P 16. (a) Prove that there is a polyhedron whose faces consist of 6 squares and (say f hexagons where f may be greater than any given number.

(b) What is the sequence of possible values of f ?

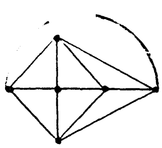
(c) What is the largest value of f for which the hexagons can all be regular.

(d) What is the largest value of f for which the hexagons can all be centrally symmetric?

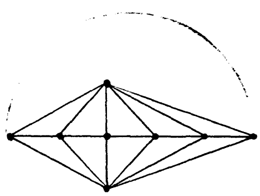
H. S. M. Coxeter, University of Toronto

Partial solution by W. Moser, University of Manitoba.

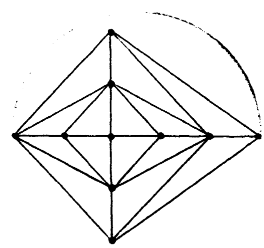
Consider the following sequence of maps M_k:



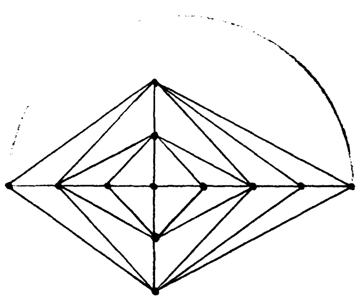
M_1



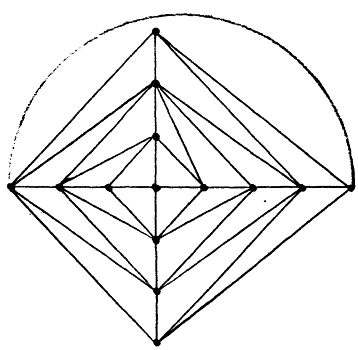
M_2



M_3



M_4



M_5

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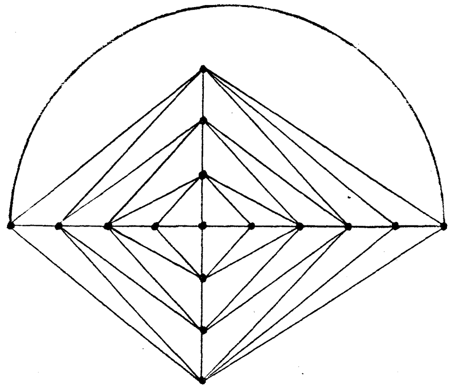

M_6

and so on.

M_k has $2k + 4$ vertices; exactly 6 of these vertices are each incident with 4 edges, and the other $2k-2$ of the vertices are each incident with 6 edges. Also, M_k has $4k + 4$ faces, all triangular. Because all the faces are triangular, it is possible to transfer the vertices of M_k to the surface of a sphere, and join them by straight edges to form convex polyhedron P_k having the same incidence relations as M_k. Truncating P_k produces a convex polyhedron which has 6 quadrilateral faces and $(4k + 4) + (2k - 2) = 6k + 2$ hexagonal faces. Thus we have proved (a) with "squares" replaced by "quadrilaterals".

Let P be a convex polyhedron. Let V, E and F be the number of vertices edges and faces of P respectively. Let V_i denote the number of vertices each incident with i edges, and F_i denote the number of faces each having i sides. Then

(1) $V - E + F = 2$

and

(2) $\sum_{i=3} iV_i = \sum_{i=3} iF_i = 2E$.

Hence for real numbers a and b satisfying

(3) $0 \leq a, \quad 0 \leq b, \quad a + b = \frac{1}{2}$,

we have, from (1) and (2),

(4) $\Sigma(1 - ai)\, F_i + \Sigma(1 - bi)\, V_i = 2$.

In particular with $a = 1/6$ and $b = 1/3$ (4) reduces to

(5) $3F_3 + 2F_4 + F_5 = 12 + F_7 + 2F_8 + 3F_9 + \ldots + 2V_4 + 4V_5 + \ldots$

If P satisfies the conditions

(6) $F_4 = 6$ and $F_i = 0, \quad i \neq 4, 6$

then (5) yields

$$2V_4 + 4V_5 + 6V_6 + \ldots = 0$$

and this implies

(7) $V_4 = V_5 = V_6 = \ldots = 0$.

Thus, (6) implies $V = V_3$. (4) now reads

$$(1-4a)6 + (1-6a)F_6 + (1-3b)V = 2$$

and this with $a = 1/3$ and $b = 1/6$ yields

(8) $V = 8 + 2F_6$.

Now suppose that in addition to (6), P satisfies the condition

(9) the hexagonal faces are regular.

Then there are at most 2 hexagonal faces meeting at a vertex. Thus, if h_i denotes the number of hexagons at the i^{th} vertex ($i = 1, 2, \ldots, V$), we have, using (8),

(10) $6F_6 = h_1 + h_2 + \ldots + h_v \leq 2V = 16 + 4F_6$

from which we deduce that $F_6 \leq 8$.

Since the truncated regular octahedron has 6 square faces and 8 hexagonal faces, we have answered part (c).

Now consider a polyhedron S with 6 parallelogram faces and f centrally symmetrical hexagonal faces. Then (see Coxeter, Regular Polytopes, p. 29), this polyhedron is necessarily a zonohedron and

(11) $2.6 + 6F_6 = 2n(n-1)$

where n is the number of zones. Furthermore, a zonohedron with n zones has at least $3n/7$ pairs of parallelogram faces (see Kelly and Moser, "On the number of ordinary lines determined by n points", Can. J. of Math., 1958, pp. 210-219).

It follows that for S,

(12) $6 \geq 2(3n/7)$.

Thus $3 \leq n \leq 7$. Relation (11) with $n = 3, 4, 5, 6, 7$ yields in turn $F_6 = 0, 2$, no solution, 8, 12. This concludes the proof of (d).

P 52. Let n be an integer > 2 and put $\omega = e^{2\pi i/n}$. Show that if $f(z)$ is regular for $|z| < A$ and satisfies the equation

(1) $$\prod_{r=0}^{n-1} f(x_0 + \omega^r x_1 + \ldots + \omega^{(n-1)r} x_{n-1})$$

$$= \prod_{r=0}^{n-1} \left\{ f(x_0) + \omega^r f(x_1) + \ldots + \omega^{(n-1)r} f(x_{n-1}) \right\},$$

where $x_0, x_1, \ldots, x_{n-1}$ are arbitrary complex numbers, then $f(z) = az$, where a is some complex constant.

L. Carlitz, Duke University

Solution by N. Kimura, University of Saskatchewan.

For convenience of notation, define

$$(3) \qquad s(r) = 1 + \omega^r + \omega^{2r} + \ldots + \omega^{(n-1)r}, \quad r = 0, 1, 2, \ldots, n-1$$

Then $s(r) = \begin{cases} n & \text{if } r = 0 \\ 0 & \text{otherwise} . \end{cases}$

Letting $x_0 = x_1 = \ldots = x_{n-1}$ in (1), we have

$$(f(0))^n = \prod_{r=0}^{n-1} (s(r)\, f(0)) = 0 ,$$

because one factor contains $s(1)$ which is zero by (3). Thus

$$(4) \qquad c_0 = f(0) = 0 .$$

Let $x_0 = x + y$, $x_1 = x_2 = \ldots = x_{n-1} = x$ in (1), where x and y are any complex numbers in the circle $|z| < A/(n+1)$ so that (1) has meaning. Then we have

$$\prod_{r=0}^{n-1} f(s(r)x + y) = \prod_{r=0}^{n-1} \left\{ f(x+y) - f(x) + s(r)f(x) \right\}, \quad \text{or}$$

$$(f(nx+y)f(y))^{n-1} = (f(x+y) - f(x))^{n-1}(f(x+y) - f(x) + nf(x)) .$$

Dividing both members by y^{n-1} (excepting the case $y = 0$), and letting $y \to 0$, we have

(5) $a^{n-1} f(nx) = nf(x) (f'(x))^{n-1}$, for $|x| < A/(n+1)$, where

(6) $a = c_1 = f'(0)$.

If $a = 0$, then $f(x)(f'(x))^{n-1} = 0$, and therefore

$$(d/dx)(f(x)^2)^{n-1} = 0 ,$$

which implies that $f(x)$ itself must be a constant such that $f(0) = 0$ by (4). Thus $f(z) = 0 = az$ for $a = 0$.

We assume that $a \neq 0$, in what follows.

It is easy to check that $f(z) = az$ satisfies the condition (1). We shall prove that there is no other solution of (1). To this end, let b be the second non-zero coefficient of the Taylor expansion of $f(z)$, in other words,

(7) $b = c_p \neq 0$, and $c_j = 0$ for $1 < j < p$; i.e.,

(8) $f(z) = az + bz^p + \ldots$

Then for $|x| < A/(n+1)$, $f(nx) = anx + bn^p x^p + \ldots$, or

(9) $a^{n-1} f(nx) = a^n nx + a^{n-1} bn^p x^p + \ldots$;

and $(f'(x))^{n-1} = (a + bpx^{p-1} + \ldots)^{n-1}$

$= a^{n-1} + a^{n-2} bp(n-1)x^{p-1} + \ldots$, or

(10) $nf(x)\{f'(x)\}^{n-1} = a^n nx + a^{n-1} bn(pn - p + 1)x^p + \ldots$

Now from (5), (9) and (10) above, we have

(11) $a^{n-1} bn^p = ba^{n-1} n(pn - p + 1)$.

Since $a \neq 0$, $b \neq 0$ and $n \neq 0$,

(12) $n^{p-1} = p(n-1) + 1$.

But $n^{p-1} = (1 + (n - 1))^{p-1}$

$$= 1 + (p - 1)(n - 1) + \frac{1}{2}(p - 1)(p - 2)(n - 1)^2 + \ldots$$

Therefore from (12)

$$\frac{1}{2}(p - 1)(p - 2)(n - 1)^2 < n - 1 ,$$

or

$$(p - 1)(p - 2)(n - 1) \leq 2.$$

From this, since $n > 2$ or $n - 1 \geq 2$, the only possibility is that $p = 2$ or $p = 3$. But $p = 2$ implies from (12) that $n = 1$ which contradicts the assumption that $n > 2$. Similarly, $p = 3$ implies from (12) that either $n = 1$ or $n = 2$. Thus there exists no such b satisfying (7), in other words, there is no solution of (12). This proves that there is no solution of (5) other than $f(z) = az$.

Also solved by the proposer.

Comment by the proposer. Rosenbaum and Segal, Math. Gaz., 44, (1960), 97-105, have shown that in the case $n = 2$ the only regular solutions of (1) are $f(z) = az$ and $f(z) = A \sinh cz$ where A, c are complex constants. The case $n = 3$ is treated in a paper of the proposer, Amer. Math. Monthly, 68, (1961), 753-756.

P 53. For real α, β, γ, δ and $(\alpha x + \beta y)(\gamma x + \delta y) = ax^2 + bxy + cy^2$ prove that $\max(a, b, c) \geq 4/9\ (\alpha + \beta)(\gamma + \delta)$.

L. Moser and J. R. Pounder,
University of Alberta

Solution by W. J. Blundon, Memorial University of Newfoundland. The problem is equivalent to proving for real, a, b, c the inconsistency of the inequalities

$$5a < 4b + 4c$$

$$5b < 4c + 4a$$

$$5c < 4a + 4b$$

$$4ac \leq b^2 .$$

Case I (b = 0). Addition of the first and third inequalities gives $a + c < 0$, whereas the second gives $a + c > 0$. Note that the fourth inequality is not used.

Case II (b > 0). Substitution of $a = \frac{1}{2} bX$, $c = \frac{1}{2} bY$ gives

$$-5X + 4Y + 8 > 0$$

$$2X + 2Y - 5 > 0$$

$$4X - 5Y + 8 > 0$$

$$XY \leq 1 .$$

The first three inequalities of this set define the interior of the triangle with vertices at (1/2, 2), (2, 1/2), (8, 8). It is clear that this triangle lies outside the region defined by $XY \leq 1$. The inequalities are therefore inconsistent. Further, if > is replaced by $\geq$, equality will occur only for $(X, Y) = (1/2, 2)$ or $(2, 1/2)$. Thus equality in the original relations occurs only when $b = c = 4a$ or $b = a = 4c$.

Case III (b < 0). The inequalities are now those of Case II with the sense reversed in the first three inequalities. It follows from the properties of convex polygonal sets that the set of points defined by the transformed inequalities is now empty. Thus the inequalities are again inconsistent. Note that, as in Case I, the fourth inequality is not used.

Also solved by A. Feldmar, W. Moser and the proposers.

P 55. Let P be a regular polygon and S a concentric sphere. Prove that the sum of the squares of the distances from a variable point of S to the vertices of P is a constant.

L. Moser, University of Alberta

Solution by G. D. Chakerian and J. D. Dixon, California Institute of Technology. More generally, let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be any points of n-space whose centroid $\Sigma\vec{v}_i = 0$. Let S be a sphere centred at 0. Then for any point $\vec{x}$ of S, the sum of the squares of the k distances from $\vec{x}$ to the $\vec{v}_i$ is

$$\Sigma|\vec{x} - \vec{v}_i|^2 = k|\vec{x}|^2 + \Sigma|\vec{v}_i|^2 - 2\vec{x}\cdot\Sigma\vec{v}_i$$

$$= k|\vec{x}|^2 + \Sigma|\vec{v}_i|^2 ,$$

which is a constant.

Also solved by N. Kimura, A. Makowski, C. J. Scriba, L. P. Wood, and the proposer.

P 56. If $x \neq 0$ prove that

$$y + y^2 = x + x^2 + x^3$$

has no solutions in integers.

W. J. Blundon, Memorial University of Newfoundland

Solution by the proposer. Let $x = ab$ and $y - x = ac$, where $(b, c) = 1$. Then $c(ac + 2ab + 1) = a^2b^3$. Now $(b^3, c) = 1$ and $(ac + 2ab + 1, a^2) = 1$. Therefore $c = a^2$ and $b^3 = ac + 2ab + 1 = a^3 + 2x + 1$. Writing the last equation in the form

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$$(a - b)^3 + 3(a - b)x + 2x + 1 = 0 ,$$

and putting $3(a - b) + 2 = z$, we have, on elimination of $a - b$,

$$27x = - 19/z - 12 + 6z - z^2 .$$

Since z is an integer, there are four possibilities.

(1) $z = 1$, whence $27x = -26$, which is impossible.

(2) $z = 19$, whence $27x = - 260$, which is impossible.

(3) $z = - 19$, whence $x = - 18$. This gives $ab = - 18$, $a - b = - 7$, so that $(a + b)^2 = - 23$ which is impossible.

(4) $z = - 1$, whence $x = 0$, which is contrary to hypothesis.

Also solved by L. Carlitz, J. D. Dixon, N. Kimura, A. Makowski, and E. Rosenthall.

Remark by A. Makowski. A proof for $x > 0$ was given by me and A. Schinzel, Mathesis 68, (1959), 128-142. The equation is also treated by O. Gross, Math. Mag. 34 (1961), 259-267.

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