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

for every $G \in \mathcal{G}^o$ with n vertices. Thus 1.1 is proved, too.

Let us illustrate this procedure considering the bipartite graph $K_{2,3}$ (see figure 2).

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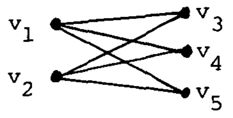

Figure 2

The resulting G_o-representation is

$$\begin{pmatrix} 0&0&1&1&1&1 \\ 1&0&1&1&1&1 \\ 1&1&0&0&0&1 \\ 1&1&1&0&1&0 \\ 1&1&1&1&0&0 \end{pmatrix}$$

where the i-th row represents vertex v_i $(1 \leq i \leq 5)$. Thus $\dim K_{2,3} \leq 6$ (the number of columns), however, as we see later in 4.4a, this G_o-representation is not minimal (deleting the 3rd column one gets a minimal G_o-representation).

A lower bound can be also obtained easily. Since the graph $G_o{}^m$ has 2^m-1 loopless vertices, we have

(1.4) $$\dim_{G_o} G \geq \log_2(n+1)$$

for $G \in G^o$ with n vertices.

In the present note we want to investigate the G_o-dimension, in particular we shall give better lower and upper bounds (Sections 2 and 3) and some methods (algorithms) to find "good" (that is close to minimal) G_o-representations of graphs (Section 3). In Section 4 we are going to determine the G_o-dimension for some special classes of graphs.

From the graph theoretic point of view usually operations (like sums) are of interest which are quite different from the direct product. But there is no reason to think that complexity notions like dimensions with respect to some direct-product-decomposition (for example G_o-dimension) might be of less interest. It was shown in the theory of graph algebras (which goes back to Shallon [15] and Oates-Williams [8]), that classes of graphs closed with respect to direct products, induced subgraphs and disjoint unions are exactly those which can be characterized by identities for their graph algebras (see [4], [10], [9]). Moreover, the G_o-dimension is a special case of the dimension $\dim_{C:B}A$ proposed in [13; p. 77] for arbitrary concrete categories C and special classes B of objects. Here, in [13], the general problem of investigating the various kinds of dimensions is posed.

2. Graph theoretic properties of L_m and lower bounds for dim G

Let L_m be the (undirected) graph arising from $G_o^{\,m}$ by omitting the only vertex $(1,1,\ldots,1) \in V(G_o^{\,m})$ with a loop $(m \in \{1,2,\ldots\})$. By 1.1 every (loopless!) $G \in G^o$ is isomorphic also to an induced subgraph of some graph L_m. In this section we investigate some graph theoretic properties of L_m which will lead to lower bounds for the G_o-dimension of any graph $G \in G^o$.

At first consider the number of edges. Since $G_o^{\,m}$ has 3^m directed edges one of which is a loop (note $((a_1, \ldots, a_m),(b_1, \ldots, b_m)) \in E(G_o^{\,m})$ if and only if $\forall i:(a_i,b_i) \in \{(0,1),(1,0),(1,1)\})$, we get $|E(G)| < 3^m$ if $m = \dim G$, that is

(2.1) $$\dim_{G_o} G > \log_3 2e$$

for every $G \in G^o$ with e undirected edges (= $2e$ directed edges). This bound is better than (1.4) if $e > \frac{1}{2}(n+1)^{\log_2 3}$, $(\log_2 3 < 1.0987)$. For $G = L_m$ however, the lower bound (1.4) is attained. Using the number e_m of undirected edges of L_m (it is easy to see that $2e_m = |E(L_m)| = |E(G_o^{\,m})| - 2\cdot 2^{m-1} - 1$), one gets the following condition for $m = \dim G$: $e \leq e_m = \frac{1}{2}(3^m - 1) + 2^{m-1}$. This improves (2.1) but it does not allow an explicit expression for the lower bound.

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Now, for $G \in G^O$, let $\chi(G)$, $\omega(G)$, $\alpha(G)$, $\beta(G)$ be the chromatic number, the clique number (= maximum number of vertices of an induced complete subgraph), the independence number $(= \omega(\bar{G}))$ and the clique covering number $(= \chi(\bar{G}))$, respectively (see [3]). Here $\bar{G}$ denotes the loopless complement of G: $V(\bar{G}) = V(G)$, $E(\bar{G}) = \{(a,b) \mid a \neq b \in V(G)$ and $(a,b) \notin E(G)\}$. Obviously

$$\chi(G) \geqq \omega(G), \quad \beta(G) \geqq \alpha(G) \tag{2.2}$$

for every $G \in G^O$.

By colouring each vertex $(a_1, \ldots, a_m)$ of L_m with colour i if $a_1 = \ldots = a_{i-1} = 1$ and $a_i = 0$, we see $\chi(L_m) \leqq m$. On the other hand the vertices $(0,1,1,\ldots,1)$, $(1,0,1,\ldots,1)$, ..., $(1,1,\ldots,1,0)$ form an m-clique which proves $m \leqq \omega(L_m)$; hence by (2.2):

$$\chi(L_m) = \omega(L_m) = m . \tag{2.3}$$

Since $\chi(G) \leqq \chi(L_m) = m$ for any subgraph G of L_m (with $m = \dim G$) we get

PROPOSITION 2.4. $dim_{G_O} G \geqq \chi(G) \geqq \omega(G)$ *for* $G \in G^O$. □

The set $\{(a_1, \ldots, a_m) \mid a_1 = 0\} \subseteqq V(L_m)$ is independent in L_m (that is, there is no edge between its vertices), hence $\alpha(L_m) \geqq 2^{m-1}$. On the other hand, each of the $\frac{1}{2} \cdot 2^m$ "complementary pairs" $\{(a_1, \ldots, a_m), (b_1, \ldots, b_m)\}$ with $a_i + b_i = 1$ $(i = 1, \ldots, m)$ is an undirected edge of G_O^m, and they together cover all vertices of G_O^m (and so of L_m, too), that is $2^{m-1} \geqq \beta(L_m)$; thus, by (2.2),

$$\alpha(L_m) = \beta(L_m) = 2^{m-1} . \tag{2.5}$$

Since $\beta(G) \leqq \beta(L_m)$ for any subgraph G of L_m we have:

PROPOSITION 2.6. $dim_{G_O} G \geqq 1 + log_2 \beta(G) \geqq 1 + log_2 \alpha(G)$ *for* $G \in G^O$. □

3. Tight bounds for the G_o-dimension

DEFINITION 3.1. *Let* $G = (V,E)$ *be a graph. A family* $K = \{V_j \mid j \in J\}$ *of subsets* $V_j \subseteq V$ *is called an edge covering of* G *by cliques if for every* $j \in J$ *the induced subgraph* $G(V_j)$ *is a clique and if every edge* $(a,b) \in E$ *is contained in at least one such clique (that is,* $E = \bigcup_{j \in J} E(G(V_j))$ *). Let* $\beta'(G)$ *denote the least number of cliques covering the edges of* $G \in \mathcal{G}^O$ *. Moreover, let* $T(G) = \{v \in V \mid \forall w \in V \setminus \{v\}: (v,w) \in E\}$ *be the set of all vertices adjacent to every other vertex and let* $\tau(G) = |T(G)|$.

THEOREM 3.2. *Let* $G = (V,E) \in \mathcal{G}^O$ *and* $V = \{v_1, \ldots, v_n\}$ *. Then*

$$\beta'(\bar{G}) + \tau(G) \leqq \dim_{G_o} G \leqq \beta'(\bar{G}) + n .$$

Proof. Concerning the lower bound, let $M = (a_{ij})_{n \times m}$ be a G_o-representation (compare 1.2) of G, $m = \dim G$, and let $v_i \in V$ be the vertex represented by the i-th row $(a_{i1}, \ldots, a_{im})$, $i = 1, \ldots, n$. Defining $V_j = \{v_i \mid a_{ij} = 0\}$ $(1 \leqq j \leqq m)$, we have $|\{V_j \mid |V_j| = 1\}| = \tau(G)$, and we shall show that $K = \{V_j \mid |V_j| \geqq 2\}$ is an edge covering of $\bar{G}$ by cliques (proving $\tau(G) + \beta'(\bar{G}) \leqq m)$. In fact, by the definitions $(v_k, v_l) \in E(\bar{G}) \iff (v_k, v_l) \notin E(G) \iff \exists j : a_{kj} = a_{lj} = 0 \iff \exists j : v_k, v_l \in V_j$. Thus all $\bar{G}(V_j)$ are cliques and every edge in $E(\bar{G})$ is contained in some $\bar{G}(V_j)$.

As to the upper bound, let $\{V_j \mid 1 \leqq j \leqq \beta'(\bar{G})\}$ be an edge covering of $\bar{G}$ by cliques. It is easy to check that the matrix $M = (a_{ij})$ with

$$a_{ij} = \begin{cases} 0 & \text{if } v_i \in V_j \quad (j \leqq \beta'(\bar{G})), \\ & \text{or if } j = i + \beta'(\bar{G}) \ (j > \beta'(\bar{G})), \\ 1 & \text{otherwise} \end{cases}$$

$$(1 \leqq i \leqq n,\ 1 \leqq j \leqq n + \beta'(\bar{G}))$$

is a G_o-representation of G (the i-th row represents v_i). □

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Remark 3.3. The matrix M just defined splits in a natural way into a matrix M' consisting of the first $\beta'(\bar{G})$ columns and the matrix M'' consisting of the last n columns. Note that M' reflects the adjacency relation for the vertices of G, whereas M'' does not change the relation but forces all rows of the matrix to be different. Let $T(G) = \{v_1, \ldots, v_{\tau(G)}\}$. Then we must have $a_{ij} = 1$ for $1 \leqq i \leqq \tau(G)$, $1 \leqq j \leqq \beta'(\bar{G})$, and the first $\beta'(\bar{G}) + \tau(G)$ columns of M form a matrix M^* which is an extension of M' and ensures that all $v \in T(G)$ correspond to different rows. Hence, if the rows of M' or M^*, respectively, are pairwise different, then already M' or M^* is a G_o-representation of G (which implies $\dim G = \beta'(G)$ or $\dim G = \beta'(G) + \tau(G)$, respectively).

For a given graph $G = (V,E)$ let $S_1(v) = \{v' \in V \mid (v,v') \in E\}$ be the 1-sphere (the neighbours) of a vertex $v \in V$. Then:

PROPOSITION 3.4. *Let $G \in G^o$ be a graph such that different non-adjacent vertices have different 1-spheres. Then*

$$\dim_{G_o} G = \beta'(\bar{G}) + \tau(G) .$$

In particular, if in addition $T(G) = \emptyset$ then $\dim_{G_o} G = \beta'(\bar{G})$.

Proof. It is straightforward that the assumption is a sufficient condition that the rows of M^* (as defined in 3.3) are pairwise different for any given edge covering of $\bar{G}$ by cliques. □

In view of 3.4, the lower bound $\beta'(\bar{G}) + \tau(G)$ is exact for a large class of graphs. In contrast to this, for every $n \in \{1,2, \ldots\}$, there is only one graph for which the upper bound $\beta'(\bar{G}) + n$ is exact, namely the complete graph K_n with n vertices ($\beta'(\bar{K}_n) = 0$, $\tau(K_n) = n$, compare with 4.1a), since otherwise, if $E(\bar{G}) \neq \emptyset$, one can delete at least one column in M''. For $G = K_n$ upper and lower bound coincide.

Of course it would be nice to have an efficient algorithm which determines the G_o-dimension of a graph. But this problem is NP-hard. We have:

THEOREM 3.5. *The problem* G_o-*DIMENSION* $= \{(G,k) \mid G \in G^o$ *and* $\dim_{G_o} G \leq k$, $k \in \{1,2,\ldots\}\}$ *is NP-complete.*

Proof. Obviously, G_o-*DIMENSION* is in *NP* . To show the *NP*-completeness we reduce the *NP*-complete problem *COVERING BY CLIQUES* = $\{(G,t) \mid G \in G^o$ and there is an edge covering of G by t cliques, $t \in \{1,2,\ \ldots\}\}$ (Problem GT17 in [2]) to G_o-*DIMENSION*. Given a graph $G=(V,E)$ with $V=\{v_1, \ldots, v_n\}$, construct the $2n$-vertex graph $H=(V',E')$ where $V'=\{v_1,\ldots, v_n,w_1, \ldots, w_n\}$, $E'=\{(v_i,v_j) \mid (v_i,v_j) \notin E\} \cup \{(v_i,w_j) \mid i \neq j\}$. Then

$$\bar{H} = (V', E \cup \{(v_i,w_i) \mid 1 \leq i \leq n\} \cup \{(w_i,w_j) \mid i \neq j\})$$

and the only clique in $\bar{H}$ covering an edge (v_i,w_i) is $\bar{H}(\{v_i,w_i\})$ itself, whereas all edges (w_i,w_j) are covered by the clique $\bar{H}(\{w_i \mid 1 \leq i \leq n\})$. Furthermore, H fulfils the assumption of Proposition 3.4 $(\tau(H) = 0)$. This implies

$(G,k) \in$ *COVERING BY CLIQUES* if and only if $(H,k+n+1) \in G_o$-*DIMENSION*. □

At the end of this section we are going to present a method of approximating the G_o-dimension of a graph more precisely under the assumption that a minimal edge covering of $\bar{G}$ by cliques is given.

Thus let $G=(V,E) \in G^o$ and let $M'=(a_{ij})_{n \times \beta'(\bar{G})}$ be the matrix corresponding to a minimal edge covering of $\bar{G}$ by cliques (see the proof of 3.2, 3.3). Let $V=\{v_1, \ldots,v_n\}$ and let the i-th row $w_i=(a_{i1}, \ldots,a_{i\beta'(\bar{G})})$ of M' represent the vertex v_i . Then M' defines an equivalence relation $\sim_{M'}$ on V by

$$v_i \sim_{M'} v_j \ :\Leftrightarrow \ w_i = w_j \ .$$

Note that the equivalence class $[v_i]_{\sim_{M'}}$ is an independent set (anticlique) of G if and only if w_i contains a component a_{ik} equal to 0 ; otherwise, for $w_{i_o} = (1,1,\ldots,1)$ we get the clique

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$[v_{i_0}]_{\sim_{M'}} = T(G)$. It is our aim to extend M' to a G_0-representation of G adding as few as possible new columns. First it is straightforward (in order to reach the lower bound in 3.2), that for any vertex $v_j \in T(G)$ one must add one column containing only 1's except one 0 in the j-th row (compare with 3.3). We get the matrix M^* (see 3.3) with $\beta'(\bar{G}) + \tau(G)$ columns. Now, in view of Proposition 3.4 and its proof, we ask what is to be done if the assumption $S_1(v) \neq S_1(v')$ fails to hold. For two non-adjacent vertices v,v' we have $S_1(v) = S_1(v')$ if and only if $\{v,v'\}$ is a so-called autonomous set of G .

We recall: For $G = (V,E) \in G^0$, a set $A \subseteq V$ is called autonomous in G (in the sense of for example [1], [6]) if, for $a \in A$ and $b \in V \setminus A$, $(a,b) \in E$ implies $(a',b) \in E$ for all $a' \in A$. An equivalence relation $\sim$ on V is called congruence if all equivalence classes are autonomous sets. Then $G = (V,E)$ induces a well-defined graph $G/\sim = (V/_\sim , \tilde{E})$ for which $[a]_\sim$ and $[b]_\sim$ are adjacent if and only if the representatives a and b are adjacent in G . There exists a sufficiently well-developed decomposition theory for graphs (with respect to autonomous subgraphs and congruences, see [6]), some results of which probably could be used to determine the G_0-dimension of a graph in terms of its "factors". We shall not go into details here and will consider only a very special case in section 4.

Let us return to M' and $\sim_{M'}$. Since, by construction, $(v_i,v_j) \in E \iff (w_i,w_j) \in E(G_0^{\beta'(\bar{G})})$, the equivalence classes $[v_i]_{\sim_{M'}}$ are autonomous sets and $\sim_{M'}$ is a congruence. Let G' denote the factor graph $G/\sim_{M'} = (V/\sim_{M'}, \tilde{E})$. Since $T(G)$ is an equivalence class, that is $T(G) \in V/\sim_{M'}$, we can consider the induced subgraph $G^* = G'(V(G')\setminus\{T(G)\})$. Choose some colouring $\{W_i \mid i = 1,\dots, \chi(G^*)\}$ of the vertices of G^* with $\chi(G^*)$ colours (W_i is the set of elements of $V/\sim_{M'}$ with colour i). Moreover, for each i let λ_i be the maximal cardinality among the elements of W_i (note that the elements of W_i

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are equivalence classes of $\sim_{M'}!), 1 \leq i \leq \chi(G^*)$.

THEOREM 3.6. *For* $G \in G^o$ *we have*

$$\beta'(\bar{G}) + \tau(G) \leq \dim_{G_o} G \leq \beta'(\bar{G}) + \tau(G) + \sum_{i=1}^{\chi(G^*)} \lceil \log_2 \lambda_i \rceil .$$

($\lceil x \rceil$ *denotes the least integer not less than* x).

Proof. The lower bound was found in 3.2. In order to prove the upper bound we construct a G_o-representation $M = (a_{ij})$ as follows. The first $\beta'(\bar{G}) + \tau(G)$ columns are to form the matrix M^* (as considered above). Adding new columns one has to distinguish still the vertices of each equivalence class of $\sim_{M'}$ except $T(G)$. Since - by construction! - each W_i consists of cliques of $\bar{G}$ (= equivalence classes of $\sim_{M'}$ = independent sets of G), which are pairwise non-adjacent in G (since they are in the same colour class), one can distinguish all equivalence classes in W_i simultaneously (that is with the same columns). In order to distinguish the vertices of a clique of $\bar{G}$ with λ elements one has to add $\lceil \log_2 \lambda \rceil$ new columns to M^* . This completes the proof. □

Problem 3.7. The upper bound in 3.6 depends 1) on the W_i's, that is on the choice of the edge covering of $\bar{G}$ by cliques (yielding M' and M^*) and it depends 2) on the λ_i's, that is on the chosen (vertex-) colouring of G^* . Does there exist, for any graph $G \in G^o$, an edge covering of $\bar{G}$ by cliques and a vertex-colouring of the corresponding G^* such that the upper bound in 3.6 equals dim G (that is does the above construction lead to a minimal G_o-representation)? We have no counter-example.

4. Examples

In this section we investigate some families of graphs with respect to their G_o-dimension and present some tools for doing this. In particular we shall determine the exact G_o-dimension of several graphs applying the results of Section 2 and 3.

PROPOSITION 4.1. *Let* K_n *be the complete graph with* n *vertices* $(n=1,2,\ldots)$. *Then*

a) $\dim_{G_o} K_n = n$,

b) $\dim_{G_o} \bar{K}_n = 1 + \lceil \log_2 n \rceil$

Proof. a) follows from 3.4 $(\beta'(\bar{K}_n)=0,\ \tau(K_n)=n)$. b) follows from 2.6 (lower bound, $\beta(\bar{K}_n)=n$) and 3.6 (upper bound, $\beta'(K_n)=1$, $\tau(\bar{K}_n)=0$, $\bar{K}_n^* = K_1$, $\chi(\bar{K}_n^*)=1$, $\lambda_1=n$). □

Recall that the join $G+H$ of two disjoint graphs $G,H \in G^o$ is given by $V(G+H)=V(G)\cup V(H)$ and $E(G+H)=E(G)\cup E(H)\cup V(G)\times V(H) \cup V(H)\times V(G)$ (every vertex of G is adjacent to every vertex of H).

THEOREM 4.2. *For* $G,H \in G^o$, $\dim_{G_o}(G+H) = \dim_{G_o} G + \dim_{G_o} H$.

Proof. Let the matrix M be a G_o-representation of $G+H$. By definition of $G+H$ all zeros in each column either belong to rows representing vertices of G or to rows representing vertices of H. Therefore the set of rows of M splits into two disjoint classes such that the matrix built from each of them represents one of G and H, that is, $\dim(G+H) \geqq \dim G + \dim H$. Conversely, two G_o-representations of G and H can easily be arranged to a G_o-representation of $G+H$ proving the opposite inequality. □

The join $G_1+\ldots+G_k$ is a special case of the decomposition of graphs into "sums" of autonomous sets (see Section 3 and [6]). Now let $K_{n_1,\ldots,n_k}$ be the complete k-partite graph consisting of k disjoint independent sets with cardinalities $n_1,\ldots,n_k$ and all edges joining vertices from different sets, that is, $K_{n_1,\ldots,n_k} = \bar{K}_{n_1}+\ldots+\bar{K}_{n_k}$. By 4.1b and 4.2 we have:

COROLLARY 4.3. $\dim_{G_o} K_{n_1,\ldots,n_k} = k + \sum_{i=1}^{k} \lceil \log_2 n_i \rceil$. *In particular, the* G_o*-dimension of a bipartite graph* $K_{m,n}$ *is*

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$$\dim_{G_o} K_{m,n} = 2 + \lceil \log_2 m \rceil + \lceil \log_2 n \rceil . \qquad \square$$

EXAMPLES 4.4. a) dim $K_{2,3} = 5$ (by 4.3), see Figure 2.

b) dim $*_n = 2 + \lceil \log_2 n \rceil$ (by 4.2 and 4.1), where $*_n$ denotes the star with n edges, $*_n = K_1 + \bar{K}_n$.

c) dim $P = 5$ (by 3.4, $\beta'(\bar{P}) = 5$), where P denotes the well-known Petersen graph.

THEOREM 4.5. *Let* C_n *denote the undirected cycle with* n *vertices. Then*

$$\text{(i)} \quad \dim C_n = n \quad \text{for} \quad n = 3,5,7,$$

$$\text{(ii)} \quad \dim C_n = \frac{n+4}{2} \quad \text{for} \quad n = 4,6,8,$$

$$\text{(iii)} \quad \dim C_n \leqq \frac{n+5}{2} \quad \text{for odd} \quad n \geqq 9 ,$$

$$\text{(iv)} \quad \dim C_n \leqq \frac{n+2}{2} \quad \text{for even} \quad n \geqq 10 .$$

Proof. The graphs $C_3 = K_3$ and $C_4 = K_{2,2}$ are covered by 4.1a and 4.3. Thus we can assume $n \geqq 5$. Since non-adjacent vertices of C_n have different 1-spheres and $\tau(C_n) = 0$ we get $\dim_{G_o} C_n = \beta'(\bar{C}_n)$ by 3.4. Thus the proof reduces to the determination of $\beta'(\bar{C}_n)$. Let $V(C_n) = V = \{1,2,\dots, n\} = V_o \cup V_1$, where V_o (V_1 , respectively, contains all even (odd, respectively) numbers. For $i \in V(C_n)$ let $S_1(i) = \{i-1, i+1\}$ (mod n) be the two neighbours. Further define

$$X_i = \{i\} \cup V_o \setminus S_1(i) \quad \text{for} \quad i \in V_1 .$$

1) Let n be even. Then X_i $(i \in V_1)$ and V_1 induce $\frac{n}{2} + 1$ cliques in $\bar{C}_n$, which for $n \geqq 10$ cover all edges of $\bar{C}_n$. To see this, consider an edge e of $\bar{C}_n$ joining two even vertices $(\in V_o)$. Since $n \geqq 10$, there is a path of length $\geqq 6$ in C_n connecting these vertices. Consequently, there is an odd vertex $v \in V_1$ on this path such that $e \in \bar{C}_n(X_v)$. All other edges of $\bar{C}_n$ (joining even and odd vertices)

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are covered by X_i or V_1 trivially (by construction). This proves (iv). For even $n \leqq 8$, $X_i (i \in V_1)$, V_1, V_0 is an edge covering by cliques. This proves the $\leqq$-part of (ii). Since no two edges $14, 25, 36, 13, 24$ belong to a common clique of $\bar{C}_6$ we have $\beta'(\bar{C}_6) \geqq 5$. Analogously, $14, 72, 58, 36, 15, 26$ yield $\beta'(\bar{C}_8) \geqq 6$. This proves the $\geqq$-part of (ii) and finishes the proof of (ii).

2) Let n be odd. Then $X_i (i \in V_1)$, $V_1 \setminus \{1\}$ and $V_1 \setminus \{n\}$ induce $\frac{n+1}{2} + 2$ cliques in $\bar{C}_n$. By the same argument as for even n, they cover all edges of $\bar{C}_n$ provided that $n-1 \geqq 10$, that is, $n \geqq 11$. A direct examination shows that these cliques do the job also for $n = 9$. Thus (iii) is proved. Finally, for $n = 5, 7$ one easily finds $\beta'(\bar{C}_n) = n$, which completes the proof of (i). □

Remark. The G_0-dimension of C_n does not grow with n in every case. The upper bounds in 4.5 are attained for $n = 9, 10, 12$. It is an open problem whether the bounds are exact for other $n \geqq 11$.

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