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On definable groups and D-groups in certain fields with a generic derivation

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Abstract. We continue our study from Peterzil et al. (2022, *Preprint*, arXiv:2208.08293) of finite-dimensional definable groups in models of the theory T_∂, the model companion of an o-minimal $\mathcal{L}$-theory T expanded by a generic derivation ∂ as in Fornasiero and Kaplan (2021, *Journal of Mathematical Logic* 21, 2150007).

We generalize Buium's notion of an algebraic D-group to $\mathcal{L}$-definable D-groups, namely (G, s), where G is an $\mathcal{L}$-definable group in a model of T, and $s : G \to \tau(G)$ is an $\mathcal{L}$-definable group section. Our main theorem says that every definable group of finite dimension in a model of T_∂ is definably isomorphic to a group of the form

$$(G, s)^\partial = \{g \in G : s(g) = \nabla g\},$$

for some $\mathcal{L}$-definable D-group (G, s) (where $\nabla(g) = (g, \partial g)$).

We obtain analogous results when T is either the theory of p-adically closed fields or the theory of pseudo-finite fields of characteristic 0.

1 Introduction

1.1 Background and motivation

Let us begin with some motivation and background for the general reader.

The notion of an algebraic D-group is due in full generality to Buium [4] and belongs entirely to algebraic geometry. It can be described as follows: let (K, ∂) be a field of characteristic 0 equipped with a derivation $\partial : K \to K$. Then a (connected) algebraic D-group over (K, ∂) is a (connected) algebraic group G over K, together with a lifting of ∂ to a derivation $\partial' : O_K(G) \to O_K(G)$ of the structure sheaf $O_K(G)$ of K, commuting with (or respecting) co-multiplication. When G is affine, we can replace the structure sheaf by the coordinate ring of G. There are other equivalent descriptions (as given later). One can think of an algebraic D-group as an algebraic group over a differential field, which is equipped with a certain kind of order-one differential equation. When the base field is $(\mathbb{C}(t), d/dt)$, a geometric description of an algebraic D-group is a family $\mathcal{G} \to S$ of complex algebraic groups over the affine line (with finitely many points removed), together with a suitable "Ehresmann

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connection," namely lift of the vector field on S corresponding to d/dt to a vector field on the total space $\mathcal{G}$, respecting multiplication in the obvious sense. One of the main points of [4] is to show that the category of algebraic D-groups over K is the "same" as the category of finite-dimensional differential algebraic groups over K in the sense of Kolchin [12]. An algebraic D-group, (G, s) over (K, ∂) is associated with the solution set of the corresponding differential equation in a "universal" differential field (L, ∂) extending K (which will be a subgroup of $G(L)$). Moreover, Buium [4] has an exhaustive study of these finite-dimensional differential algebraic groups, yielding an account of (cases of) function field Mordell–Lang in characteristic 0. This was built on by Hrushovski [8] using model-theoretic methods and generalized to positive characteristic.

The connection with model theory is that finite-dimensional differential groups, in the sense of Kolchin above, are precisely the finite-dimensional groups *definable* in differentially closed fields $(K, +, \cdot, \partial)$, and various nontrivial model-theoretic results come into play (see [2]). Moreover, algebraic groups over algebraically closed fields K are precisely *definable* groups in algebraically closed fields $(K, +, \cdot)$ (see [22]).

The purpose of this paper is to generalize the relationships between algebraic groups, algebraic D-groups, and finite-dimensional differential algebraic groups, to other categories, sometimes provided by model theory. In one of these categories, Nash groups replace algebraic groups. The category of Nash groups lies properly in between the categories of real algebraic groups and (real) Lie groups. The model-theoretic connection is that these are precisely the groups definable in real closed fields. Likewise p-adic Nash groups, also treated in this paper are those definable in p-adically closed fields. So, among other things, we introduce the notion of a Nash D-group over a real closed differential field, likewise for p-adic analogues, and relate them to finite-dimensional definable groups in real closed (p-adically closed) fields equipped with a "generic" derivation. We also consider the case of pseudo-finite fields K, where definable groups are, up to a quotient by a finite normal subgroup, a finite index subgroup of $G(K)$, for G an algebraic group over K (see [9]).

From this point on, the paper is somewhat more technical, and assumes some knowledge of basic model theory. We will be repeating in different formalisms, the notions discussed in the last paragraphs.

1.2 The setting and main result

In [15], we initiated a study of definable groups in closed ordered differential fields (see [20]), and more generally in differential expansions of o-minimal structures, p-adically closed fields, pseudo-finite fields of characteristic 0, or topological fields which are models of an open theory (as in [13]).

In all of the above settings, we start with a suitable theory T in a language $\mathcal{L}$, where T expands the theory of fields. We add a symbol ∂ to the language to get $\mathcal{L}_\partial = \mathcal{L} \cup \{\partial\}$. The $\mathcal{L}_\partial$-theory $T \cup$ "∂ is a (compatible) derivation" will have a model companion which we call T_∂.

The main theorems in [15] said that in all of these cases, if Γ is a finite-dimensional group in a model of T_∂, then there is an $\mathcal{L}$-definable group G and an $\mathcal{L}_\partial$-definable group embedding of Γ into G.

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Here, we mostly follow the setting suggested by Fornasiero and Kaplan [6], where we start with an $\mathcal{L}$-theory T of an o-minimal expansion of a real closed field K, expand it in the language $\mathcal{L}_\partial$ to the theory T^* of a T-compatible derivation ∂, and let T_∂ be the model companion of T^*.

In [4], Buium introduced the notion of an algebraic D-group, namely a pair (G, s), where G is an algebraic group and $s : G \to \tau(G)$ a rational group section into the prolongation of G. In the setting of DCF_0 (differentially closed fields of characteristic zero), it was shown (see [18, Corollary 4.2] and [4]) that every finite-dimensional definable group is definably isomorphic to

$$(G, s)^\partial = \{g \in G : s(g) = \nabla(g)\},$$

$(\nabla(g) = (g, \partial g))$.

Our goal here is to obtain analogous tools and theorems in the setting of T_∂. We first associate to every $\mathcal{L}$-definable C^1-manifold V, with respect to K, its prolongation, the bundle $\tau(V)$. We then note, as in the algebraic case, that when G is an $\mathcal{L}$-definable group over a differentially closed subfield of K, then so is $\tau(G)$, and the projection $\pi : \tau(G) \to G$ a group homomorphism. An $\mathcal{L}$-definable D-group is then a pair (G, s) with G an $\mathcal{L}$-definable group and $s : G \to \tau(G)$ an $\mathcal{L}$-definable group section. Our main theorem (see Theorem 4.6) is the following.

Theorem *Let Γ be a finite-dimensional $\mathcal{L}_\partial$-definable group in a model of T_∂. Then there exists an $\mathcal{L}$-definable D-group (G, s) such that Γ is definably isomorphic to*

$$(G, s)^\partial = \{g \in G : s(g) = \nabla(g)\}.$$

When T is a model complete theory of large fields in the language of fields (plus maybe constants), Tressl [21] shows that the theory of models of T equipped with a derivation has a model companion. He also gave a uniform (in T) axiomatization of the model companion. Here, we treat two special cases: the case of p-adically closed fields and of pseudo-finite fields. We develop the notions of $\tau(G)$ and (G, s), for an $\mathcal{L}$-definable group G and prove the exact analogue of the above theorem for T_∂-definable groups (see Theorem 4.12). Along the way, we prove a p-adic analogue of an o-minimal theorem of Fornasiero and Kaplan (see Proposition A.1 in the Appendix).

When K is a pseudo-finite field, we prove that every $\mathcal{L}_\partial$-definable group Γ is virtually isogenous to $H_0 \cap (H, s)^\partial$, where (H, s) is an algebraic D-group over K and H_0 a finite index subgroup of H (see Theorem 4.13).

Remark 1.1 The case of an arbitrary (not necessarily finite dimensional) $\mathcal{L}_\partial$-definable group will be treated in a subsequent paper jointly with Silvain Rideau-Kikuchi.

1.3 Preliminaries

We refer to Section 2 of [15] for all conventions and basic notions. Briefly, we always work in a sufficiently saturated structure and use the fact that o-minimal structures (and later, p-adically closed fields) are geometric structures in the sense of [9], to define $\dim_{\mathcal{L}}(a/k)$ as the $acl_{\mathcal{L}}$-dimension of a over k. The dimension of an $\mathcal{L}$-definable set $X \subseteq K^n$ is defined as the maximal $\dim_{\mathcal{L}}(a/k)$, for $a \in X$ (or equivalently via cell

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decomposition). If we have $\dim(a/B) = \dim X$, for $a \in X$ an $\mathcal{L}$-definable set over B (written also as $\mathcal{L}(B)$-definable), then we say that *a is generic in X over B*. A definable $Y \subseteq X$ is said to be *large in X* if $\dim(X \setminus Y) < \dim X$ (equivalently, Y contains every generic element of X over the parameters defining X, Y).

For a tuple $a = (a_1, \ldots, a_n)$, we let $\partial a = (\partial a_1, \ldots, \partial a_n)$. To define the $\mathcal{L}_\partial$-dimension, for $a \in K^n$ and $k \subseteq K$ a differential subfield, we let $\dim_\partial(a/k) = \dim_{\mathcal{L}}(a, \partial a, \ldots, \partial^n a, \ldots /k)$ (possibly infinite). The $\mathcal{L}_\partial$-dimension of an $\mathcal{L}_\partial$-definable set $X \subseteq K^n$ over k is the maximum $\dim_\partial(a/k)$, as a varies in X.

2 Manifolds, tangent spaces, and tangent bundles

We fix an o-minimal expansion of a real closed field K in a language $\mathcal{L}$. All definability in this section is in the o-minimal structure, allowing parameters.

We first recall the basic definition of a differentiable manifold and its tangent bundle in the o-minimal setting (for differentiability in this context, see [23, Section 7]).

Notation Let $U \subseteq K^r \times K^n$ be an open definable set, and $f : U \to K^m$ a definable C^1-map, written as $f(x, y)$, $f = (f_1, \ldots, f_m)$. Given $(a, b) \in U$, we let $(D_x f)_{(a,b)} : K^r \to K^m$, and $(D_y f)_{(a,b)} : K^n \to K^m$ denote the corresponding K-linear maps defined as follows: $(D_x f)_{(a,b)}$ is the $m \times r$ matrix of partial derivatives

$$\left(\frac{\partial f_i}{\partial x_j}(a,b)\right)_{1 \le i \le m, 1 \le j \le r},$$

and $(D_y f)_{(a,b)}$ is the $m \times n$ matrix

$$\left(\frac{\partial f_i}{\partial y_t}(a,b)\right)_{1 \le i \le m, 1 \le t \le n}.$$

Then, $(Df)_{(a,b)}$ is the $m \times (r+n)$-matrix

$$\left((D_x f)_{(a,b)}, (D_y f)_{(a,b)}\right).$$

For a C^1 map $f : V \to W$ between open subsets of K^n and K^m, respectively, we write $Df : V \times K^n \to W \times K^m$, for the map

$$(a, u) \mapsto (f(a), (Df)_a \cdot u),$$

where $(Df)_a \cdot u = (\sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(a) u_i)_{j=1}^m$.

2.1 Definable manifolds and their tangent bundles

Definition 2.1 *An $\mathcal{L}$-definable C^1manifold of dimension r, with respect to K*, is a topological Hausdorff space M, together with a finite open cover $M = \bigcup_{i=1}^n W_i$, and homeomorphisms $\phi_i : V_i \to W_i$, where $V_i \subseteq K^r$ is a definable open set, such that $V_{i,j} = \phi_i^{-1}(W_i \cap W_j)$ is a definable open subset of V_i, and each map $\phi_{i.j} = \phi_j^{-1} \circ \phi_i : V_{i,j} \to V_{j,i}$ is a definable C^1-map (between definable open subsets of K^r).

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The collection $\{(V_i, W_i, \phi_i)_{i\in I}\}$ is *an atlas* for M. We identify the manifold with the quotient of the disjoint union $\bigsqcup_i V_i$ by the equivalence relation $a \sim_M b \Leftrightarrow b = \phi_{i,j}(a)$. We say that the manifold *is defined over* A if the sets and functions in the atlas are all defined over A.

If we omit the C^1 requirement from the transition maps, then the manifold is called *a definable topological (or C^0) manifold.*

If M, N are $\mathcal{L}$-definable C^1(or C^0)-manifolds, then so is $M \times N$, with the natural atlas.

Definition 2.2 For M an $\mathcal{L}$-definable C^1-manifold of dimension r given as above, we let the *tangent space of* M, $T(M)$ be the quotient of $\bigsqcup_{i=1}^n V_i \times K^r$ by the equivalence relation, denoted by $\sim_{T(M)}$, given via the maps:

$$D\phi_{i,j} : V_{i,j} \times K^r \to V_{j,i} \times K^r \ ; \ D\phi_{i,j}(c,u) = (\phi_{i,j}(c), D(\phi_{i,j})_c \cdot u).$$

We then write

$$T(M) = \bigsqcup_i V_i \times K^n \ / \sim_{T(M)},$$

and denote (equivalence classes of) elements in $T(M)$ by $[a,u]$, $a \in \bigsqcup_i V_i$, $u \in K^r$.

Note that if $M = U \subseteq K^r$ is a definable open set with the identity atlas, then $T(M) = U \times K^r$.

The following are easy to verify.

Lemma 2.3 *Assume that M and N are $\mathcal{L}$-definable, C^1-manifolds, given by atlases $(W_i, V_i, \phi_i)_{i\in I}$ and $(U_j, Z_j, \psi_j)_{j\in J}$. If $f : M \to N$ is a C^1-map (read through the charts), then there is a well-defined continuous map $Df : T(M) \to T(N)$ satisfying, whenever the elements are in the appropriate V_i and U_j,*

$$Dh([a,u]) = [(f(a), D(\psi_j^{-1} \circ f \circ \phi_i)_a \cdot u)].$$

Proof In fact, the map $(a,u) \mapsto (f(a), D(\psi_j^{-1} \circ f \circ \phi_i)_a \cdot u)$ induces a well-defined map from $T(M)$ into $\bigsqcup_{j\in J} U_j \times K^{\dim N}$. The quotient by $\sim_{T(N)}$ gives the desired map. ∎

Lemma 2.4 (1) *For M, N $\mathcal{L}$-definable, C^1-manifolds, $T(M \times N) = T(M) \times T(N)$.*
(2) *(Chain rule) For $f : M \to N$ and $h : N \to S$ two $\mathcal{L}$-definable C^1-maps between $\mathcal{L}$-definable C^1 manifolds, $D(h \circ f) = D(h) \circ D(f)$.*

Summarizing, we have the following lemma.

Lemma 2.5 *(T, D) is a functor from the category of definable C^1-manifolds to the category of definable C^0-manifolds. It, moreover, preserves products.*

3 Adding a derivation

Let T be a complete, model complete theory of an o-minimal expansion of a real closed field K, in a language $\mathcal{L}$. The following definition is due to Fornasiero and Kaplan [6].

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Definition 3.1 A derivation $\partial : K \to K$ is called *T-compatible* if for every $\emptyset$-definable C^1 map $f : U \to K$, for $U \subseteq K^m$ open, for all $a \in U$, we have

$$\partial f(a) = Df_a \cdot \partial a.$$

(Here, $\partial(a_1, \ldots, a_m) = (\partial a_1, \ldots, \partial a_m)^t$.)

Note that if $a \in dcl(\emptyset)$ and ∂ is T-compatible, then $\partial(a) = 0$.

Fornasiero and Kaplan [6] show that the $\mathcal{L}_\partial$ theory $T\cup$ "∂ is a compatible derivation" has a model companion, which we call T_∂. We assume from now on that ∂ is a T-compatible derivation on K, and work in models of T_∂. See [6, Proposition 2.8 and Lemma 2.9] for instances where the compatibility condition holds.

We observe the following claim.

Claim 3.2 *Assume that $M = \bigsqcup_i V_i / \sim_M$ is a $\emptyset$-definable manifold. Then, for $a \in M$, ∂a is a well-defined element of $T(M)_a$. Namely, if $a_i \sim_M a_j$, then $(a_i, \partial a_i) \sim_{T(M)} (a_j, \partial a_j)$.*

Proof This is easy to verify, using the compatibility of ∂. ∎

3.1 The definition of f^∂ on an open set

The following theorem of Fornasiero and Kaplan, which follows easily from their [6, Lemma A.3], plays an important role here: in the Appendix, we prove the analogous result, Proposition A.1, for p-adically closed fields, and the proof could be modified to give an alternative proof in the o-minimal setting as well.

Fact 3.3 *Assume that $g : W \to K^r$ is an $\mathcal{L}(\emptyset)$-definable partial function on some open $W \subseteq K^n \times K^m$, and $b \in \pi_2(W) \subseteq K^m$ is $dcl_{\mathcal{L}}$-independent. If $g(x, b)$ is a C^1-map on $W^b = \{a \in K^n : (a, b) \in W\}$, then for every $a \in W^b$, the function g is a C^1-function (of all variables) at (a, b).*

As a corollary, one obtains the following.

Fact 3.4 *If $f(x)$ is an $\mathcal{L}(A)$-definable C^1-function on an open subset of K^n, then there is a $dcl_{\mathcal{L}(\emptyset)}$-independent tuple $b \subseteq A$, and an $\mathcal{L}(\emptyset)$-definable C^1-function $g(x, y)$ on an open subset of $K^n \times K^{|b|}$ such that $f(x) = g(x, b)$.*

Definition 3.5 For $U \subseteq K^n$ open and $f : U \to K^r$ an $\mathcal{L}$-definable C^1-map (possibly over additional parameters), let

$$f^\partial(a) = \partial f(a) - (Df)_a \partial a.$$

Notice that if f is $\emptyset$-definable, then $f^\partial(a) = 0$. For the following, see also [6, Lemma 2.12].

Lemma 3.6 *If $f : U \to K^r$ is an $\mathcal{L}$-definable C^1 map, over a differential field k, then f^∂ is $\mathcal{L}$-definable over k, and continuous on U.*

Proof By Fact 3.4, we may write $f(x) = g(x, b)$, for $b \in K^m$ which is $\mathcal{L}(\emptyset)$-independent, and g which is a C^1 map, $\mathcal{L}(\emptyset)$-definable. By the compatibility of ∂,

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$$\partial f(a) = \partial g(a,b) = (Dg)_{(a,b)}(\partial a, \partial b) =$$

$$= (D_x g)_{(a,b)}\partial a + (D_y g)_{(a,b)}\partial b = (Df)_a \partial a + (D_y g)_{(a,b)}\partial b.$$

It follows that $f^\partial(a) = \partial f(a) - (Df)_a \partial a = (D_y g)_{(a,b)}\partial b$, and since $b \in k$, then so is ∂b. Also, because g is a C^1-function, f^∂ is continuous. ∎

Remark 3.7 When $p(x) = \sum_m a_m x^m$ is a polynomial over k, then $p^\partial(x)$ is a polynomial over k of the same degree:

$$p^\partial(x) = \sum_m \partial a_m x^m.$$

For $a \in K^n$, we let $\nabla(a) = (a, \partial a)$, and for $r \in \mathbb{N}$, $\nabla^r(a) = (a, \partial a, \dots, \partial^r a)$. We also need the following.

Lemma 3.8 *Assume that $k \subseteq K$ is a differential field, $a \in K^m, c \in K^n$, and $c \in dcl_{\mathcal{L}}(k,a)$. Then $\nabla(c) \in dcl_{\mathcal{L}}(k, \nabla(a))$. If, in addition, c and a are $\mathcal{L}$-interdefinable over k, then $\nabla(a)$ and $\nabla(c)$ are $\mathcal{L}$-interdefinable over k.*

Proof Assume first that a is $\mathcal{L}$-generic in K^m over k. Then, $c = f(a)$ for f an $\mathcal{L}$-definable over k and C^1 at a. We have $\partial f(a) = (Df)_a \partial a + f^\partial(a)$, where, by Lemma 3.6, $f^\partial(x)$ is $\mathcal{L}$-definable over k. So, if we let $h(x,u) = (f(x), (Df)_x u + f^\partial(x))$, then $h(\nabla(a)) = \nabla(c)$, so $\nabla(c) \in dcl_{\mathcal{L}}(k,a)$.

Given a general $a \in K^m$, we can write it, up to permutation of coordinates, as (a_1, a_2) where $a_1 \in K^{m_1}$ is $\mathcal{L}$-generic over k and $a_2 \in dcl_{\mathcal{L}}(a_1)$. Then $c \in dcl_{\mathcal{L}}(k, a_1)$, so by what we saw, $\nabla(c) \in dcl_{\mathcal{L}}(k, \nabla(a_1)) \subseteq dcl_{\mathcal{L}}(k, \nabla(a))$.

Finally, it clearly follows that if a and c are $\mathcal{L}$-interdefinable over k, then so are $\nabla(a)$ and $\nabla(c)$. ∎

3.2 Prolongation of functions on open sets

Here and below, we make use of Marker's account [14] of prolongations in the algebraic setting.

Definition 3.9 For $U \subseteq K^r$ open and $f : U \to K^n$ an $\mathcal{L}$-definable C^1-map, we let $\tau(f) : U \times K^r \to K^n \times K^n$ be defined as

$$\tau(f)(a,u) = (f(a), (Df)_a \cdot u + f^\partial(a)) = (f(a), (Df)_a \cdot (u - \partial a) + \partial f(a)).$$

Using Lemma 3.6 (the $\mathcal{L}$-definability of f^∂) and the $\mathcal{L}$-definability of Df, we have the following lemma.

Lemma 3.10 *If f is a C^1-map, $\mathcal{L}$-definable over a differential field k, then $\tau(f)$ is continuous and $\mathcal{L}$-definable over the same k.*

Using the second equality in the definition of $\tau(f)$ and the chain rule for D, we immediately obtain the following lemma.

Lemma 3.11 *If $f : U \to V$ and $h : V \to W$ are definable C^1-functions on open sets, then*

$$\tau(h \circ f) = \tau(h) \circ \tau(f).$$

3.3 The definition of $\tau(M)$ and $\tau(f)$ for definable manifolds

Definition 3.12 Assume that $M = \bigsqcup_i V_i / \sim_M$ is an $\mathcal{L}$-definable C^1-manifold of dimension n. Then the prolongation of M is defined as

$$\tau(M) := \bigsqcup_i V_i \times K^n / \sim_{\tau(M)},$$

where $(a_i, u) \sim_{\tau(M)} (a_j, v)$ if $\tau(\phi_{i,j})(a_i, u) = (a_j, v)$.

By Lemma 3.10, $\tau(M)$ is an $\mathcal{L}$-definable C^0-manifold.

The following is easy to verify.

Lemma 3.13 *Assume that $M = \bigsqcup_i V_i / \sim_M$ is an $\mathcal{L}$-definable C^1-manifold. Then*

$$(a_i, u) \sim_{T(M)} (a_j, v) \Leftrightarrow (a_i, u + \partial a_i) \sim_{\tau(M)} (a_j, v + \partial a_j).$$

In particular, the map

$$\sigma_M : (a, u) \mapsto (a, u + \partial a)$$

*induces a well-defined $\mathcal{L}_\partial$-definable bijection **over** M, between $T(M)$ and $\tau(M)$.*

Using the above lemma, we see that for $a \in M$, the element $(a, \partial a) \in \tau(M)$ is well defined (e.g., as $\sigma_M(a, 0)$). We thus have a well-defined map $\nabla : M \to \tau(M)$, given in coordinates by $\nabla_M(a) = (a, \partial a)$.

Definition 3.14 Assume that M and N are $\mathcal{L}$-definable C^1-manifolds, $f : M \to N$ an $\mathcal{L}$-definable C^1 map. Then *the prolongation of f*, $\tau(f) : \tau(M) \to \tau(N)$, is defined by

$$\tau(f) := \sigma_N \circ Df \circ \sigma_M^{-1}.$$

The following is easy to verify.

Lemma 3.15 *Assume that M and N as above are given via the atlases $\{(V_i, W_i, \phi_i)_{i \in I}\}$ and $\{(U_j, Z_j, \psi_j)\}$, respectively, with $\dim M = r$ and $\dim N = n$. If $f : M \to N$ is an $\mathcal{L}$-definable C^1-map, then, for $(a, u) \in V_i \times K^r$, we have*

$$\tau(f)([a, u]) = [\tau(\psi_j^{-1} \circ f \circ \phi_i)(a, u)].$$

Lemma 3.16 *Let M, N be $\mathcal{L}$-definable C^1-manifolds defined over a differential field k.*

(1) *If $f : M \to N$ is $\mathcal{L}$-definable over k, then so is $\tau(f) : \tau(M) \to \tau(N)$, and $\tau(f)$ is continuous.*
(2) *If $f : M \to N$ and $h : N \to S$ are $\mathcal{L}$-definable C^1 maps between $\mathcal{L}$-definable C^1-manifolds, then $\tau(h \circ f) = \tau(h) \circ \tau(f)$.*
(3) *We have $\tau(M \times N) = \tau(M) \times \tau(N)$. Moreover, if $\pi_1 : M \times N \to N$ and $\pi_1 : \tau(M) \times \tau(N) \to \tau(M)$ are the projection maps on the first coordinates, then $\tau(\pi_1) = \pi_1 \circ \tau$.*
(4) *We have $\nabla_N \circ f = \tau(f) \circ \nabla_M$.*

Proof (1) By Lemma 3.15, the result reduces to the $\mathcal{L}$-definability of each $\tau(\psi_j^{-1} \circ f \circ \phi_i)$, and therefore follows from Lemma 3.6. (2) follows from Lemma 3.11. (3) and (4) are easy to verify. ∎

As a corollary, we have the following lemma.

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Lemma 3.17 *τ is a functor from the category of definable C^1-manifolds to definable C^0 manifolds, which moreover preserves products.*

4 $\mathcal{L}_\partial$-definable groups

4.1 Prolongation of $\mathcal{L}$-definable groups, D-groups, and Nash D-groups

Let G be an $\mathcal{L}$-definable group of dimension m. By [16], it admits the structure of an $\mathcal{L}$-definable C^0-manifold. Since we shall be using the particular construction, in its C^1-version, we repeat the details below for future use (see [16, Lemmas 2.4 and Proposition 2.5]).

Fact 4.1 *There exist a group topology t on G, a large $\mathcal{L}$-definable set $W \subseteq G$, a definable open $V \subseteq K^m$, and an $\mathcal{L}$-definable homeomorphism σ from V (with the K^m-topology) and W (with the t-topology), and there are $g_1, \ldots, g_k \in G$, such that:*

(i) $G = \bigcup_j g_j W$.

(ii) The maps $\phi_i : V \to g_i W : x \mapsto g_i \sigma(x)$ endow (G, t) with a definable C^1-atlas.

(iii) The group G is a C^1-group with respect to this atlas, namely the group operations of G are C^1 maps.

To be precise, [16, Proposition 2.5] states the result in the C^0 category. However, as is commented in [16, Remark 2.6], if the underlying o-minimal structure is the field $(\mathbb{R}, <, +, \cdot)$, then one can obtain an analytic atlas for G, making it an analytic group. If we work, as we do here, in an o-minimal structure over an arbitrary real closed field, where definable functions are piecewise C^1 [23, Theorem 6.3.2], then exactly the same proof would yield the above C^1-atlas, making G a C^1-group. Moreover, every other definable C^1-atlas on G which makes it into a C^1 group yields the same C^1-structure, namely the identity map is a diffeomorphism of the two (this follows from the fact that definable functions are generically C^1). In addition, if G was definable over A, then since the notion of a C^1-atlas is first-order, one can obtain a corresponding C^1-atlas for G which is also defined over A.

Remark 4.2 The result above, from [16], has already been used in several other settings (e.g. [17]). As we shall be using it again in the p-adic setting, we point out that Fact 4.1 holds (with the exact same proof) under the following assumptions: the group $G \subseteq K^n$, is definable is an $\mathcal{L}$-expansion of a topological field K (namely, a field with a definable basis for a Hausdorff, non-discrete, field topology), which is a geometric structure, and in addition admits a C^1-cell decomposition.

From now on, one we endow every $\mathcal{L}$-definable group G with its canonical C^1-structure.

By purely categorical reasons, using Lemmas 2.5 and 3.17, we have (see [14, Section 2] for the same construction in algebraic groups) the following lemma.

Lemma 4.3 *Let G be a definable group, endowed with its canonical C^1-structure, and let $m : G \times G \to G$ be the group product. Then*

$$\langle T(G); Dm\rangle \quad \textit{and} \; \langle \tau(G); \tau(m)\rangle$$

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are $\mathcal{L}$-definable C^0-groups (namely, topological groups with respect to an $\mathcal{L}$-definable C^0-atlas)), and the function $[a,u] \mapsto a$ is in both cases an $\mathcal{L}$-definable group homomorphism from $T(G)$ and $\tau(G)$ onto G.

The map $a \mapsto [a,0] : G \to T(G)$ is an $\mathcal{L}$-definable group section and $\nabla_G : G \to \tau(G)$ is an $\mathcal{L}_\partial$-definable group section.

Definition 4.4 Assume that G is an $\mathcal{L}$-definable group, and $s : G \to \tau(G)$ is an $\mathcal{L}$-definable group section. Then the pair (G,s) is called *an $\mathcal{L}$-definable D-group*.

Remark 4.5 When $T = RCF$ is the theory of real closed fields, every definable group admits the structure of a Nash group with respect to K. Namely, the underlying manifold and group operations are semialgebraic over K and either real analytic, when $K = \mathbb{R}$, or C^∞ in general (see discussion in [10], based on [1]). In this case, every definable homomorphism between such groups is a Nash map; thus, $\pi : T(G) \to G$ and $\pi : \tau(G) \to G$ are Nash maps, and an $\mathcal{L}$-definable section $s : G \to \tau(G)$ is a Nash map. We call a D-group (G,s) in this case *a Nash D-group*.

Our goal is to prove the following theorem.

Theorem 4.6 *Let T_∂ be the model companion of a complete, model complete, o-minimal theory T, with a T-compatible derivation ∂. Assume that Γ is an $\mathcal{L}_\partial$-definable group of finite $\mathcal{L}_\partial$-dimension. Then there exists an $\mathcal{L}$-definable D-group (G,s) and an $\mathcal{L}_\partial$-definable group embedding $\Gamma \to G$ whose image is*

$$(G,s)^\partial = \{g \in G : s(g) = \nabla_G(g)\}.$$

We first recall our result [15, Theorem 6.8]. We shall be using the following version.

Theorem 4.7 *If Γ is a finite-dimensional $\mathcal{L}_\partial$-defined group in a model of T_∂, then it can be $\mathcal{L}_\partial$-definably embedded in an $\mathcal{L}$-definable group $G \subseteq K^n$ such that:*

(i) Every $\mathcal{L}$-generic type $p \vdash G$ is realized by some $\gamma \in \Gamma$.

(ii) There are $\mathcal{L}$-definable sets $X_1,\dots,X_r \subseteq G$, $G = \bigcup_{i=1}^r X_i$, and $\mathcal{L}$-definable functions $s_i : X_i \to K^n$ such that for each $\mathcal{L}$-generic $a \in X_i$, $a \in \Gamma$ iff $\partial a = s_i(a)$. (Recall that for $a = (a_1,\dots,a_n)$, $\partial a = (\partial a_1,\dots,\partial a_n)$.

Note that as a corollary of above we may obtain $X_1',\dots,X_k' \subseteq G$ pairwise disjoint, all of the same dimension as $\dim G$, satisfying (ii), but instead of $G = \bigcup X_i'$ we have $\dim(G\setminus \bigcup X_i') < \dim G$. Indeed, we replace the original X_i by $X_i' = X_i \setminus \bigcup_{j<i} X_j$ and remove all X_i' whose dimension is smaller than $\dim G$.

In fact, we shall prove a more precise version of Theorem 4.6.

Theorem *Assume that Γ and G satisfy (i) and (ii) of Theorem 4.7. If we endow G with its C^1-structure, then there exists an $\mathcal{L}$-definable $s : G \to \tau(G)$, such that $\Gamma = (G,s)^\partial$, where $(G,s)^\partial = \{g \in G : s(g) = \nabla(g)\}$.*

We first prove a general fact about groups in geometric structures.

Proposition 4.8 *Let G be a definable group in a geometric structure, and let $S \subseteq G$ be a definable subset. Assume that for every generic pair $(a,b) \in S \times S$, we have $a \cdot b \in S$ and for every generic $a \in S$ we have $a^{-1} \in S$.*

Then there is a definable $S_0 \subseteq S$ such that $S_0 \cdot S_0$ is a subgroup of G and S_0 is a large subset of both S and $S_0 \cdot S_0$.

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Proof We let

$$S_1 = \{s \in S : \text{ the set } \{t \in S : s \cdot t \in S \,\&\, t \cdot s \in S\} \text{ is large in } S\}.$$

By definability of dimension in geometric structures, S_1 is definable. By our assumptions, S_1 contains all generic elements of S; thus, by our assumptions, $S_0 := S_1 \cap S_1^{-1}$ is also large in S. We claim that $S_0 \cdot S_0$ is a subgroup of G.

We need to prove that for every $a, b, c, d \in S_0$, we have $abc^{-1}d^{-1} \in S_0 \cdot S_0$. We fix $g \in S$ generic over a, b, c, d, and consider

$$abc^{-1}d^{-1} = (abg)(g^{-1}c^{-1}d^{-1}).$$

Since $b \in S_0$, the set $\{t \in S : bt \in S\}$ is large in S, defined over b, and therefore contains g. Thus, bg is in S and by our choice, it is in fact generic in S over a, c, d, so in particular belongs to S_0. Thus, $a(bg) \in S$, and again generic there over c, d, so belongs to S_0. Similarly, $g^{-1}c^{-1}d^{-1} \in S_0$, so $abc^{-1}d^{-1} \in S_0 \cdot S_0$. Let $H := S_0 \cdot S_0$.

To see that S_0 is a large subset of H, we fix $g \in S_0$ and $h \in S$ generic over g (so $h \in S_0$). Then $gh, gh^{-1} \in S$ and generic there so in S_0. It follows that $g \in S_0 \cdot S_0$ and $h \in S_0^{-1} \cdot S_0 = S_0 \cdot S_0$.

Hence, $S_0 \subseteq H$ and every generic $h \in S$ over g is in H, so S_0 is large in H. ∎

We are now ready to prove Theorem 4.6.

We first apply Theorem 4.7 and the subsequent corollary and deduce the existence of pairwise disjoint $\mathcal{L}$-definable $X_1, \dots, X_r \subseteq G \subseteq K^n$, each of dimension equal to $m = \dim G$, such that $X = \bigsqcup_j X_j$ is a large subset of G and on each j, we have an $\mathcal{L}$-definable $s_j : X_j \to K^n$, such that for g generic in X_j, we have $g \in \Gamma \Leftrightarrow \partial g = s_j(g)$. We let $s : X \to K^n$ be the union of the s_j's.

Next, we apply Fact 4.1, and fix an $\mathcal{L}$-definable large $W \subseteq G$, $V \subseteq K^m$ open, $\sigma : V \to W$ a homeomorphism, and $g_1, \dots, g_k \in G$, such that the maps $\phi_i : V \to g_i W : x \mapsto g_i\sigma(x)$ endow G with a definable C^1-manifold structure, and make G into a C^1-group.

By intersecting W with the relative interior of X in G, we may assume that $W = X$.

Claim 4.9 *There exists an $\mathcal{L}$-definable $\hat{s} : V \to K^m$, such that for every $\mathcal{L}$-generic $a \in V$, $\hat{s}(a) = \partial a \Leftrightarrow s(\sigma(a)) = \partial\sigma(a)$.*

Proof Every $a \in V$ is $\mathcal{L}$-interdefinable with $\sigma(a)$, so by Lemma 3.8, $\nabla(a)$ and $\nabla(\sigma(a))$ are $\mathcal{L}$-interdefinable over k. By compactness, there exists an $\mathcal{L}$-definable (partial) bijection $h : W \times K^n \to V \times K^m$, such that for each generic $a \in V$, $h(\nabla(\sigma a)) = \nabla(a)$. Let

$$\hat{s}(a) = \pi_2(h(\sigma(a)), s(\sigma(a))),$$

where $\pi_2 : V \times K^n \to K^n$ is the projection onto the second coordinate.

Now, if $s(\sigma(a)) = \partial(a)$, then $(\sigma(a), s(\sigma(a))) = \nabla(\sigma(a))$, so

$$\hat{s}(a) = \pi_2(h(\nabla\sigma(a))) = \pi_2(\nabla(a)) = \partial a.$$

The converse follows from the invertibility of h. ∎

Going back to G, we now endow G with a finite C^1-atlas $(V_i, g_i W, \phi_i)_{i\in I}$, where $V_i = V$ for all i, and identify G with $\bigsqcup V_i/\sim_M$. We also identify Γ with the group $\bigsqcup \phi_i^{-1}(\Gamma \cap g_i W)/\sim_M$. Notice that each $g_i W/\sim_M$ is large in G, and by Claim 4.9,

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there is an $\mathcal{L}$-definable $\hat{s} : V \to K^m$ such that, for generic $a \in V$, $\hat{s}(a) = \partial a$ if and only if $s(\sigma(a)) = \partial\sigma(a)$. Thus, by our assumption, for every generic $g \in G$, $g \in \Gamma \Leftrightarrow \hat{s}(g) = \partial g$. For simplicity, from now on, we use s instead of $\hat{s}$ and let $X = dom(s)$, an $\mathcal{L}$-definable large subset of G.

Consider the $\mathcal{L}$-definable C^1-group $\tau(G)$ as before, and the associated $\mathcal{L}$-definable homomorphism $\pi : \tau(G) \to G$, together with an $\mathcal{L}_\partial$-definable group section $\nabla_G : G \to \tau(G)$. The map s can be replaced by $x \mapsto (x, s(x))$, so we may think of it as a function from X into $\tau(X) = X \times K^m$ with $\pi \circ s(x) = x$.

In addition, we still have for every generic $g \in X$, $g \in \Gamma \Leftrightarrow s(g) = \nabla_G(g)$. By our assumptions, every generic $\mathcal{L}$-type of X contains an element of Γ; hence, the $\mathcal{L}$-definable set $X_0 = \{x \in X : s(x) \in \tau(X)\}$ is large in X, so without loss of generality, $X = X_0$. Let S be the graph of $s|X_0$.

We claim that S satisfies the assumptions of Proposition 4.8: indeed, assume that (a, b) is generic in S^2. Namely, $a = (g, s(g))$ and $b = (h, s(h))$, for (g, h) generic in $X \times X$. We need to prove that $ab \in S$.

By [15, Lemma 6.7], applied to the function $(s, s) : X \times X \to \tau(X \times X)$, there exists $(x, y) \in X \times X$, realizing the same $\mathcal{L}$-type as (g, h) such that $\nabla_{G\times G}(x, y) = (s(x), s(y))$. But then, by our assumptions, $(x, y) \in \Gamma \times \Gamma$, so $xy \in \Gamma$. Because xy is still $\mathcal{L}$-generic in G, we have $xy \in X$. Thus, we have

$$s(xy) = \nabla_G(xy) = \nabla_G(x)\nabla_G(y) = s(x)s(y)$$

(where the middle equality follows from the fact that ∇_G is a group homomorphism). Since $tp_{\mathcal{L}}(x, y) = tp_{\mathcal{L}}(g, h)$, we also have $s(gh) = s(g)s(h)$, hence $ab = (gh, s(gh))$, is in S.

We similarly prove that for a generic in S, we have $a^{-1} \in S$; thus, S satisfies, indeed, the assumption of Proposition 4.8.

Hence, there exists an $\mathcal{L}$-definable $S_0 \subseteq S$, such that S_0 is a large subset of the group $H = S_0 \cdot S_0$. Since S_0 is large in H, for every generic $(g, s(g)) \in H$, we have $\pi^{-1}(g) \cap H$ is a singleton, which implies that $ker(\pi|H) = \{1\}$, and hence H is the graph of a function. Also, since the group $\pi(H)$ is large in G, it necessarily equals to G.

We therefore found an $\mathcal{L}$-definable group-section $\hat{s} : G \to \tau(G)$, making $(G, \hat{s})$ into a D-group. In addition, $x \in \Gamma \Leftrightarrow \hat{s}(x) = \nabla_G(x)$, for all x generic in G.

It is left to see that

$$\Gamma = (G, \hat{s})^\partial = \{x \in G : \hat{s}(x) = \nabla_G(x)\}.$$

Let $X_0 = \pi(S_0)$ and $\Gamma_0 = X_0 \cap \Gamma$. By the definition of S, $\Gamma = \{x \in \pi(S) : S(x) = \nabla_G(x)\}$, so $\Gamma_0 = \{x \in X_0 : \hat{s}(x) = \nabla_G(x)\}$. We claim that $\Gamma_0 \cdot \Gamma_0 = \Gamma$.

Indeed, let $\gamma \in \Gamma$, and pick g generic in X_0 over γ. By the geometric axioms, there exists $\gamma_1 \equiv_{\mathcal{L}(\gamma)} g$ such that $\hat{s}(\gamma_1) = \nabla_G(\gamma_1)$, namely $\gamma_1 \in \Gamma_0$. It follows that $\gamma \cdot \gamma_1^{-1}$ is $\mathcal{L}$-generic in G over γ and hence in X_0, namely in Γ_0. Hence, $\gamma \in \Gamma_0 \cdot \Gamma_0$.

It follows that for all $\gamma \in \Gamma$, we have $\hat{s}(\gamma) = \nabla_G(\gamma)$. To see the converse, assume that $\hat{s}(x) = \nabla_G(x)$, and choose $\gamma \in \Gamma_0$ generic over x. We then have $\hat{s}(\gamma) = \nabla_G(\gamma)$, and $x \cdot \gamma$ generic in X_0. Because $\hat{s}$ is a homomorphism,

$$\hat{s}(x\gamma) = \hat{s}(x)\hat{s}(\gamma) = \nabla_G(x)\nabla_G(\gamma) = \nabla_G(x\gamma).$$

It follows that $x\gamma \in \Gamma$ and hence so is x. This ends the proof of Theorem 4.6.

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4.2 The case of p-adically closed fields

Let K be a p-adically closed field, namely a field which is elementarily equivalent to a finite extension of $\mathbb{Q}_p$. The field admits a definable valuation, which we may add to the field language and call this language $\mathcal{L}$.

We shall use multiplicative notation for the valuation map $||: K \to \{0\} \cup vK$. Namely,

$$|0| = 0 < vK \ , ||: K^* \to (vK, \cdot) \text{ a group homomorphism}$$

and

$$\forall x, y \in K \ |x + y| \leq \max\{|x|, |y|\}.$$

For $a = (a_1, \ldots, a_n) \in K^n$, we write $\|h\| = \max\{|a_i| : i = 1, \ldots, n\}$. Since K is a geometric structure, we use the acl-dimension below.

Definition 4.10 For $U \subseteq K^m$ open, a map $f : U \to K^n$ is called *differentiable at* $a \in U$ if there exists a K-linear map $T : K^m \to K^n$ such that for all $\varepsilon \in vK$ there is $\delta \in vK$, such that for all $h \in K^m$, if $\|h\| < \delta$, then

$$\|f(a + h) - f(a) + T(h)\| < \varepsilon\|h\|.$$

The linear map T can be identified with Df_a the $n \times m$ matrix of partial derivatives of f. We identify $M_{n\times m}(K)$ with $K^{n\cdot m}$. The function f is called *continuously differentiable on* U, or C^1, if it is differentiable on U and the map $x \mapsto Df_a$ is continuous.

Differentiable maps satisfy the chain rule, by the usual proof (see, for example, [19, Remark 4.1] for a proof in $\mathbb{Q}_p$).

Toward our main result, we first note that p-adically closed fields satisfy the assumptions in Remark 4.2: indeed, these are geometric fields with a definable Hausdorff, non-discrete topology. Let us see that they admit C^1-cell decomposition (we could not find a precise reference for that in the literature).

First, one can read-off analytic cell decomposition in finite extensions of $\mathbb{Q}_p$ from Scowcroft and van den Dries [24, Sections 4 and 5]. More explicitly, the result is stated in [5, Theorem 3.3] (as mentioned there, the theorem works for the Macintyre language, as well as the subanalytic one). Since being C^1 is a definable property, one may conclude a C^1-cell decomposition for definable sets in arbitrary elementarily equivalent structures, i.e., p-adically closed fields.

Thus, as we commented in Remark 4.2, the result of Fact 4.1 holds in this setting as well and in particular, every $\mathcal{L}$-definable group admits an $\mathcal{L}$-definable C^1-manifold which makes it into a C^1-group.

We now endow K with a derivation, denoted by ∂. By Tressl's work (see [21, Theorem 7.2]), the theory of p-adically closed fields with a derivation has a model companion T_∂. In our one derivation case, (Tressl deals with several commuting derivations), one can axiomatize T_∂ with the following geometric axioms (see, for instance, [15, Fact 5.7(ii)]): whenever (V, s) is an irreducible D-variety over K with a smooth K-point and U is a Zariski open subset of V defined over K, then there is $a \in U(K)$ such that $(a, s(a)) = \nabla(a)$. (Recall that a D-variety (V, s) defined over K

is a K-variety V equipped with a rational section s defined over K from V to $T(V)$ [15, Definition 2.4].)

Now, exactly as in the work of Fornasiero and Kaplan for real closed fields [6, Lemmas 2.4 and 2.7 and Proposition 2.8], every nontrivial derivation is compatible with the theory of p-adically closed fields, namely compatible with every $\mathcal{L}(\emptyset)$-definable C^1 map, as in Definition 3.1. We briefly review the details.

As in [6, Lemma 2.4], it is enough to verify compatibility of $\emptyset$-definable C^1-functions in neighborhoods of $acl_{\mathcal{L}}$-independent points in K^n (we use here the fact that $acl_{\mathcal{L}} = dcl_{\mathcal{L}}$). By quantifier elimination, the graph of every $\mathcal{L}(\emptyset)$-definable function $g: U \to K$, for $U \subseteq K^n$ open, is given implicitly by $\{(x,y) : x \in U \& f(x,y) = 0\}$, for $f(x,y)$ an irreducible polynomial over $\mathbb{Z}$. Now, if $a \in U$ is $\mathcal{L}$-generic over $\emptyset$, then $(\partial f/\partial y)(a, g(a)) \neq 0$ and as in [6, Lemma 2.7], ∂ is compatible with g.

In order to develop the rest of the theory as in the o-minimal case, we prove in the Appendix (see Proposition A.1) that definable functions in p-adically closed fields satisfy the analogue of Fact 3.4.

Proposition 4.11 *Given an $\mathcal{L}(\emptyset)$-definable $W \subseteq K^n \times K^m$ and an $\mathcal{L}(\emptyset)$-definable $g : W \to K$, if $(a,b) \in W$, $\dim(b/\emptyset) = m$, W^b is open and $g(x,b)$ is a C^1-function on W^b, then $(a,b) \in Int(W)$ and g is a C^1-function at (a,b).*

Now, the category of K-differentiable manifolds M and their associated functors T and τ can be developed identically to Sections 1 and 2. This allows us to associate to every definable group G the definable groups $T(G)$ and $\tau(G)$, such that the natural projections onto G are group homomorphisms. If G is a C^1-group, then $T(G)$ and $\tau(G)$ are C^0-groups.

By a p-adic D-group, we mean a pair (G,s) where G is an $\mathcal{L}$-definable C^1-group and $s : G \to \tau(G)$ an $\mathcal{L}$-definable homomorphic section (i.e., $\pi \circ s = id$).

As before, we define in models of T_∂, given a D-group (G,s),

$$(G,s)^\partial = \{g \in G : s(g) = \nabla_G(g)\}.$$

In order to prove our main theorem in the p-adically closed field, we let T be the theory of p-adically closed fields and let Γ be a finite-dimensional $\mathcal{L}_\partial$-definable group in $K \models T_\partial$. Since p-adically closed fields are large geometric fields, we may apply [15, Theorem 5.11] to conclude that Theorem 4.7 holds in this setting as well. Namely, Γ embeds into an $\mathcal{L}$-definable group G, with the additional $\mathcal{L}$-definable X_i's as in the theorem. Now we repeat word-for-word the proof in the o-minimal setting (see also Remark 4.2) to conclude the following theorem.

Theorem 4.12 *Let T be the theory of p-adically closed fields, and let Γ be a finite-dimensional L_∂-definable group in $K \models T_\partial$. Then there exists an $\mathcal{L}$-definable D-group (G,s) such that Γ is definably isomorphic to $(G,s)^\partial$.*

4.3 The case of pseudo-finite fields

Let $\mathcal{L}$ be the language of rings, and let $C = (c_{i,n})_{n\in\mathbb{N}, i<n}$ be an infinite countable set of new constants. Let T be the $\mathcal{L}(C)$-theory of pseudo-finite fields of characteristic 0, namely the theory of pseudo-algebraically closed fields plus the scheme of axioms

saying, for every $n \in \mathbb{N}$, that there is a unique extension of degree n, and that the polynomial

$$X^n + c_{n-1,n}X^{n-1} + \cdots + c_{0,n}$$

is irreducible.

Since T is a model-complete theory of large fields, one can apply the Tressl machinery and so the theory of differential expansions of models of T has a model-companion [21, Corollary 8.4], which has been axiomatized [21, Theorem 7.2] (in case of expansions by a single derivation, one obtains a geometric axiomatization [3, Lemma 1.6]). Recall that since T has almost q.e. (see [3, Remark 1.4(2)]), the theory T_∂ does too [3, Definition 1.5 and Lemma 2.3], [21, Theorem 7.2(iii)].

Let $\mathcal{U}$ be our sufficiently saturated model of T_∂, a differential expansion of a pseudo-finite field, and let $\bar{\mathcal{U}} \supseteq \mathcal{U}$ be a saturated model of DCF_0 extending it. We work over a small submodel $(K, \partial) \models T_\partial$.

We briefly review the construction of the algebraic prolongation $\tau(V) \subseteq \bar{\mathcal{U}}^n \times \bar{\mathcal{U}}^n$ of an irreducible algebraic variety (see [14] for details).

Assume that the ideal $I(V)$ is generated by polynomials $p_1, \ldots, p_m$, over K, and let $P : \bar{\mathcal{U}}^n \to \bar{\mathcal{U}}^m$ be the corresponding polynomial map $P(x) = (p_1(x), \ldots, p_m(x))$. The definition of DP and $\tau(P)$ is defined as before using the formal derivative of polynomials (see also Remark 3.7). Then

$$T(V) = \{(x,u) \in \bar{\mathcal{U}}^{2n} : P(x) = 0 \,\&\, (DP)_x \cdot u = 0\}$$

and

$$\tau(V) = \{(a,u) \in \bar{\mathcal{U}}^{2n} : a \in V \,\&\, \tau(P)(a,u) = 0\}.$$

Both are algebraic varieties over K. For $a \in V(\bar{\mathcal{U}})$, $a \mapsto \partial a$ is a section of $\pi : \tau(V) \to V$, and we have

$$\tau(V) = \{(a,u) \in \bar{\mathcal{U}}^{2n} : a \in V \,\&\, u - \partial a \in T(V)_a\}.$$

So, $\tau(V)_a$ is an affine translate of the vector space $T(V)_a \subseteq \bar{\mathcal{U}}^n$. In particular, $\dim(\tau(V)_a) = \dim V$.

As described in [14], the above constructions of $T(V)$ and $\tau(V)$ can be extended to abstract, not necessarily affine, algebraic varieties (which are covered by finitely many affine algebraic varieties). Furthermore, if H is an algebraic group, then $T(H)$ and $\tau(H)$ are algebraic groups with the property that the map ∇_H is now a group morphism [14, Section 2].

Our goal is to prove the following theorem.

Theorem 4.13 *Let T be the theory of pseudo-finite fields, and let Γ be a finite-dimensional definable group in $K \models T_\partial$. Then there exists a K-algebraic D-group (H, s) such that Γ is virtually definably isogenous over K to the K-points of $(H, s)^\partial$.*

By "Γ and $(H,s)^\partial(K)$ are virtually isogenous," we mean the following: there exist an $\mathcal{L}_\partial$-definable subgroup $\Gamma_0 \subseteq \Gamma$ of finite index and an $\mathcal{L}_\partial$-definable homomorphism $\sigma : \Gamma_0 \to H$ with finite kernel, whose image has finite index in the K-points of $(H,s)^\partial$.

We first need the following lemma.

Lemma 4.14 *Let W_1, W_2 be irreducible algebraic varieties over a differential field K, and a, b generic tuples in W_1 and W_2, respectively, over K. Suppose that a and b are field-theoretically interalgebraic over K (in particular,* $\dim W_1 = \dim W_2$*), and let $W \subseteq W_1 \times W_2$ be the (irreducible) variety over K with generic point (a, b). Then $\tau(W)_{(a,b)}$ is the graph of a bijection over $K(a, b)$, $\alpha : \tau(W_1)_a \to \tau(W_2)_b$.*

Proof This basically follows from the fact that the projection from W to W_1, W_2 is generically étale. However, for completeness, we include a proof. By dimension considerations, the affine space $\tau(W)_{(a,b)}$ projects onto both $\tau(W_1)_a$ and $\tau(W_2)_b$.

Fix some coordinate a_1 of the tuple a. By our interalgebraicity assumption, a_1 is in the field-theoretic algebraic closure of $K(b)$. Let $q(x)$ be the minimal monic polynomial of a_1 over $K(b)$. $q(x) = x^n + f_{n-1}(b)x^{n-1} + \cdots + f_1(b)x + f_0(b)$, where the $f_i(b)$ are K-rational functions of the tuple b. After getting rid of denominators, we can rewrite $q(x)$ as $q_n(b)x^n + q_{n-1}(b)x^{n-1} + \cdots + q_1(b)x + q_0(b)$ where the q_i are polynomials over K.

Hence, $r(x, y)$: $q_n(y)x^n + q_{n-1}(y)x^{n-1} + \cdots + q_0(y)$ is a polynomial (over K) in $I_K(W)$.

Hence, $(\partial r/\partial x)(a_1, b)(u_1) + \sum_j (\partial r/\partial y_j)(a_1, b)(v_j) + r^{\partial}(a_1, b) = 0$ for $(u_1, \ldots, v_1, \ldots, v_j, \ldots) \in \tau(W)_{a,b}$.

By the minimality of $r(x, b)$, we have $\partial r/\partial x(a_1, b) \neq 0$, and hence

$$u_1 = \left(\sum_j (\partial r/\partial y_j)(a_1, b)(v_j) + r^{\partial}(a_1, b))/(\partial r/\partial x)(a_1, b) \right).$$

Thus, $u_1 \in dcl(K, a, b, v)$. We similarly prove that u and v are inter-definable over $K(a, b)$. Since $\tau(W)_{(a,b)}$ is a translate of a linear space, containing (u, v) whose dimension equals $\dim W_1 = \dim W_2$, it must be the graph of a bijection. ■

We now return to the proof of Theorem 4.13.

By [15, Theorem 5.11], there exist an $\mathcal{L}$-definable group G and a definable group embedding : $\sigma : \Gamma \to G$. Furthermore, every generic $\mathcal{L}$-type of G is realized by an element in $\sigma(\Gamma)$ and, in addition, there is a covering of G by finitely many $\mathcal{L}(K)$-definable sets X_i, $i = 1, \ldots, m$, and for each X_i, there is a K-rational function $s_i : X_i \to \mathcal{U}^n$ such that for every $a \in X_i(\mathcal{U})$ which is $\mathcal{L}$-generic in X_i over K, we have

$$a \in \sigma(\Gamma) \Leftrightarrow \partial(a) = s_i(a).$$

For simplicity, we assume now that $\sigma = id$, so $\Gamma \subseteq G$.

We may take each X_i to be Zariski dense in a K-variety V_i. We are only interested in those V_i whose Zariski dimension is maximal, call it d, so in particular, every algebraic type in V_i over K, of dimension d, is realized in X_i in $\mathcal{U}$, so by the axioms also realized by some $a \in X_i(\mathcal{U})$ with $\partial a = s_i(a)$, and hence, by the above, also realized in Γ. Finally, each X_i can be taken to be the $\mathcal{U}$-points of $W_i = Reg(V_i) := V_i \backslash Sing(V_i)$, namely the $\mathcal{U}$-points of a smooth quasi affine K-variety.

We now apply [9, Theorem C] in the structure $\mathcal{U}$: there exist a connected algebraic group H over K, $\mathcal{L}$-definable subgroups of finite index, $G_0 \subseteq G$ and $H_0 \subseteq H(\mathcal{U})$, and an $\mathcal{L}$-definable surjective homomorphism $f : G_0 \to H_0$ whose kernel is finite, all defined

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over K. The group H, as an algebraic group over K, has an associated K-algebraic group $\tau(H)$. Notice that $acl_{\mathcal{L}}$ equals the field acl and we have $\dim H = d$.

Let $\Gamma_0 = \Gamma \cap G_0$, a subgroup of Γ of finite index. Since $f(G_0)$ is Zariski dense in H, so is $f(\Gamma_0)$. As we shall now see, we can endow H with the structure of a D-group, (H, s) such that

$$f(\Gamma_0) = \{h \in H_0 : s(h) = \nabla_H(h)\}.$$

Claim 4.15 *For every $g \in \Gamma_0$, $tr.deg(\nabla_H f(g)/K) \le \dim(H)$.*

Proof We first prove the result for $g \in \Gamma_0$ such that $tr.deg(g/K) = d$.

Since $tr.deg(g/K) = d$, there exists a K-algebraic quasi-affine variety W_i as above such that g is generic in W_i over K. The $\mathcal{L}$-definable function f takes values in H, and because $acl_{\mathcal{L}}$ is the same as the field acl, there exists an algebraic correspondence $C_i \subseteq W_i \times H$ over K, such that $(g, f(g))$ is field-generic in C_i. It follows that $(\nabla_{W_i}(g), \nabla_H f(g)) \in \tau(C_i) \subseteq \tau(W_i) \times \tau(H)$.

By Lemma 4.14, $\tau(C_i)_{(g,f(g))}$ induces an (algebraic) bijection over K between $\tau(W_i)_g$ and $\tau(H)_{f(g)}$. In particular, $\nabla_{W_i}(g)$ and $\nabla_H f(g)$ are interalgebraic over K. By our construction, $\nabla_{W_i}(g) = s_i(g)$; hence, g and $\nabla_H f(g)$ are interalgebraic over K (notice that g is algebraic over $\nabla_{W_i}(g)$). Hence, $tr.deg(\nabla_H f(g)/K) = tr.deg(g/K) = d$.

Assume now that g is an arbitrary element of Γ_0, and let $h \in \Gamma_0$ be such that $tr.deg(h/K, g, \nabla_H f(g)) = d$.

Since $tr.deg(hg/K) = d$ and $hg \in \Gamma_0$, it follows from the above that

$$d = tr.deg(\nabla_H f(hg)/K)) = tr.deg(\nabla_H f(h) \cdot \nabla_H f(g)/K),$$

and therefore

$$tr.deg(\nabla_H f(h) \cdot \nabla_H f(g)/\nabla_H f(h), K) \le d.$$

The elements $\nabla_H f(h) \cdot \nabla_H f(g)$ and $\nabla_H f(g)$ are interalgebraic over K and $\nabla_H f(h)$, and thus $tr.deg(\nabla_H f(g)/\nabla_H f(h), K) \le d$.

We know that h and $\nabla_{X_i}(h)$ (and hence also $\nabla_H f(h)$) are interalgebraic over K (as witnessed by s_i), and as h and $\nabla_H(f(g))$ are independent over K, it follows that $\nabla_H f(h)$ and $\nabla_H f(g)$ are independent over K. Therefore,

$$tr.deg(\nabla_H f(g)/K) = tr.deg(\nabla_H f(g)/\nabla_H f(h), K) \le d. \qquad \blacksquare$$

We now consider the subgroup $\nabla_H f(\Gamma_0)$ of $\tau(H)$ and let $S \subseteq \tau(H)$ be its Zariski closure, an algebraic subgroup of $\tau(H)$. By the claim above, $\dim(S) \le \dim H$, but since S contains $\nabla_H f(h)$ for $\mathcal{L}$-generic $h \in \Gamma_0$, we have $\dim(S) = \dim(H)$. Consider the projection $\pi : \tau(H) \to H$, a group homomorphism, and its restriction to S. Since H is connected, we have $\pi(S) = H$, and hence $ker(\pi) \cap S$ is a finite subgroup of $\tau(H)_e$. However, $\tau(H)_e = T(H)_e$ is a vector space over K, a field of characteristic 0, thus torsion-free. Hence, $ker(\pi) \cap S$ is trivial, so $\pi : S \to H$ is a group isomorphism. It follows that S can be viewed as a group section $s : H \to \tau(H)$. Since S is the Zariski closure of $\nabla_H(f(\Gamma_0))$, we have for every $g \in \Gamma_0$, $\nabla_H(f(g)) = s(f(g))$.

Recall that $H_0 = f(G_0)$ is an $\mathcal{L}$-definable subgroup of finite index of $H(\mathcal{U})$.

Claim 4.16 $f(\Gamma_0) = (H_0, s)^{\partial} = \{h \in H_0 : \nabla_H(h) = s(h)\}$.

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Proof We only need to prove the $\supseteq$ inclusion.

We first prove that for every $h \in (H_0, s)^\partial$, if $tr.deg(h/K) = d$, then $h \in f(\Gamma_0)$. Indeed, since $h \in H_0$ is $\mathcal{L}$-generic in H over K, there exists $g \in G_0$, necessarily $\mathcal{L}$-generic in G over K, such that $h = f(g)$. By our assumptions, there exists $g' \in \Gamma$, such that g' and g realize the same $\mathcal{L}$-type over K. Thus, g' is in G_0 so also in Γ_0. In addition, $f(g')$ and $f(g) = h$ must realize the same $\mathcal{L}$-type over K. Because S is the Zariski closure of $\nabla_H f(\Gamma_0)$, it follows that $f(g') \in (H_0, s)^\partial$.

Since $\nabla_H(h) = s(h)$ and $\nabla_H(f(g')) = s(f(g'))$, it follows that for every $n \in \mathbb{N}$, there is an $\mathcal{L}(K)$-definable function s_n such that

$$\nabla^n_H(h) = s_n(h)\,,\ \nabla^n_H(f(g')) = s_n(f(g'))$$

(recall that $\nabla^n_H(g) = (g, \partial g, \ldots, \partial^n g)$). Because h and $f(g')$ realize the same $\mathcal{L}$-type over K, we may conclude that for every n, $\nabla^n_H(h)$ and $\nabla^n_H(f(g')))$ realize the same $\mathcal{L}$-type over K, and therefore

$$\mathrm{tp}_{\mathcal{L}(K)}(\nabla^\infty_H(h)) = \mathrm{tp}_{\mathcal{L}(K)}(\nabla^\infty_H(f(g'))).$$

$(\nabla^\infty_H(g) = (g, \partial g, \ldots, \partial^n g, \ldots).)$

By [21, 7.2(iii)], every $\mathcal{L}_\partial(K)$-formula is equivalent in T_∂ to a Boolean combination of formulas of the form $(x, \partial x, \ldots, \partial^n x) \in Y$, where Y is an $\mathcal{L}(K)$ definable set. Thus, it follows from the above that h and $f(g')$ realize the same $\mathcal{L}_\partial$-type over K, and therefore $h \in f(\Gamma_0)$, as needed.

This proves that every $h \in (H_0, s)^\partial$ with $tr.deg(h/K) = d$ belongs to $f(\Gamma_0)$. However, every $h \in (H_0, s)^\partial$ can be written as $h = h_1 h_2$, with $h_1, h_2 \in (H_0, s)^\partial$ and $tr.deg(h_1/K) = tr.deg(h_2/K) = d$. Indeed, pick $h_1 \in (H_0, s)^\partial$ with $tr.deg(h_1/hK) = d$ and $h_2 = h_1^{-1}h$). Thus, every $h \in (H_0, s)^\partial$ belongs to $f(\Gamma_0)$. This ends the proof of Theorem 4.13. ∎

A Appendix

We fix K a p-adically closed field. All definability below is in the language $\mathcal{L}$ of K. Our goal is to prove the following p-adic analogue of Fornasiero–Kaplan's theorem (see [6, A.3]).

Proposition A.1 *Let K be a p-adically closed field. Assume that $g : W \to K^r$ is an $\mathcal{L}(A)$-definable partial function on some definable $W \subseteq K^n \times K^m$, and $b \in \pi_2(W) \subseteq K^m$ is $acl_{\mathcal{L}}$-independent over A.*

If $W^b = \{a \in K^n : (a, b) \in W\}$ is open and $g(x, b)$ is a C^1-map on W^b, then for every $a \in W^b$, $(a, b) \in Int(W)$ and the function g is a C^1-map (of all variables) in a neighborhood of (a, b).

We shall use the following three important properties of p-adically closed fields (as well as o-minimal structures and some other geometric structures).

Fact A.2 *Fix $A \subseteq K^{eq}$.*

1. *Given $a \in K^m$ and $b \in K^n$, if $U \ni a$ is a (definable) open set in K^m, then there exists a definable open V, $a \in V \subseteq U$, such that* $\dim(b/A[V]) = \dim(b/A)$ *(we use $[V]$ for the canonical parameter of V) (see [7, Corollary 3.13] or [11, Lemma 4.30]).*

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2. *Assume that $X \subseteq K^{m+n}$ is definable over A, $a \in K^m$, $b \in K^n$, and $(a,b) \in X$. Assume further that $X^b = \{x \in K^m : (x,b) \in X\}$ is finite. Then there exists a definable open $W \ni (a,b)$ (possibly over additional parameters) such that $X \cap W$ is the graph of a definable map from K^m to K^n (this follows from cell decomposition in p-adically closed fields).*
3. *If $U \subseteq K^m$ is open and $f : U \to K^n$ is an A-definable map, then f is C^1 at every $a \in U$ with $\dim(a/A) = m$ (see [24]).*

An immediate corollary of the first two is the following.

Fact A.3 *For $a \in K^m$ and $b \in K^n$ and $A \subseteq K^{eq}$, if $a \in acl(b,A)$, then there exists $A_1 \supseteq A$ such that $a \in dcl(bA_1)$ and $\dim(b/A_1) = \dim(b/A)$.*

Proof of Proposition A.1 We first prove a continuous version.

Lemma A.4 *Assume that $g : W \to K^r$ is an $\mathcal{L}(A)$-definable partial function on some definable $W \subseteq K^m \times K^n$, and $b \in \pi_2(W) \subseteq K^n$ is $acl_{\mathcal{L}}$-independent over A. If $W^b \subseteq K^m$ is open and $g(x,b)$ is continuous on W^b, then $(a,b) \in Int(W)$ and for every $a \in X^b$, the function g is continuous at (a,b).* ■

Proof We need the following claim.

Claim A.5 *Assume that $X \subseteq K^m \times K^n$ is an A-definable set, $(a,b) \in X$, and b is $acl_{\mathcal{L}}$-independent in K^n over A. If $a \in Int(X^b)$, then $(a,b) \in Int(X)$.* ■

Proof Applying Fact A.2(1), there is a definable open $V \ni a$ such that $a \in V \subseteq X_b$ and $\dim(b/A[V]) = \dim(b/A)$.

Because b is generic in K^n over $A[V]$, it remains generic in $Y = \{b' \in K^n : V \subseteq X_{b'}\}$. It follows that $\dim(Y) = n$ and $b \in Int(Y)$, so $(a,b) \in V \times Int(Y) \subseteq Int(X)$. ■

To prove the lemma, let $V \subseteq K^r$ be an open neighborhood of $g(a,b)$. By Fact A.2(1), we may replace V by V_1, $g(a,b) \ni V_1 \subseteq V$, with b generic in K^n over $A[V_1]$. Consider the set $X = \{(x,y) \in W : f(x,y) \in V_1\}$. We need to see that $(a,b) \in Int(X)$. Since $f(x,b)$ is continuous at a, we have $a \in Int(X^b)$, and hence by the above claim, $(a,b) \in Int(X)$.

We now return to the proof of Proposition A.1. Just like in [6], we first reduce to the case where $a = 0 \in K^m$ and $g(0,y) \equiv 0$.

After permuting a, we may write it as $a = (a_1, a_2)$ where (a_2, b) is acl-independent over A and $a_1 \in acl(a_2 bA)$. Since $W^b \subseteq K^m$ is open, the set $W^{(a_2,b)}$, obtained by fixing additional parameters, is open in $K^{m-|a_2|}$. Similarly, $x_1 \mapsto f(x_1, a_2, b)$ is still C^1 at a_1 (since $f(x,b)$ was C^1 at a). Thus, by replacing b with (a_2, b) and a with a_1, we may assume that $a \in acl(bA)$. By Fact A.3, we may add parameters to A while preserving the genericity of b, such that $a \in dcl(bA)$. We still use A for this new parameter set. Thus, $b = \alpha(a)$ for an A-definable function α. Since b is generic in $dom(\alpha)$, then α is continuously differentiable at b. Without loss of generality, $dom(\alpha) = \pi_1(W)$.

Consider the local C^1-diffeomorphism $\bar{\alpha} : (x,y) \mapsto (x - \alpha(y), y)$. It sends W to a set $\bar{W}$ and (a,b) to $(0,b)$, so by Fact A.5, $(0,b) \in Int(\bar{W})$. The pushforward of g via $\bar{\alpha}$ is $\bar{g}(x,y) = g(x + \alpha(y), y)$. The map $\bar{g}(x,b)$ is still C^1 on $\bar{W}^b$, and it is sufficient to

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prove that $\bar{g}$ is C^1 at $(0, b)$. So, we may replace g with $\bar{g}$, W with $\bar{W}$, and (a, b) with $(0, b)$. We still use g and W for the sets. Finally, since b is generic in K^n, it follows from Fact A.2(3) that the function $g(0, y)$ is C^1 in a neighborhood of b, so we may replace g with $g(x, y) - g(0, y)$, and thus assume that $g(0, y) \equiv 0$, and in particular, $D_y g_{(0,b)} = 0$, so $Dg_{(0,b)} = (D_x g_{(0,b)}, 0) \in M_{r\times(m+n)}(K)$.

To simplify notation below, we view $x \in K^n$ both as a row and a column vector, depending on context. Thus, for, say, $(x, y) \in K^m \times K^n$, we write $Dg_{(a,b)} \cdot (x, y)$, instead of $Dg_{(a,b)}(x, y)^t$.

Notice that in order to show that g is differentiable at $(0, b)$, we need to show that for every $\varepsilon \in \nu K$, the point $(0, b)$ belongs to the interior of the set of $(x, y) \in K^m \times K^n$, such that

$$\|g(x, y) - g(0, b) - Dg_{(0,b)} \cdot (x - 0, y - b)\| < \varepsilon\|(x, y - b)\|,$$

which, since $g(0, b) = 0$ and $D_y g_{(0,b)} = 0$, equals

$$\left\{(x, y) \in K^m \times K^n : \|g(x, y) - D_x g_{(0,b)} \cdot x\| < \varepsilon\|(x, y - b)\|\right\}. \tag{A.1}$$

We fix $\varepsilon \in \nu K$. By our assumption that $g(x, b)$ is differentiable at 0, it follows that 0 is in the interior of

$$\left\{x \in K^m : \|g(x, b) - D_x g_{(0,b)} \cdot x\| < \varepsilon\|x\|\right\}.$$

By Claim A.5, $(0, b)$ is in the interior of

$$\left\{(x, y) \in K^m \times K^n : \|g(x, y) - D_x g_{(0,y)} \cdot x\| < \varepsilon\|x\|\right\};$$

hence, there exists $\delta_1 \in \nu K$ such that if $\|(x, y - b)\| < \delta_1$, then

$$\|g(x, y) - D_x g_{(0,y)} \cdot x\| < \varepsilon\|x\|.$$

In order to prove that $(0, b)$ is in the interior of the set in (A.1), we write

$$g(x, y) - D_x g_{(0,b)} \cdot x = g(x, y) - D_x g_{(0,y)} \cdot x + (D_x g_{(0,y)} - D_x g_{(0,b)}) \cdot x. \tag{A.2}$$

Claim A.6 *There is $\delta_2 \in \nu K$, such that for all $(x, y) \in K^m \times K^n$, if $\|y - b\| < \delta_2$, then*

$$\|(D_x g_{(0,y)} - D_x g_{(0,b)}) \cdot x\| < \varepsilon\|x\|.$$

Proof We first observe that for every $A = (a_{i,j}) \in M_n(K)$, if for all $i, j, |a_{i,j}| < \varepsilon$, then for all $x \in K^n$, we have $\|A \cdot x\| < \varepsilon\|x\|$.

Consider the map $G : K^n \to M_{r\times n}(K)$, given by $G(y) = D_x g_{(0,y)}$ (we identify the space on the right with K^{rn}). It is definable over A and hence continuous at b. Thus, there exists $\delta_2 \in \nu K$ such that whenever $\|y - b\| < \delta_2$, then $\|G(y) - G(b)\| = \|D_x g_{(0,y)} - D_x g_{(0,b)}\| < \varepsilon$. The result follows from our above observation. ∎

If we now take $\delta = \min\{\delta_1, \delta_2\}$, for δ_2 as in the above claim, then for all $(x, y) \in K^m \times K^n$ with $\|(x, y - b)\| < \delta$, we have, using (A.2),

$$\|g(x, y) - D_x g_{(0,b)} \cdot x\| \le \max\{\|g(x, y) - D_x g_{(0,y)} \cdot x\|, \|(D_x g_{(0,y)} - D_x g_{(0,b)}) \cdot x\|\}$$

$$< \varepsilon\|x\| \le \varepsilon\|(x, y - b)\|.$$

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This ends the proof that $g(x, y)$ is differentiable at (a, b). Since $a \in W^b$ was arbitrary, it follows that for all $x \in W^b$, $g(x, y)$ is differentiable at (x, b). Consider now the map $G : (x, y) \mapsto Dg_{(x,y)}$. Since $g(x, b)$ is C^1 on W^b, the map $G(x, b)$ is continuous on W^b, and therefore by Lemma A.4, G is continuous at (a, b). Thus, g is C^1 at (a, b). This ends the proof of Proposition A.1.

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