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Characterization of Low-pass Filters on Local Fields of Positive Characteristic

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Abstract. In this article, we give necessary and sufficient conditions on a function to be a low-pass filter on a local field K of positive characteristic associated with the scaling function for multiresolution analysis of $L^2(K)$. We use probability and martingale methods to provide such a characterization.

1 Introduction

A function $\psi \in L^2(\mathbb{R})$ is said to be a *wavelet* if its integer translations and dyadic dilations $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ form an orthonormal basis for $L^2(\mathbb{R})$, where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j,k \in \mathbb{Z}$. One way to construct a wavelet is through the multiresolution analysis (MRA). An MRA is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$, satisfying the following conditions:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(b) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(c) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;
(d) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(e) there exists a function $\varphi \in V_0$, called a scaling function , such that

$$\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$$

forms an orthonormal basis for V_0.

Using condition (e) of the MRA, we can write

$$\frac{1}{2}\varphi\Big(\frac{1}{2}x\Big) = \sum_{k\in\mathbb{Z}} \alpha_k \varphi(x+k), \tag{1.1}$$

where $\alpha_k = \frac{1}{2}\int_{\mathbb{R}} \varphi(\frac{1}{2}x)\overline{\varphi(x+k)}dx$.

Taking the Fourier transform of equation (1.1), we get $\widehat{\varphi}(2\xi) = \widehat{\varphi}(\xi)m_0(\xi)$, where $m_0(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{ik\xi}$, is a 2π periodic function in $L^2(\mathbb{T})$ called the *low-pass filter* associated with the scaling function φ.

A. Cohen [10] and W. Lawton [25] independently gave the necessary and sufficient conditions for a trigonometric polynomial to be a low-pass filter of an MRA on $L^2(\mathbb{R})$. Later, Hernández and Weiss [19] gave a characterization of low-pass filters by using

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Cohen's approach. They considered certain smooth classes of low-pass filters. Then Papadakis, Ŝikić, and Weiss [26] gave a complete characterization by assuming only the Hölder condition at the origin instead of smoothness condition. Furthermore, San Antolín [28] generalized it to a general dilation matrix. The probabilistic approach of this characterization was discussed by Dobrić, Gundy, and Hitczenko [12].

In 2000, R. F. Gundy [18] gave necessary and sufficient conditions for an arbitrary periodic function to be a low-pass filter. His technique is also useful if we consider that the translates of scaling function form a Riesz basis instead of an orthonormal basis for V_0. E. Curry [11] extended this result for multivariable wavelets.

The characterization of wavelets and MRA wavelets on local fields of positive characteristic has been discussed in [8]. We gave this characterization by using affine and quasi-affine frames [6]. Characterization of scaling functions from which we can construct wavelets on such a field has been provided in the article [4]. In this article, we give the characterization of low-pass filter for local fields of positive characteristic.

A field K equipped with a topology is called a local field if both the additive and multiplicative groups of K are locally compact abelian groups. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$): zero characteristic and positive characteristic. The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Khrennikov, Shelkovich, and Skopina [21] constructed a number of scaling functions generating an MRA of $L^2(\mathbb{Q}_p)$. But later on in [2], Albeverio, Evdokimov, and Skopina proved that all of these scaling functions lead to the same Haar MRA. Some wavelet bases for $L^2(\mathbb{Q}_p)$ different from the Haar system were constructed in [14] and [1]. These wavelet bases were obtained by relaxing the basis condition in the definition of an MRA. Recently, Evdokimov and Skopina [13] proved that no orthogonal wavelet basis for $L^2(\mathbb{Q}_p)$ exists that is not generated by Haar MRA.

Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristic are similar, their wavelet and MRA theory are quite different. Lang [22–24] constructed several examples of compactly supported wavelet for the Cantor dyadic group. Farkov constructed many examples for Vilenkin groups [15–17].

The concept of wavelets on local fields was developed by J. J. Benedetto and R. L. Benedetto [9]. Jiang, Li, and Jin [20] gave the definition of an MRA for local fields K of positive characteristic and have constructed the corresponding orthonormal wavelet. The work of Shukla and Vyas [29] is preceded by [20]. We refer the reader to [3, 5, 7] for some other aspect of wavelet theory on such a field.

The algebraic structure of a local field K of positive characteristic is similar to that of the real number field and the translation set $\{u(k) : k \in \mathbb{N}_0\}$ of K is a countable discrete subgroup of K (see Proposition 2.5). This is analogous to the fact that the translation set $\mathbb{Z}$ of $\mathbb{R}$ is a countable discrete subgroup of $\mathbb{R}$. But, unlike the real line, it is not true in general that $u(k) + u(l) = u(k + l)$ for nonnegative integers k and l (see Section 2 for details). This problem does not show up in the Euclidean case. We have to deal with issues related to this problem separately.

The article is organized as follows. Section 2 contains a brief introduction to local fields and Fourier analysis on such a field. In Section 3, we give some definitions and state the main theorem of this article, which gives necessary and sufficient conditions for a function to be a low-pass filter on local fields of positive characteristic. In the last section, we continue the proof of our main result by using probability and martingale methods.

2 Preliminaries on Local Fields

Let K be a field and a topological space. Then K is called a *locally compact field* or a *local field* if both K^+ and K^* are locally compact abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively.

If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. So by a local field, we mean a field K that is locally compact, non-discrete, and totally disconnected.

We use the notation of the book by Taibleson [30]. Proofs of all the results stated in this section can be found in [27, 30].

Let K be a local field. Since K^+ is a locally compact abelian group, we choose a Haar measure dx for K^+. If $\alpha \neq 0, \alpha \in K$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* or *valuation* of α. We also let $|0| = 0$.

The map $x \to |x|$ has the following properties:

(a) $|x| = 0$ if and only if $x = 0$;
(b) $|xy| = |x||y|$ for all $x, y \in K$;
(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the *ultrametric inequality*. It follows that

$$|x + y| = \max\{|x|, |y|\} \text{ if } |x| \neq |y|.$$

The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. It is the unique maximal compact subring of K. Define $\mathfrak{P} = \{x \in K : |x| < 1\}$. The set $\mathfrak{P}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$. It is principal and prime.

Since K is totally disconnected, the set of values $|x|$ as x varies over K is a discrete set of the form $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. Hence, there is an element of $\mathfrak{P}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{P}$. Such an element is called a *prime element* of K. Note that $\mathfrak{P} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$, as an ideal in $\mathfrak{D}$.

It can be proved that $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{P}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{P}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. For a proof of this fact we refer the reader to [30].

For a measurable subset E of K, let $|E| = \int_K \chi_E(x) dx$, where χ_E is the characteristic function of E and dx is the Haar measure of K normalized so that $|\mathfrak{D}| = 1$. Then it is easy to see that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$ (see [30]). It follows that if $x \neq 0$ and $x \in K$, then $|x| = q^k$ for some $k \in \mathbb{Z}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{P} = \{x \in K : |x| = 1\}$. $\mathfrak{D}^*$ is the group of units in K^*. If $x \neq 0$, we can write $x = \mathfrak{p}^k x'$, with $x' \in \mathfrak{D}^*$.

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Recall that $\mathfrak{D}/\mathfrak{P} \cong GF(q)$. Let $\mathcal{U} = \{a_i : i = 0, 1, \dots, q-1\}$ be any fixed full set of coset representatives of $\mathfrak{P}$ in $\mathfrak{D}$. Let $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| \le q^{-k}\}$, $k \in \mathbb{Z}$. These are called *fractional ideals*. Each $\mathfrak{P}^k$ is compact and open and is a subgroup of K^+ (see [27]).

If K is a local field, then there is a nontrivial, unitary, continuous character χ on K^+. It can be proved that K^+ is self dual (see [30]).

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{P}^{-1}$. We can find such a character by starting with any nontrivial character and rescaling. We will define such a character for a local field of positive characteristic. For $y \in K$, we define $\chi_y(x) = \chi(yx)$, $x \in K$.

Definition 2.1 If $f \in L^1(K)$, then the Fourier transform of f is the function $\widehat{f}$ defined by

$$\widehat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\,dx.$$

Note that

$$\widehat{f}(\xi) = \int_K f(x)\overline{\chi(\xi x)}\,dx = \int_K f(x)\chi(-\xi x)\,dx.$$

Similar to the standard Fourier analysis on the real line, one can prove the following results.

(a) The map $f \to \widehat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\widehat{f}\|_\infty \le \|f\|_1$.
(b) If $f \in L^1(K)$, then $\widehat{f}$ is uniformly continuous.
(c) If $f \in L^1(K) \cap L^2(K)$, then $\|\widehat{f}\|_2 = \|f\|_2$.

To define the Fourier transform of function in $L^2(K)$, we introduce the functions Φ_k. For $k \in \mathbb{Z}$, let Φ_k be the characteristic function of $\mathfrak{P}^k$.

Definition 2.2 For $f \in L^2(K)$, let $f_k = f\Phi_{-k}$ and

$$\widehat{f}(\xi) = \lim_{k\to\infty} \widehat{f_k}(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\overline{\chi_\xi(x)}\,d\xi,$$

where the limit is taken in $L^2(K)$.

We have the following theorem (see [30, Theorem 2.3]).

Theorem 2.3 *The Fourier transform is unitary on* $L^2(K)$.

A set of the form $h + \mathfrak{P}^k$ will be called a *sphere* with centre h and radius q^{-k}. It follows from the ultrametric inequality that if S and T are two spheres in K, then either S and T are disjoint or one sphere contains the other. Also, note that the characteristic function of the sphere $h + \mathfrak{P}^k$ is $\Phi_k(\cdot - h)$ and that $\Phi_k(\cdot - h)$ is constant on cosets of $\mathfrak{P}^k$.

Let χ_u be any character on K^+. Since $\mathfrak{D}$ is a subgroup of K^+, the restriction $\chi_u|_{\mathfrak{D}}$ is a character on $\mathfrak{D}$. Also, as characters on $\mathfrak{D}$, $\chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. That is, $\chi_u = \chi_v$ if $u + \mathfrak{D} = v + \mathfrak{D}$ and $\chi_u \ne \chi_v$ if $(u + \mathfrak{D}) \cap (v + \mathfrak{D}) = \phi$. Hence, if $\{u(n)\}_{n=0}^\infty$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then $\{\chi_{u(n)}\}_{n=0}^\infty$ is a list

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of distinct characters on $\mathfrak{D}$. It was proved in [30] that this list is complete. That is, we have the following proposition.

Proposition 2.4 *Let $\{u(n)\}_{n=0}^{\infty}$ be a complete list of (distinct) coset representatives of $\mathfrak{D}$ in K^+. Then $\{\chi_{u(n)}\}_{n=0}^{\infty}$ is a complete list of (distinct) characters on $\mathfrak{D}$. Moreover, it is a complete orthonormal system on $\mathfrak{D}$.*

Given such a list of characters $\{\chi_{u(n)}\}_{n=0}^{\infty}$, we define the Fourier coefficients of $f \in L^1(\mathfrak{D})$ as

$$\widehat{f}(u(n)) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}dx.$$

The series $\sum_{n=0}^{\infty} \widehat{f}(u(n))\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\int_{\mathfrak{D}} |f(x)|^2 dx = \sum_{n=0}^{\infty} \left|\widehat{f}(u(n))\right|^2.$$

Also, if $f \in L^1(\mathfrak{D})$ and $\widehat{f}(u(n)) = 0$ for all $n = 0, 1, 2, \dots$, then $f = 0$ almost everywhere.

These results hold irrespective of the ordering of the characters. We now proceed to impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. Note that $\Gamma = \mathfrak{D}/\mathfrak{P}$ is isomorphic to the finite field $GF(q)$ and $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, \dots, \epsilon_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\epsilon_j\}_{j=0}^{c-1} \cong GF(q)$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}_0$ such that $0 \le n < q$, we have

$$n = a_0 + a_1 p + \dots + a_{c-1} p^{c-1}, \quad 0 \le a_k < p, \quad k = 0, 1, \dots, c-1.$$

Define

$$u(n) = (a_0 + a_1\epsilon_1 + \dots + a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}. \tag{2.1}$$

Note that $\{u(n) : n = 0, 1, \dots, q-1\}$ is a complete set of coset representatives of $\mathfrak{D}$ in $\mathfrak{P}^{-1}$. Now, for $n \ge 0$, write

$$n = b_0 + b_1 q + b_2 q^2 + \dots + b_s q^s, \quad 0 \le b_k < q, \quad k = 0, 1, 2, \dots, s,$$

and define

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \dots + u(b_s)\mathfrak{p}^{-s}. \tag{2.2}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$, but it follows that

$$u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s) \quad \text{if } r \ge 0, k \ge 0 \quad \text{and} \quad 0 \le s < q^k.$$

In the following proposition we list some properties of $\{u(n)\}$ that will be used later. For a proof, we refer the reader to [4].

Proposition 2.5 *For $n \in \mathbb{N}_0$, let $u(n)$ be defined as in (2.1) and (2.2). Then*

(i) *$u(n) = 0$ if and only if $n = 0$. If $k \ge 1$, then $|u(n)| = q^k$ if and only if $q^{k-1} \le n < q^k$;*
(ii) *$\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}$;*

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(iii) *for a fixed* $l \in \mathbb{N}_0$, *we have* $\{u(l) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$.

For brevity, we will write $\chi_n = \chi_{u(n)}$ for $n \in \mathbb{N}_0$. As mentioned before, $\{\chi_n : n \in \mathbb{N}_0\}$ is a complete set of characters on $\mathfrak{D}$.

Let K be a local field of characteristic $p > 0$ and let $\epsilon_0, \epsilon_1, \dots, \epsilon_{c-1}$ be as above. We define a character χ on K as follows (see [3]):

$$\chi(\epsilon_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \neq 1. \end{cases}$$

Note that χ is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{P}^{-1}$.

In order to be able to define the concepts of multiresolution analysis and wavelet on local fields, we need analogous notions of translation and dilation. Since $\bigcup_{j\in\mathbb{Z}} \mathfrak{p}^{-j}\mathfrak{D} = K$, we can regard $\mathfrak{p}^{-1}$ as the dilation (note that $|\mathfrak{p}^{-1}| = q$), and since $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\{u(n) : n \in \mathbb{N}_0\}$ can be treated as the translation set. Note that it follows from Proposition 2.5 that the translation set form a subgroup of K^+.

A function f on K will be called *integral-periodic* if

$$f\big(x + u(k)\big) = f(x) \text{ for all } k \in \mathbb{N}_0.$$

3 Low-pass Filters

Similar to $\mathbb{R}^n$, wavelets can be constructed from a multiresolution analysis. We define an MRA on local fields as follows (see [20]).

Definition 3.1 Let K be a local field of characteristic $p > 0$, let $\mathfrak{p}$ be a prime element of K, and let $u(n) \in K$ for $n \in \mathbb{N}_0$ be as defined above. An MRA of $L^2(K)$ is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(K)$ satisfying the following properties:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;
(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iv) $f \in V_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(v) there is a function $\varphi \in V_0$, called the *scaling function*, such that

$$\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$$

forms an orthonormal basis for V_0.

Let φ be a scaling function for an MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$. For $f \in L^2(K)$, we define $f_{j,k}(x) = q^{j/2} f(\mathfrak{p}^{-j}x - u(k))$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$.

Since $\varphi \in V_0 \subset V_1$, and $\{\varphi_{1,k} : k \in \mathbb{N}_0\}$ is an orthonormal basis in V_1, we have

$$\varphi(x) = \sum_{k\in\mathbb{N}_0} h_k q^{1/2} \varphi(\mathfrak{p}^{-1}x - u(k)), \tag{3.1}$$

where $h_k = \langle \varphi, \varphi_{1,k} \rangle$ and $\{h_k : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$. Taking Fourier transforms, we get

$$\widehat{\varphi}(\xi) = q^{-1/2} \sum_{k\in\mathbb{N}_0} h_k \overline{\chi_k(\mathfrak{p}\xi)} \widehat{\varphi}(\mathfrak{p}\xi) = m(\mathfrak{p}\xi)\widehat{\varphi}(\mathfrak{p}\xi), \tag{3.2}$$

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where $m(\xi) = q^{-1/2} \sum_{k \in \mathbb{N}_0} h_k \overline{\chi_k(\xi)}$ is an integral-periodic function, called the *low-pass filter associated with the scaling function* φ.

We have the following relation for such a low-pass filter m (see [4]):

$$\sum_{l=0}^{q-1} |m(\xi + \mathfrak{p}u(l))|^2 = 1 \text{ almost every } \xi \in K.$$

We define two operators A and B on $L^\infty(\mathfrak{D})$ and $L^1 \cap L^\infty(K)$, respectively, by

$$\begin{aligned} Af &= \sum_{l=0}^{q-1} |m(\mathfrak{p}(\cdot + u(l)))|^2 f(\mathfrak{p}(\cdot + u(l))), \\ Bf &= |m(\mathfrak{p}\cdot)|^2 f(\mathfrak{p}\cdot). \end{aligned}$$

Since m is a low-pass filter corresponding to the scaling function φ, then by (3.2) $|\widehat{\varphi}(\xi)|^2$ is a fixed point of the operator B. For a scaling function φ, let us denote $S_\varphi(\xi) = \sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))|^2$. We have

(3.3)

$$\begin{aligned} S_\varphi(\xi) &= \sum_{k \in \mathbb{N}_0} \left|\widehat{\varphi}(\xi + u(k))\right|^2 = \sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \left|\widehat{\varphi}\big(\xi + u(l + qk)\big)\right|^2 \\ &= \sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \left|\widehat{\varphi}\big(\xi + u(l) + \mathfrak{p}^{-1}u(k)\big)\right|^2 \\ &= \sum_{l=0}^{q-1} \sum_{k \in \mathbb{N}_0} \left|\widehat{\varphi}\big(\mathfrak{p}\xi + \mathfrak{p}u(l) + u(k)\big)\right|^2 \left|m\big(\mathfrak{p}\xi + \mathfrak{p}u(l) + u(k)\big)\right|^2 \\ &= \sum_{l=0}^{q-1} \left|m\big(\mathfrak{p}\xi + \mathfrak{p}u(l)\big)\right|^2 S_\varphi\big(\mathfrak{p}(\xi + u(l))\big) \qquad \text{(since } m \text{ is integral-periodic)} \\ &= AS_\varphi(\xi). \end{aligned}$$

Therefore, $S_\varphi(\xi)$ is a fixed point of the operator A.

Definition 3.2 Let $g \in L^1 \cap L^\infty(K)$. A function f is almost everywhere g-continuous at the origin if

$$\lim_{j \to \infty} \frac{f(\mathfrak{p}^j \xi)}{|g(\mathfrak{p}^j \xi)|^2}$$

exists and is constant almost everywhere. This limit is denoted by $\frac{f(0)}{|g(0)|^2}$.

Definition 3.3 $D_\infty(\widehat{\varphi})$ is the space of functions $h(\xi)$ satisfying

(i) both $h(\xi)$ and $h^{-1}(\xi)$ belong to $L^\infty(\mathfrak{D})$.

(ii) $h(\xi)$ is almost everywhere $\widehat{\varphi}$-continuous at the origin and $\frac{h(0)}{|\widehat{\varphi}(0)|^2} = 1$.

Note that if $\varphi(x)$ is a scaling function then $S_\varphi(\xi)$ is almost everywhere $\widehat{\varphi}$-continuous at the origin. In fact, $S_\varphi(\xi) \in D_\infty(\widehat{\varphi})$. Using this weak form of continuity, Gundy [18] has given a characterization of low-pass filter for dyadic dilations. E. Curry [11] has generalized this characterization for the multivariable case.

Definition 3.4 We call a function φ a *pre-scaling function associated with an* MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ if its translates $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ form a Riesz basis for V_0.

Let H be a closed subspace of $L^2(K)$. A system $\{f_k : k \in \mathbb{N}_0\}$ of functions in $L^2(K)$ is said to be a *Riesz basis* of H if for any $f \in H$, there exists a sequence $\{a_k : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$ such that $f = \sum_{k\in\mathbb{N}_0} a_k f_k$ with convergence in $L^2(K)$ and

$$C_1 \sum_{k\in\mathbb{N}_0} |a_k|^2 \le \Big\| \sum_{k\in\mathbb{N}_0} a_k f_k \Big\|_2^2 \le C_2 \sum_{k\in\mathbb{N}_0} |a_k|^2, \tag{3.4}$$

where the constants C_1 and C_2 are independent of f.

Remark 3.5 (i) Note that if we take $C_1 = C_2 = 1$, then the Riesz basis is an orthonormal basis for H.

(ii) A function $\varphi \in L^2(K)$ that satisfies the refinement equation (3.1) for some scalars $\{h_k\}_{k\in\mathbb{N}_0}$ but need not satisfy the Riesz basis property (3.4) is called a *refinement function*. So, every pre-scaling function is a refinement function.

In [4], we proved that if the discrete translates of a scaling function form a Riesz basis of the core subspace V_0 of $L^2(K)$, then there exists another function φ_1 such that $\{\varphi_1(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis of V_0.

We have the following lemma for integral-periodic unimodular functions on K. This lemma will be helpful for proving our main result.

Lemma 3.6 *Let μ be an integral-periodic unimodular function on K. That is,*

(i) $\mu(\xi) = \mu(\xi + u(k))$ *almost everywhere for every* $k \in \mathbb{N}_0$, *and*
(ii) $|\mu(\xi)| = 1$ *almost everywhere on* K.

Then there is a unimodular function t on K such that

$$\mu(\xi) = t(\mathfrak{p}^{-1}\xi)\overline{t(\xi)} \quad a.e.\ on\ K. \tag{3.5}$$

Proof Let $Q_j = \{x \in K : |x| = q^j\}$. Observe that $K \smallsetminus \{0\} = \cup_{j\in\mathbb{Z}} Q_j$. Let t be any measurable unimodular function defined on Q_0. For example, we can take $t(\xi) = 1$ for all $\xi \in Q_0$.

Consider $\xi \in Q_1$; then $|\mathfrak{p}\xi| = q^{-1}|\xi| = 1$. This implies $\mathfrak{p}\xi \in Q_0$. Hence, $t(\mathfrak{p}\xi)$ is well defined for $\xi \in Q_1$. Define

$$t(\xi) = t(\mathfrak{p}\xi)\mu(\mathfrak{p}\xi). \tag{3.6}$$

We now proceed inductively. Suppose that t is defined for $Q_1, Q_2, \dots, Q_{n-1}$ so that equation (3.5) satisfies for $\bigcup_{j=0}^{n-1} Q_j$. Define t by (3.6) if $\xi \in Q_n$. Hence, the induction is complete.

Similarly, if $\xi \in Q_{-1}$, then $\mathfrak{p}^{-1}\xi \in Q_0$. Hence, $t(\mathfrak{p}^{-1}\xi)$ is defined. Using (3.6), we define

$$t(\xi) = t(\mathfrak{p}^{-1}\xi)\overline{\mu(\xi)}. \tag{3.7}$$

Again using induction we can define t by equation (3.7) for Q_j, $j \le -1$.

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Therefore, we define $t(\xi)$ for $\xi \in Q_j$, $j \neq 0$, by

$$t(\xi) = \begin{cases} t(\mathfrak{p}\xi)\mu(\mathfrak{p}\xi), & \text{for } \xi \in Q_j,\ j \geq 1, \\ t(\mathfrak{p}^{-1}\xi)\overline{\mu(\xi)}, & \text{for } \xi \in Q_j\ j \leq -1. \end{cases} \tag{3.8}$$

Thus, (3.5) follows from (3.8) if we set $t(0) = 1$. ∎

We are now ready to present our main theorem, which gives necessary and sufficient conditions of a function to be a low-pass filter for a local field K of positive characteristic.

Theorem 3.7 *Let m be a low-pass filter associated with a pre-scaling function φ. Then the following hold.*

(i) *m is integral-periodic, $m \in L^2(\mathfrak{D})$, and $|m(\xi)|^2$ is almost everywhere $\widehat{\varphi}$-continuous at the origin with*

$$\lim_{j\to\infty} |m(\mathfrak{p}^j\xi)| = 1 \quad a.e.$$

(ii) *The operators A and B have nontrivial fixed points, $S_\varphi(\xi) \in L^\infty(\mathfrak{D})$ and $|\widehat{\varphi}|^2 \in L^1 \cap L^\infty(K)$, respectively.*

(iii) *The fixed point S_φ of operator A is the unique function in the class $D_\infty(\widehat{\varphi})$.*

Conversely, if a function m satisfies (i), (ii), *and* (iii), *then m is a low-pass filter associated with a pre-scaling function φ for an MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$.*

Proof First we prove the converse part.

Suppose that the operator B has a fixed point $|\widehat{\varphi}(\xi)|^2$. The fixed point $S_\varphi(\xi)$ of the operator A is the unique function in $D_\infty(\widehat{\varphi})$. Then by [4, Proposition 3.5], the ratio $|\widehat{\varphi}|/S_\varphi^{1/2}$ is a scaling function for an MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$. The low-pass filter corresponding to this scaling function is

$$m_0(\xi) = |m(\xi)|\Big(\frac{S_\varphi(\xi)}{S_\varphi(\mathfrak{p}^{-1}\xi)}\Big)^{1/2}.$$

This leads us to define

$$\widetilde{m}_0(\xi) = m(\xi)\Big(\frac{S_\varphi(\xi)}{S_\varphi(\mathfrak{p}^{-1}\xi)}\Big)^{1/2}.$$

Note that $\widetilde{m}_0(\xi) = \operatorname{sgn} m(\xi) m_0(\xi)$

By Lemma 3.6, we can write $\operatorname{sgn} m(\xi) = t(\mathfrak{p}^{-1}\xi)\overline{t(\xi)}$, where t is a unimodular function on K. Define

$$\begin{aligned} \widehat{\varphi}(\xi) &:= t(\xi)|\widehat{\varphi}(\xi)| = t(\xi)\overline{t(\mathfrak{p}\xi)}t(\mathfrak{p}\xi)|m(\mathfrak{p}\xi)\widehat{\varphi}(\mathfrak{p}\xi)| \\ &= \operatorname{sgn} m(\mathfrak{p}\xi)|m(\mathfrak{p}\xi)|\widehat{\varphi}(\mathfrak{p}\xi) = m(\mathfrak{p}\xi)\widehat{\varphi}(\mathfrak{p}\xi). \end{aligned}$$

Since $t(\xi)$ is a unimodular function, all the conditions of [4, Theorem 5.1] are satisfied, and hence, $\varphi(\xi)$ is a required pre-scaling function for an MRA.

Now let $m(\xi)$ be a low-pass filter associated with a pre-scaling function φ for an MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$. By definition, the operator B has a fixed point $|\widehat{\varphi}|^2$. And also from (3.3), S_φ is a fixed point of the operator A. Furthermore, $S_\varphi^{-1} \in L^2(\mathfrak{D})$ (see [4, Lemma 3.4]). This implies that the function $\gamma(x)$, defined by

$$|\widehat{\gamma}(\xi)|^2 = \frac{|\widehat{\varphi}(\xi)|^2}{S_\varphi(\xi)},$$

is a scaling function for the same MRA (see [4, Proposition 3.5]) and that

$$\sum_{k \in \mathbb{N}_0} \left| \widehat{\gamma}\big(\xi + u(k)\big)\right|^2 = 1.$$

By the characterization of scaling function, we have

$$1 = \lim_{j\to\infty} |\widehat{\gamma}(\mathfrak{p}^j \xi)|^2 = \lim_{j\to\infty} \frac{|\widehat{\varphi}(\mathfrak{p}^j\xi)|^2}{S_\varphi(\mathfrak{p}^j\xi)} \text{ a.e.}$$

This shows that $S_\varphi(\xi)$ is almost everywhere $\widehat{\varphi}$-continuous at zero. It only remains to prove that S_φ is the unique function in the class $D_\infty(\widehat{\varphi})$.

4 Proof of the Uniqueness

In this section, we want to prove that $S_\varphi(\xi)$ is a unique function in $D_\infty(\widehat{\varphi})$. Suppose $h(\xi)$ is another such function. We claim that $S_\varphi(\xi) = h(\xi)$ for almost every ξ. Since $\gamma(\xi)$ is a scaling function of an MRA, it is obvious that the Fourier transform of γ at $\xi = 0$ is 1. Also, we have $\sum_{k\in\mathbb{N}_0} |\widehat{\gamma}(\xi + u(k))|^2 = 1$ for almost every $\xi \in \mathfrak{D}$ and $\lim_{j\to\infty} |\widehat{\gamma}(\mathfrak{p}^j\xi)|^2 = 1$ for almost every ξ on K. Therefore, we can interpret $|\widehat{\gamma}(\xi + u(k))|^2$, $k \in \mathbb{N}_0$, as a probability distribution on $\mathbb{N}_0$ for almost every $\xi \in \mathfrak{D}$.

Let μ be the low-pass filter associated with the scaling function γ. Then

$$\mu(\xi) = \frac{\widehat{\varphi}(\mathfrak{p}^{-1}\xi)}{S_\varphi(\mathfrak{p}^{-1}\xi)} \cdot \frac{S_\varphi(\xi)}{\widehat{\varphi}(\xi)} = m(\xi)\frac{S_\varphi(\xi)}{S_\varphi(\mathfrak{p}^{-1}\xi)}.$$

Let $M(\xi) = |\mu(\xi)|^2$. Notice that $M(\xi)$ is an integral-periodic function and satisfies $M(0) = 1$ and

$$\sum_{l=0}^{q-1} M\big(\xi + \mathfrak{p}u(l)\big) = 1, \quad \text{a.e. } \xi \in \mathfrak{D}. \tag{4.1}$$

Every non-negative integer $k \in \mathbb{N}_0$ can be expressed uniquely as

$$k = \sum_{j=1}^{\infty} \omega_j(k) q^{j-1}, \quad 0 \le \omega_j(k) \le q-1.$$

We identify k with the sequence $(0, \omega_1(k), \omega_2(k), \dots)$ and define $\omega_0(k) = 0$. The integer zero is identified with the sequence zero. Note that each such sequence has finitely many non zero terms.

Let $D = \{1, 2, \dots, q-1\}$ and $D_0 = D \cup \{0\}$. Let $\Omega = D_0^{\mathbb{N}}$ be the set of sequences. We identify $\mathbb{N}_0$ with the subset of Ω consisting of finite sequences. Fix $k \in \mathbb{N}_0$. For $N \ge 1$, let $\mathbf{k}_N = \{\omega : \omega_i = \omega_i(k), 0 \le i \le N\}$ be a finite cylinder in Ω.

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For each $\xi \in \mathfrak{D}$, we define probability Q_ξ^N on the set of all such cylinders as follows. For $0 \le k \le q^N - 1$, we set

$$Q_\xi^N(k) = \prod_{j=1}^{N} M\big(\mathfrak{p}^j(\xi + u(k))\big). \tag{4.2}$$

Lemma 4.1

$$\sum_{0 \le k \le q^N - 1} Q_\xi^N(k) = 1 \tag{4.3}$$

Proof We will prove this lemma by using induction on N. Define conditional probability by

$$M\big(\mathfrak{p}^j(\xi + u(k))\big) = Q_\xi\big(\omega_j(k) \| \omega_{j-1}, \dots, \omega_1\big).$$

Equation (4.3) can also be written as $Q_\xi^N(\mathbf{k}_N) = 1$.

For $N = 1$,

$$Q_\xi^1(k) = M(\mathfrak{p}(\xi + u(k))) = Q_\xi(\omega_1(k)).$$

Using equation (4.1), we can easily see that the result is true for $N = 1$.

$$Q_\xi^1(\mathbf{k}_1) = \sum_{\omega_1 \in D_0} Q_\xi(\omega_1(k)) = \sum_{k=0}^{q-1} M(\mathfrak{p}(\xi + u(k))) = 1 \quad \text{a.e.} \quad \xi.$$

Assume that it is true for $N-1$, *i.e.,* $Q_\xi^{N-1}(\mathbf{k}_{N-1}) = 1$. Now we want to prove it is true for N. We write

$$\begin{aligned} Q_\xi^N(k) &= \Big(\prod_{j=1}^{N-1} M\big(\mathfrak{p}^j(\xi + u(k))\big)\Big) \times M\big(\mathfrak{p}^N(\xi + u(k))\big) \\ &= Q_\xi^{N-1}(k) \times Q_\xi\big(\omega_N(k) \| \omega_{N-1}, \dots, \omega_1\big), \\ Q_\xi^N(\mathbf{k}_N) &= Q_\xi^{N-1}(\mathbf{k}_{N-1}) \times Q_\xi\big(\omega_N(\mathbf{k}_N) \| \omega_{N-1}, \dots, \omega_1\big) \end{aligned}$$

where,

$$\begin{aligned} &Q_\xi\big(\omega_N(\mathbf{k}_N) \| \omega_{N-1}, \dots, \omega_1\big) \\ &\quad = \sum_{\omega_N=0}^{q-1} M\big(\mathfrak{p}^N\big(\xi + u(\omega_1) + \mathfrak{p}^{-1}u(\omega_2) + \dots + \mathfrak{p}^{-N+1}u(\omega_N)\big)\big) \\ &\quad = \sum_{\omega_N=0}^{q-1} M\big(\mathfrak{p}^N\xi + \mathfrak{p}^N u(\omega_1) + \mathfrak{p}^{N-1}u(\omega_2) + \dots + \mathfrak{p}u(\omega_N)\big). \end{aligned}$$

Note that the summation is only on ω_N as $\omega_1, \dots, \omega_{N-1}$ are given. Again using (4.1), we get

$$Q_\xi\big(\omega_N(\mathbf{k}_N) \| \omega_{N-1}, \dots, \omega_1\big) = 1$$

Hence, the induction is complete. ∎

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Therefore, Q_ξ^N, $N \geq 1$, specifies a probability. By the basic Kolmogorov theorem, the family Q_ξ^N extends to a probability say P_ξ on the Borel sets of Ω. If we assume that infinite product of (4.2) exists, then we have

$$1 = \sum_{k\in\mathbb{N}_0} |\widehat{\gamma}(\xi + u(k))|^2 = \sum_{k\in\mathbb{N}_0} \lim_{N\to\infty} \prod_{j=1}^{N} M\big(\mathfrak{p}^j(\xi + u(k))\big)$$
$$= \sum_{k\in\mathbb{N}_0} \lim_{N\to\infty} Q_\xi^N(k) \quad \text{for a.e. } \xi.$$

Hence, Q_ξ^N is tight in the Prokorov sense on the set of finite sequence. Therefore, P_ξ is concentrated on finite sequences. We say $P_\xi(\mathbb{N}_0) = 1$ for almost every ξ.

Consider $X_j(\omega(k)) = \omega_j(k)$, where $\omega_j(k) \in D_0$. Define $\xi_1(k) := \xi$ and $\xi_{j+1}(k) := \mathfrak{p}(\xi_j + u(\omega_j(k)))$.

For $0 \leq k \leq q^N - 1$, we write $k = \sum_{j=1}^N \omega_j(k) q^{j-1}$, $0 \leq \omega_j(k) \leq q - 1$. And

$$u(k) = u(\omega_1) + \mathfrak{p}u(\omega_2) + \cdots + \mathfrak{p}^{-N+1}u(\omega_N), \text{ using equation (2.2).}$$

Also, we can write

$$\begin{aligned}
\mathfrak{p}^N\big(\xi + u(k)\big) &= \mathfrak{p}^N\big(\xi + u(\omega_1) + \mathfrak{p}u(\omega_2) + \cdots + \mathfrak{p}^{-N+1}u(\omega_N)\big)\\
&= \mathfrak{p}\big(\mathfrak{p}^{N-1}\xi + \mathfrak{p}^{N-1}u(\omega_1) + \mathfrak{p}^{N-2}u(\omega_2) + \cdots + \mathfrak{p}u(\omega_{N-1}) + u(\omega_N)\big)\\
&= \mathfrak{p}\big(\xi_N + u(\omega_N)\big).
\end{aligned}$$

Now we can define the conditional probability of X_j given $X_{j-1}, \ldots, X_1$ is

$$M\big(\mathfrak{p}\big(\xi_j + u(\omega_j(k))\big)\big)$$

for each $j \geq 1$. Since P_ξ is concentrated on finite sequences for almost every ξ, hence, the sequence $\{X_j\}_{j\geq 1}$ converges to zero relative to P_ξ.

Now

$$P_\xi(\xi_{j+1} \| \xi_j, \ldots, \xi_1) = M\big(\mathfrak{p}\big(\xi_j + u(\omega_j(k))\big)\big).$$

By construction, $P_\xi(\xi_{j+1} \| \xi_j, \ldots, \xi_1) = P_\xi(\xi_{j+1} \| \xi_j)$. Thus, $\{\xi_j\}_{j\geq 1}$ is a Markov process.

Since P_ξ is concentrated on a finite sequence, hence, sequence $\{\xi_j\}_{j\geq 1}$ converges to zero.

Now we will come back to uniqueness question. Consider $r(\xi) = \frac{h(\xi)}{S_\varphi(\xi)}$. We want to show that $r(\xi) = 1$ for almost every ξ. We know that $h(\xi)$ and $S_\varphi(\xi)$ are fixed points of the operator A and $S_\varphi(\xi) = 1$ almost everywhere , hence, $r(\xi)$ satisfies

$$r(\xi) = \sum_{l=0}^{q-1} |m(\mathfrak{p}(\xi + u(l)))|^2 r(\mathfrak{p}(\xi + u(l))).$$

Therefore, the composition $r(\xi_j)$ is a martingale, *i.e.,*

$$\begin{aligned}
E\big(r(\xi_{j+1}) \| r(\xi_j), \ldots, r(\xi_1)\big) &= E\big(r\big(\mathfrak{p}(\xi_j + u(\omega_j))\big) \| r(\xi_j), \ldots, r(\xi_1)\big)\\
&= E\big(r(\mathfrak{p}\big(\xi_j + u(\omega_j)\big)\big) \| r(\xi_j)\big)\\
&= \sum_{\omega_j\in D_0} M\big(\mathfrak{p}(\xi_j + u(\omega_j))\big) r\big(\mathfrak{p}(\xi_j + u(\omega_j))\big) = r(\xi_j).
\end{aligned}$$

The martingale $r(\xi_j)$ is strictly positive, bounded, and converges P_ξ-almost surely to $r(0) = 1$ for almost every ξ, since $\xi_j \to 0$. By Lebesgue dominated converges theorem and for all $j \geq 1$, we get

$$r(0) = E\big(r(0)\|r(\xi_j)\big) = E\big(\lim_{n\to\infty} r(\xi_n)\|r(\xi_j)\big) = \lim_{n\to\infty} E\big(r(\xi_n)\|r(\xi_j)\big) = r(\xi_j).$$

Thus,

$$r(0) = r(\xi) = \frac{h(\xi)}{S_\varphi(\xi)}$$

for almost every ξ. This gives $h(\xi) = S_\varphi(\xi)$ for almost every ξ, which proves the uniqueness assertion of the theorem. ■

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